EC9D3 Advanced Microeconomics, Part I: Lecture 9

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Interpersonal comparisons

- We will assume that all social choice functions f satisfy welfarism, i.e. U, P and IIA, and continuity.
- Hence there is a continuous function W such that: $V(x) > V(y)$ if and only if $W(u(x)) > W(u(y))$.
- The welfare function depends only on the utility ranking, not on how the ranking comes about.

Interpersonal comparisons (2)

- Under Arrow axiom, utilities are measured along an ordinal scale, and are non-comparable across individuals.
- Specifically, the function W aggregates the preferences $(u_i)_{i=1,\ldots,n}$ if and only if W aggregates the preferences $(v_i(u_i))_{i=1,\ldots,n,}$ for all increasing transformation $v_i(u_i)$, for any i, independently across i.
- We modify the framework to allow for cardinal comparisons of utility, and comparability across individuals.

Interpersonal comparisons (2)

- Suppose that preferences are fully comparable but measured on the ordinal scale.
- The social ranking V must be invariant to arbitrary, but common, increasing transformations v_i applied to every individual's utility function u_i.
- Specifically, the function W aggregates the preferences $(u_i)_{i=1,\ldots,n}$ if and only if W aggregates the preferences $(v_i(u_i))_{i=1,\ldots,n}$, for all increasing transformation $v_i(.)$, such that v_i' is constant across i.

Interpersonal comparisons (3)

- Suppose that preferences are fully comparable and measured on the cardinal scale.

- The social ranking V must be invariant to increasing, linear transformations $v_i(u_i) = a_i + bu_i$, where b is common to every individual.

- Specifically, the function W aggregates the preferences $(u_i)_{i=1,...,n}$ if and only if W aggregates the preferences $(v_i(u_i))_{i=1,...,n}$, for all transformation $v_i(.)$, such that $v_i(u_i) = a_i + bu_i$, with $b > 0$.

Rawlsian Form

- HE: Hammond Equality. Let u and u' be two distinct utility vectors. Suppose that $u_k = u'_k$ for all k other than i and j. If $u_i > u'_i > u'_j > u_j$, then $W(u') > W(u)$.
- Condition HE states that the society has a preference towards decreasing the dispersion of utilities across individuals.
- AN: The social rule W is anonymous if for every permutation p, $W(u_1, ..., u_N) = W(u_{p(1)}, ...u_{p(N)})$.

Rawls Theorem

Theorem

Suppose that preferences are fully comparable and measured on the ordinal scale. The social welfare function W satisfies Weak Pareto, Anonymity and Hammond Equality if and only if it takes the Rawlsian form

$$W(u) = \min\{u_1, ..., u_N\}.$$

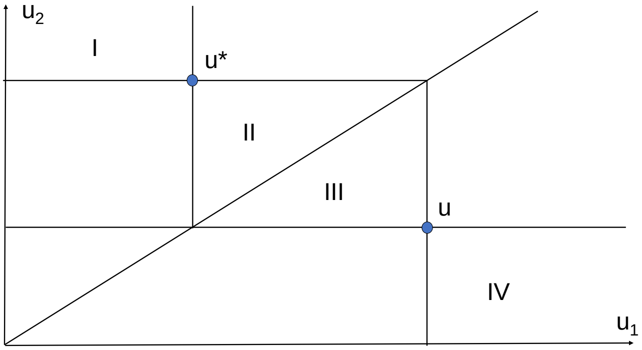
Proof. It is easy to see that the function $W(u) = \min\{u_1, ..., u_N\}$ satisfies Weak Pareto, Anonymity and Hammond equality.

To show the converse, we will see only the case for $N = 2$.

Rawls Theorem (2)

Consider a utility index u, with $u_1 > u_2$.

