Warwick University
Department of Economics

Francesco Squintani

Solutions to Assignment 1
EC9D3 Advanced Microeconomics

1. Let

$$\hat{u}(x_1, x_2) = k\ (x_1 - a)^{\alpha}(x_2 - b)^{\beta}$$

applying the logarithmic (monotonic) transformation (notice that in applying this monotonic transformation the consumption feasible set of the consumer is transformed in particular is reduced to $x_1 \geq a$ and $x_2 \geq b$) we obtain:

$$\ln k + \alpha \ln(x_1 - a) + \beta \ln(x_2 - b)$$

removing the constant $\ln k$ (also a monotonic transformation) we obtain

$$\alpha \ln(x_1 - a) + \beta \ln(x_2 - b)$$

dividing by $(\alpha + \beta)$ (a monotonic transformation, once again) we finally obtain

$$\delta \ln(x_1 - a) + (1 - \delta) \ln(x_2 - b)$$

where $\delta = \alpha/(\alpha + \beta)$.

2. We shall start from the consumer's maximization of utility problem.

(i) the consumer's utility maximization problem is:

$$\max_{\{x_1, x_2\}} \quad \delta \ln(x_1 - a) + (1 - \delta) \ln(x_2 - b)$$
$$\text{s.t.} \quad p_1 x_1 + p_2 x_2 \leq m$$

which delivers as the optimal solution the following Marshallian demands:

$$x_1(p, m) = a + \frac{\delta}{p_1}(m - ap_1 - bp_2)$$

and

$$x_2(p,m) = b + \frac{1-\delta}{p_2}(m - ap_1 - bp_2).$$

(ii) substituting the Marshallian demands just derived into the utility function we obtain the following *indirect utility function*.

$$\begin{aligned} v(p,m) &= \delta \ \ln\left[\tfrac{\delta}{p_1}(m - ap_1 - bp_2)\right] + (1-\delta)\ln\left[\tfrac{1-\delta}{p_2}(m - ap_1 - bp_2)\right] = \\ &= \ln(m - ap_1 - bp_2) + \delta\ln\delta + (1-\delta)\ln(1-\delta) - \delta\ln p_1 - (1-\delta)\ln p_2 \end{aligned}$$

(iii) the consumer's expenditure minimization problem is:

$$\begin{aligned} \min_{\{x_1,x_2\}} \quad & p_1x_1 + p_2x_2 \\ \text{s.t.} \quad & \delta\ln(x_1 - a) + (1-\delta)\ln(x_2 - b) \geq U \end{aligned}$$

which delivers as the optimal solution the following Hicksian demands:

$$h_1(p,U) = a + e^U\left[\frac{p_2\ \delta}{p_1\ (1-\delta)}\right]^{1-\delta}$$

and

$$h_2(p,U) = b + e^U\left[\frac{p_1\ (1-\delta)}{p_2\ \delta}\right]^{\delta}$$

(iv) substituting the Hicksian demands just derived in the consumer's expenditure $p_1x_1 + p_2x_2$ we derive the following expenditure function:

$$\begin{aligned} e(p,U) &= p_1\left\{a + e^U\left[\tfrac{p_2\ \delta}{p_1\ (1-\delta)}\right]^{1-\delta}\right\} + p_2\left\{b + e^U\left[\tfrac{p_1\ (1-\delta)}{p_2\ \delta}\right]^{\delta}\right\} = \\ &= a\ p_1 + b\ p_2 + [\delta + (1-\delta)]\ e^U\ \delta^{-\delta}\ (1-\delta)^{-(1-\delta)}p_1^{\delta}p_2^{(1-\delta)} = \\ &= a\ p_1 + b\ p_2 + e^U\ \delta^{-\delta}\ (1-\delta)^{-(1-\delta)}p_1^{\delta}p_2^{(1-\delta)} \end{aligned}$$

(v) The parameter a can be interpreted as the minimum feasible consumption of commodity x_1 and b the minimum feasible consumption of commodity x_2.

We need to assume that $m > p_1a + p_2b$ in order for the consumer to be in his/her consumption set.

3. Answers:

(i) The Marshallian demand function for commodity C can be obtained substituting the expressions (1) and (2) in the binding budget constraint and solving for C.

(ii) Yes they are. Consider, in fact, the value of (1) and (2) at $(\lambda p_A, \lambda p_B, \lambda p_C, \lambda m)$.

(iii) The Slutsky decomposition and the symmetry of the substitution matrix imply:

$$\begin{aligned}\frac{\partial h_A}{\partial p_B} &= \frac{\partial x_A}{\partial p_B} + \frac{\partial x_A}{\partial m} x_B = \frac{\alpha_2}{p_C} + \frac{\alpha_3}{p_C} x_B = \\ &= \frac{\partial h_B}{\partial p_A} = \frac{\partial x_B}{\partial p_A} + \frac{\partial x_B}{\partial m} x_A = \frac{\beta_1}{p_C} + \frac{\beta_3}{p_C} x_A\end{aligned}$$

or

$$\frac{\alpha_2}{p_C} + \frac{\alpha_3}{p_C} x_B = \frac{\beta_1}{p_C} + \frac{\beta_3}{p_C} x_A$$

which must hold for every A and B. Therefore if $x_A = 1$ and $x_B = 1$ we get

$$\alpha_2 + \alpha_3 = \beta_1 + \beta_3 \tag{1}$$

and if $x_A = 1$ and $x_B = 2$ we get

$$\alpha_2 + 2\alpha_3 = \beta_1 + \beta_3. \tag{2}$$

Equations (1) and (2) imply $\alpha_3 = 0$ therefore

$$\eta_{A,m} = \frac{\partial x_A}{\partial m} \frac{m}{x_A} = \frac{\alpha_3}{p_C} \frac{m}{x_A} = 0.$$

4. Answers:

(i) Substituting the values of prices and demands in the expression of the Marshallian demand for A we get:

$$\alpha + \beta + \gamma + 100 = 2$$

which implies:

$$\beta = -\alpha - \gamma - 98.$$

Moreover, from Slutsky equation for commodity A we obtain:

$$\frac{\partial h_A}{\partial p_A} = \frac{\partial x_A}{\partial p_A} + \frac{\partial x_A}{\partial m} x_A = \frac{\beta}{p_C} + \frac{10}{p_C} x_A = \beta + 20 \leq 0$$

that implies

$$\beta < -20.$$

(ii) Given that B and C are complements we obtain from:

$$\sum_{l=1}^{L} p_l \, \frac{\partial h_l}{\partial p_i} = 0$$

(for every $i = 1, \ldots, L$), that A and B are necessarily substitutes as well as A and C. Therefore, Slutsky decomposition yields:

$$\frac{\partial h_A}{\partial p_B} = \frac{\partial x_A}{\partial p_B} + \frac{\partial x_A}{\partial m} x_B = \frac{\gamma}{p_C} + \frac{10}{p_C} x_B = \gamma + 30 > 0$$

which implies $\gamma > -30$. Moreover:

$$\frac{\partial h_A}{\partial p_C} = \frac{\partial x_A}{\partial p_C} + \frac{\partial x_A}{\partial m} x_C = -\beta \frac{p_A}{p_C^2} - \gamma \frac{p_B}{p_C^2} - 10 \frac{m}{p_C^2} + \frac{10}{p_C} x_C = -\beta - \gamma - 50 > 0$$

which implies $\gamma < -50 - \beta$.