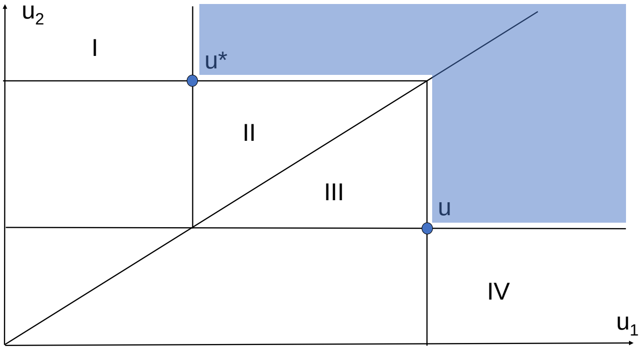
Let u^* be such that $u_1^* = u_2$ and $u_2^* = u_1$.



Rawls Theorem (3)

By anonymity, the utility profile u^* must be ranked in the same way as u: hence $W(u) = W(u^*)$.

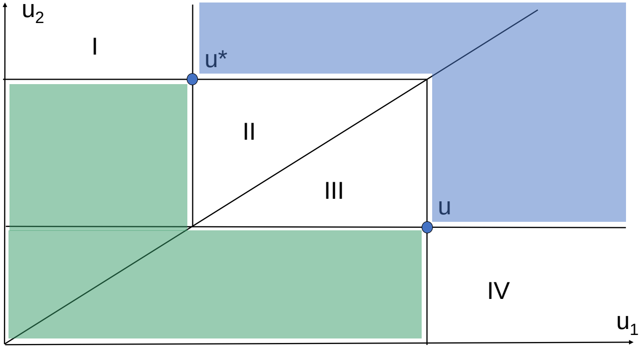
By Weak Pareto, all u' such that $u' > u$ or $u' > u^*$ must be such that $W(u') > W(u)$. The whole area in blue is s.t. $W(u') > W(u)$.



Rawls Theorem (4)

By Weak Pareto, all u' such that $u' < u$ or $u' < u^*$ must be such that $W(u') < W(u)$.

Hence the whole area in green is such that $W(u') < W(u)$.

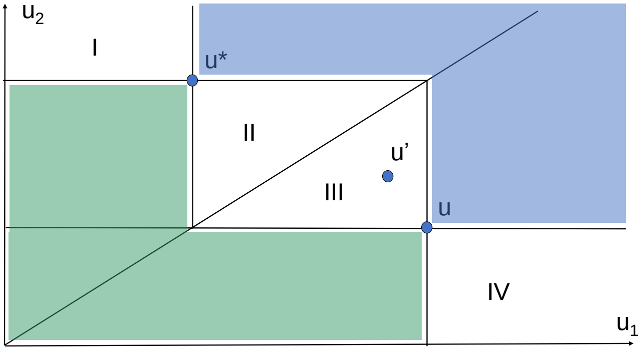


Rawls Theorem (5)

Pick a point u' in zone III. To be in III, it must be that $u_2 < u'_2 < u'_1 < u_1$.

Every linear transform v such that $v_i(u_i) = u_i$ yields:
$u_2 < v_2(u'_2) < v_1(u'_1) < u_1$.

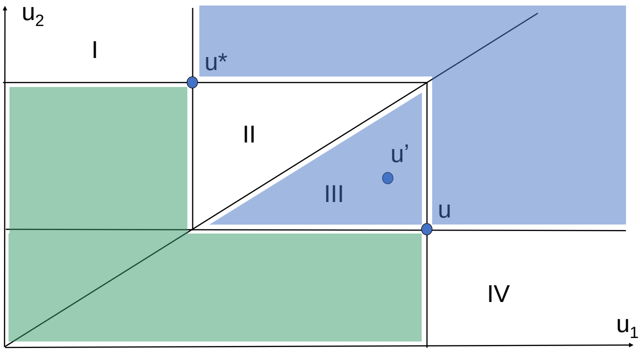
This conclude that all points in III are ranked the same way wrt to u.



Rawls Theorem (6)

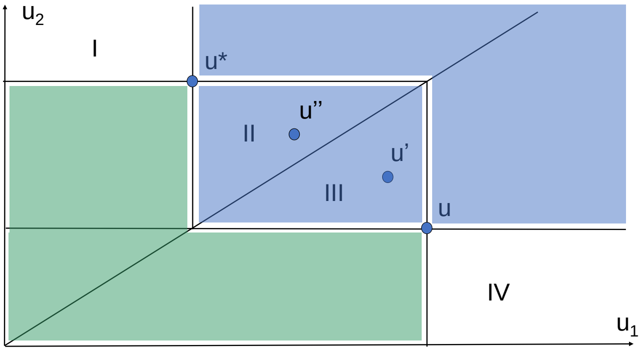
To be in **III**, it must be that $u_2 < u_2' < u_1' < u_1$

Hammond Equality implies that $W(u') > W(u)$.



Rawls Theorem (7)

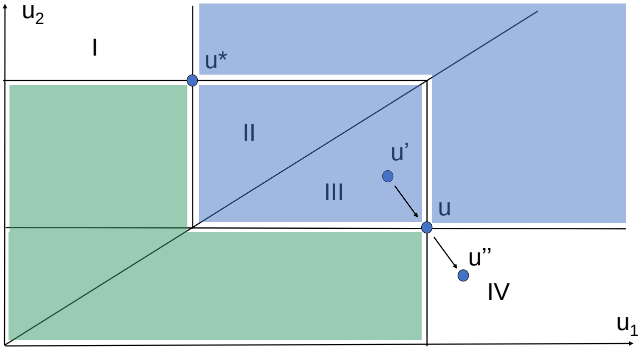
By anonymity, the ranking of each u' in III relative to u must be the same as the ranking of any utility vector u'' in II: $W(u'') > W(u)$.



Rawls Theorem (8)

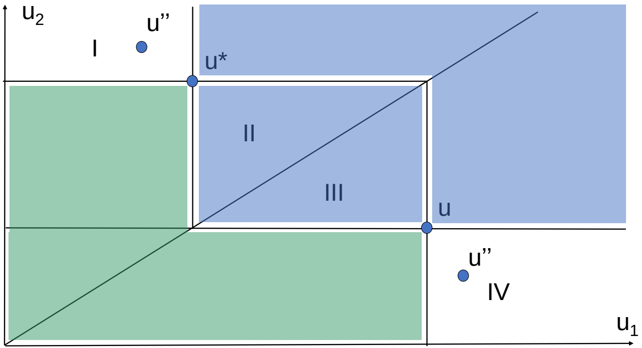
Any linear transform v such that $v_1(u_1') = u_1$, $v_2(u_2') = u_2$, yields:
$W(v(u)) < W(v_1(u_1'), v_2(u_2')) = W(u)$.

Hence all the utility vectors u'' in IV are ranked opposite to all utility vectors u' in III, relative to u.



Rawls Theorem (9)

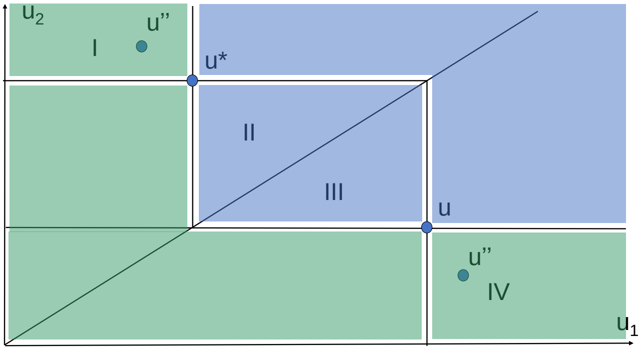
Hence $W(u) > W(u'')$ for all u'' in IV, and, by anonymity, $W(u) > W(u'')$ for all u'' in I.



Rawls Theorem (10)

We conclude that $V(II)$ and $V(III) > W(u) > V(I)$ and $V(IV)$.

We are left to consider the boundaries of these sets.

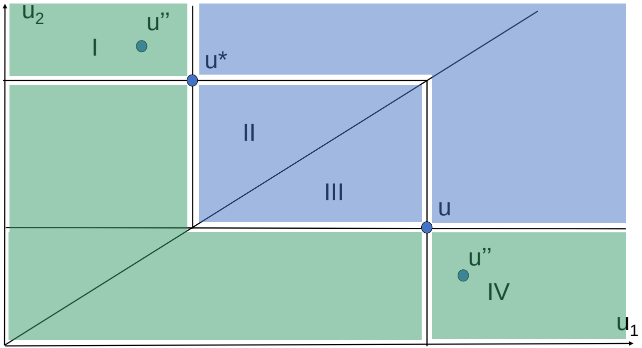


Rawls Theorem (11)

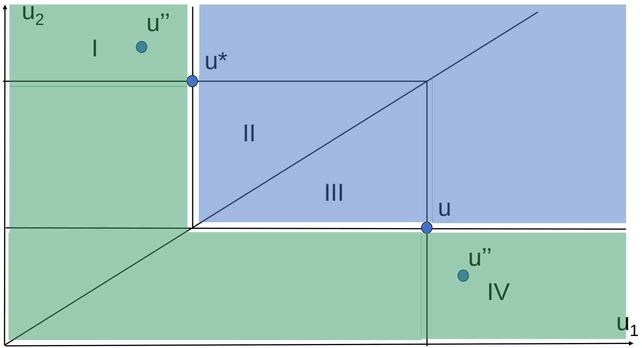
Because W is continuous, the boundaries opposite to each other, relative to u must be indifferent to u'.

Boundaries between II and III and blue set must be better than u for W.

Boundaries between I and IV and the green set must be indifferent to u.



We have obtained the Rawlsian indifference curves, where $W(u) = \min\{u_1, u_2\}$. □



Utilitarian Form

Theorem

Suppose that preferences are fully comparable and measured on the cardinal scale. The social welfare function W satisfies Weak Pareto and Anonymity if and only if it takes the utilitarian form:

$$W(u) = u_1 + ... + u_N.$$

Proof. It is easy to see that the function $W(u) = u_1 + ... + u_N$, satisfies weak Pareto and anonymity.

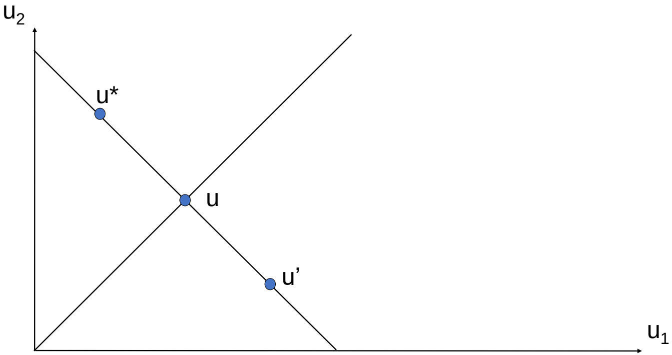
To show the converse, we will see only the case for $N = 2$.

Utilitarian Form (2)

Pick u with $u_1 = u_2$. Consider $k(u) = \{(u'_1, u'_2) : u'_1 + u'_2 = u_1 + u_2\}$.

For any u' on $k(u)$, the vector u^* s.t. $(u^*_1, u^*_2) = (u'_2, u'_1)$ is also on $k(u)$.

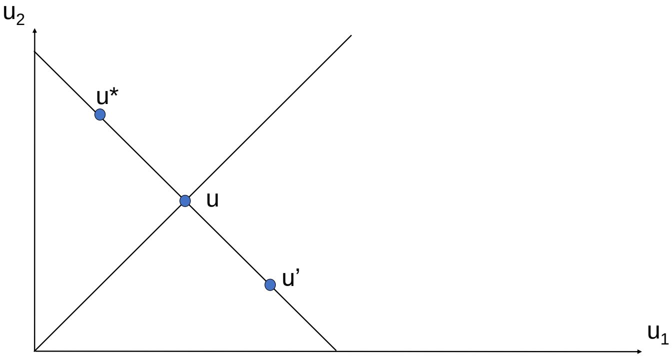
By anonymity, $W(u') = W(u^*)$.



Utilitarian Form (3)

Suppose now that $W(u) > W(u')$.

Under CS/IC, this ranking must be invariant to transformation $v_i(u_i) = a_i + bu_i$.

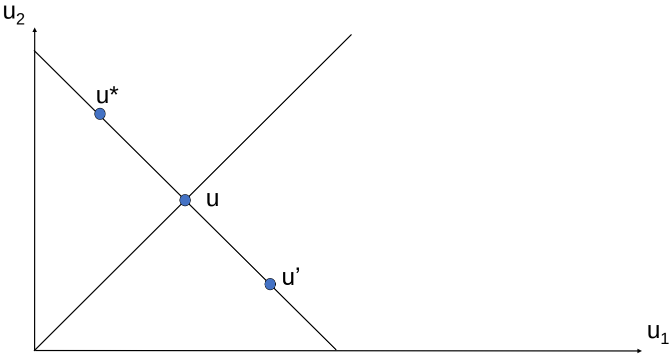


Utilitarian Form (4)

Let $v_i(u_i) = (u_i - u_i') + u_i$ for $i = 1, 2$.

Hence, $(v_1(u_1'), v_2(u_2')) = u$ and $(v_1(u_1), v_2(u_2)) = u^*$.

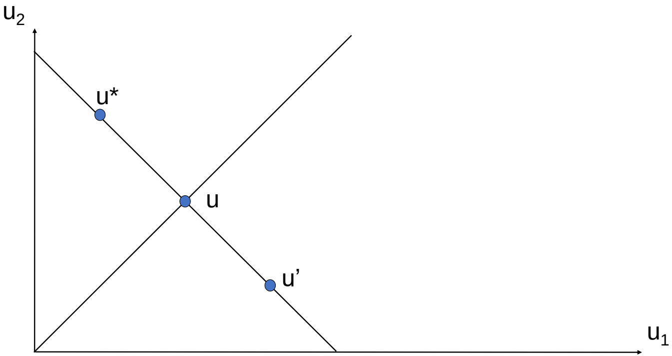
If $W(u) > W(u')$, then $W(u^*) > W(u)$, contradicting $W(u') = W(u^*)$.



Utilitarian Form (5)

If $W(u) < W(u')$, then $W(u^*) < W(u)$, contradicting $W(u') = W(u^*)$.

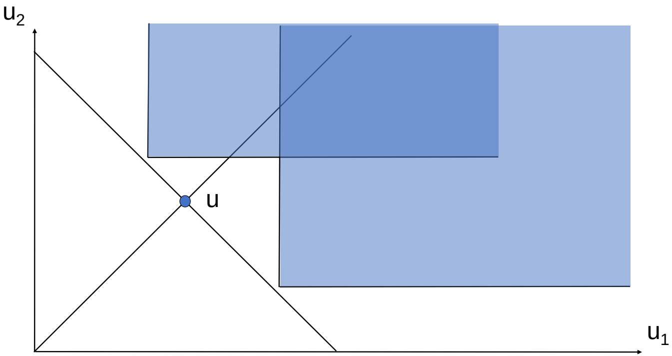
Hence we conclude that $W(u) = W(u')$ for all vectors u' on $k(u)$.



Utilitarian Form (6)

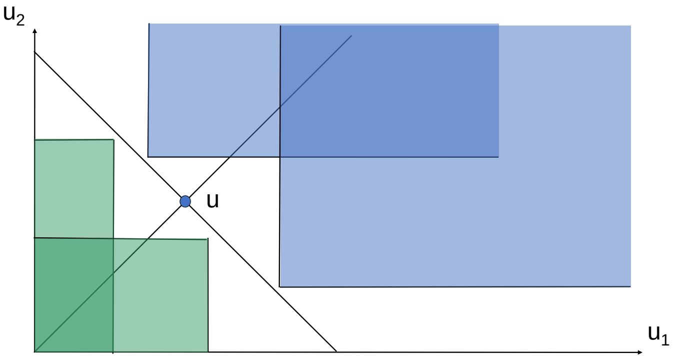
By weak Pareto, each vector u'' to the north-east of a vector u' on $k(u)$ is strictly preferred to u.

Thus $W(u'') > W(u)$ for u'' such that $u''_1 + u''_2 > u_1 + u_2$.



Utilitarian Form (7)

Similarly, $W(u'') < W(u)$ for u'' such that $u''_1 + u''_2 < u_1 + u_2$.

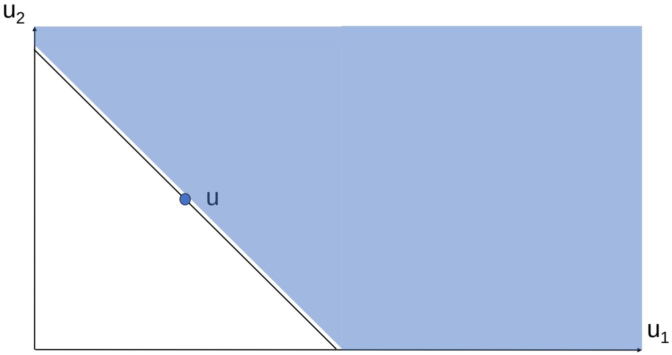


Utilitarian Form (8)

The indifference curve of any u is $k(u) = \{(u_1', u_2') : u_1' + u_2' = u_1 + u_2\}$.

Hence $W(u) = u_1 + u_2$.

Indifference curves are straight lines of slope -1.



Rawlsian and Utilitarian Forms

- The maximin Rawlsian form and the utilitarian form both belong to constant elasticity class with the formula:

$$W = (u_1^r + ... + u_N^r)^{1/r}$$

where $0 \neq r < 1$, and $s = 1/(1-r)$ is the constant elasticity of social substitution between any pair of individuals.

- As $r \rightarrow 1$, the welfare W approaches the utilitarian form.

- As $r \rightarrow -\infty$, the welfare W approaches the Rawlsian form.

A Theory of Justice

- Behind a "veil of ignorance", an individual does not know which position she will take in a society.
- Will she be rich or poor, successful or unsuccessful?
- Suppose she assigns equal probability to any of the possible economic and social identities that exist in the society
- Then, a rational evaluation would evaluate welfare according to the expected utility:

$$[u_1(x) + ... + u_N(x)]/N.$$

A Theory of Justice (2)

- This is equivalent to adopt the utilitarian criterion:

$$W(u(x)) = u_1(x) + ... + u_N(x).$$

- But the approach is also consistent with every CES form, embodying different degrees of risk aversion.
- Consider the positive transformation $v_i(x) = -u_i(x)^{-a}$ with $a > 0$.
- Suppose that $u_i(x)$ represents utility over social states "with certainty," whereas $v_i(x)$ represents utility "with uncertainty."
- In the form $v_i(x) = -u_i(x)^{-a}$, $a > 0$ represents the degree of risk aversion.

A Theory of Justice (3)

- Suppose the social welfare function is given by the expected utility:

$$W = [v_1(x) + ... + v_N(x)]/N = [-u_1(x)^{-a} - ... - u_N(x)^{-a}]/N.$$

- Welfare is equivalently represented by the monotonic transformation

$$W = (-u_1(x)^{-a} - ... - u_N(x)^{-a})^{-1/a}$$

- We obtain that any CES form is compatible with the expected utility formulation behind a veil of ignorance.
- The extreme risk aversion case of CES is

$$W = \min\{u_1(x), ..., u_N(x)\}.$$

- The Rawlsian form is concerned with the agent with the lowest utility.

Manipulation

- Suppose we have an institution (social choice function) that maps profiles of individual preferences into a social choice.

- Will individuals reveal their preferences truthfully?

- Example: incentives for strategic behavior in pairwise voting.

Methods of Manipulation

- Modification of the set of alternatives.
- The composition of the group that is authorized to decide may be changed.
- Influence on the true preferences of other individuals.
- Modification of the social choice procedure.
- Falsification of the true decision result.
- Here, we consider dishonest revelation of one's own preferences.

An Example of Manipulation

- 3 individuals $\{1, 2, 3\}$ and 3 alternatives $\{x, y, z\}$.
- Pairwise voting in stages: First x against y, then the winner against z.
- True preferences:

  $1 : x \succ_1 y \succ_1 z, \quad 2 : y \succ_2 z \succ_2 x, \quad 3 : z \succ_3 x \succ_3 y.$

  Hence, z is chosen, which is worst for individual 1.
- If 1 misreports his preferences dishonestly, and 2 and 3 reveal their true preferences, then y is chosen, which 1 prefers to z:

  $1' : y \succ_1' x \succ_1' z, \quad 2 : y \succ_2 z \succ_2 x, \quad 3 : z \succ_3 x \succ_3 y.$
- 1 has incentives to announce his preferences dishonestly.

Manipulability

- If the social choice function f selects one single outcome out of X, then f is called definitive.
- The social choice function $f(R, X)$ is manipulable if there exists one preference profile $R = (R_1, ..., R_n)$ and at least one individual i, such that there is a preference profile R_i' with $f(R_1, ..., R_{i-1}, R_i', R_{i+1}, ..., R_n, X) \succ_i f(R, X)$.
- We consider the robustness of a social choice rule against manipulation of a single individual.
- If there is no social choice rule that is robust against manipulation by single individuals, then there will be no social choice rule that is robust against manipulation by several individuals.

Gibbard-Satterthwaite Axioms

- (GS1) The social choice function $f(R, X)$ is definitive for every finite set X and for every preference profile R, and any $X = \{x, y, z\}$, $f(R, X) = x$ if and only if $xF(R, X)y$ and $xF(R, X)z$.
- (GS2) X contains at least 3 elements.
- (GS3) f is non-dictatorial.
- (GS4) If every individual prefers each element of $X \setminus \{y\}$ against y, then we will have $f(R, X) = f(R, X \setminus \{y\})$.

  This means: If y has the lowest estimation of every individual, then the social choice will not be changed by eliminating y out of X.
- (GS5) $f(R, X)$ is not manipulable.

Gibbard-Satterthwaite Theorem

Theorem

f cannot satisfy (GS1) - (GS5): If X contains at least 3 elements and if f is definitive and not manipulable, then f is dictatorial.

Proof: We show that the axioms (GS1) - (GS5) are inconsistent by showing that they imply the 4 (inconsistent) Arrow axioms.

- (GS1) implies unrestricted domain (UD).
- (GS2) is an assumption of Arrow's theorem.
- (GS3) is identical to non-dictatorship (ND).
- It remains to show that (GS1) - (GS5) imply that F is transitive and satisfies PE and IIA.

Gibbard-Satterthwaite Theorem (2)

Pareto-efficiency (PE) follows directly from (GS4):

- Suppose that $\hat{X} \subset X$ is the set $\{x, y\}$.
- If $x \succ_i y$ for all i, then we have $xF(R, \{x, y\})y$ according to (GS4).
- Thus, due to (GS1), we have: $f(R, \{x, y\}) = f(R, \{x\}) = x$.
- This is equivalent to Pareto-efficiency.

Independence of irrelevant alternatives (IIA) follows from (GS1) & (GS5).

- If R and R' differ outside of $\hat{X}$, e.g. $\hat{X} = \{x, y\}$, only for one individual i, then $f(R, \hat{X}) = f(R', \hat{X})$, by (GS1) and (GS5).
- Repeated application gives IIA.

Gibbard-Satterthwaite Theorem (3)

It remains to show: (GS1) - (GS5) imply transitivity of F.

- Intransitivity may only arise by cycles of the following kind: $x \succ_F y \succ_F z \succ_F x$.
- We assume that there exists such a cycle $x \succ_F y \succ_F z \succ_F x$ and show that this is inconsistent with (GS1) - (GS5).
- Due to (GS1), f has to select a single element out of $\{x, y, z\}$.
- Let us assume that $f(R, \{x, y, z\}) = x$.
- Let us consider preference profile R' same as R except that alternative y has the lowest rank in the preference order of individual 1.
- (GS5) is only fulfilled if $f(R', \{x, y, z\}) = f(R, \{x, y, z\}) = x$.

Gibbard-Satterthwaite Theorem (4)

- Thus, we have $f(R', \{x, y, z\}) = x$.
- Let us repeat this for all n persons, to obtain preference profile R^* such that y has the lowest ranking for all individuals.
- Then, we have $f(R^*, \{x, z\}) = x$ due to (GS4).
- According to the independence of irrelevant alternatives (IIA), this results in $f(R, \{x, z\}) = x$.
- As a consequence, $x \succ_F z$.
- Thus, there is no cycle and (GS1) - (GS5) imply transitivity of F. □

Sen's Liberal Paradox

- Sen's paradox is an extension of Arrow's theorem in which Sen misses the axiom of individual liberty (freedom of decision).
- Liberality means that a person has its own range for decisions.
- An individual i is called decisive between alternatives x and y, if i is decisive for x against y and for y against x.

  I.e., i is allowed to eliminate x or y, or none of both alternatives.
- A decision rule fulfills Sen's weak power condition (L), if and only if

  1. there is an individual i and a pair of alternatives x and y, such that i is decisive between x and y,

  2. there is an individual j and a pair of alternatives w and z, such that j is decisive between w and z.

Sen's Liberal Paradox (2)

Theorem

If X contains at least 2 alternatives and N contains at least 2 individuals, then there is no social choice function that satisfies Unrestricted Domain UD, Pareto-efficiency (PE) and Sen's weak power condition (L).

Proof: Case 1. $\{x, y\}$ and $\{w, z\}$ have 2 elements in common, i.e. $x = z$ and $y = w$ or $x = w$ and $y = z$.

- i and j are decisive between x and y according to L.
- We look at the following preference profile: $i : x \succ_i y$, $j : y \succ_j x$, and all other individuals $k \in N \setminus \{i, j\} : x \succ_k y$.
- Then, we have $x \notin f(R, \{x, y\})$, as j is decisive for y against x, and $y \notin f(R, \{x, y\})$, as i is decisive for x against y.
- This implies that $f(R, \{x, y\}) = \emptyset$, a violation of UD.

Sen's Liberal Paradox (3)

Case 2: $\{x, y\}$ and $\{w, z\}$ have exactly 1 element in common.

- Without loss of generality, we assume $x = z$.
- According to L, i is decisive between x and y, whereas j is decisive between x and w.
- We look at the following preference profile: $i : x \succ_i y \succ_i w$, $j : y \succ_j w \succ_j x$, all other individuals $k \in N \setminus \{i, j\} : x \succ_k y \succ_k w$.
- We have $w \notin f(R, \{x, y, w\})$, as y is Pareto-superior to w.
- $x \notin f(R, \{x, y, w\})$, as j is decisive for w against x.
- $y \notin f(R, \{x, y, w\})$, as i is decisive for x against y.
- This implies $f(R, \{x, y, w\}) = \emptyset$, violating UD.

Sen's Liberal Paradox (4)

Case 3: $\{x, y\}$ and $\{w, z\}$ have no element in common.

- We look at the following preference profile: $i : w \succ_i x \succ_i y \succ_i z$, $j : y \succ_j z \succ_j w \succ_j x$, all other $k \in N \setminus \{i, j\} : w \succ_k x \succ_k y \succ_k z$.
- We have $x \notin f(R, \{x, y, z, w\})$, as w is Pareto-superior to x.
- $z \notin f(R, \{x, y, z, w\})$, as y is Pareto-superior to z.
- $y \notin f(R, \{x, y, z, w\})$, as i is decisive for x against y.
- $w \notin f(R, \{x, y, z, w\})$, as j is decisive for z against w.
- This implies $f(R, \{x, y, z, w\}) = \emptyset$, violating UD.