University of Warwick, Department of Economics | Spring 2012
EC941: Game Theory | Prof. Francesco Squintani

Final Exam

Answer TWO questions. All questions carry equal weight. Time allowed 2 hours.

1. Consider the following Cournot duopoly. Firm 1 and 2 face the following inverse demand function

$$p = a - Q \equiv a - q_1 - q_2, \quad a > 0,$$

where q_i is the quantity produced by firm $i = 1, 2$. Both firms produce at the same constant marginal cost $c < a$. Both firms maximize a (convex) combination of profits and revenues. More precisely, firm i maximizes

$$\beta_i R_i(q_1, q_2) + (1 - \beta_i)\Pi_i(q_1, q_2),\ i = 1, 2$$

where R_i and Π_i represent firm i's revenues and profits respectively, and $\beta_i \in (0, 1)$. Assume that $\beta_1 > \beta_2$.

a. Prove that the amended Cournot duopoly coincides with the asymmetric Cournot duopoly in which both firms maximize profits, firm 1's marginal cost is equal to $(1 - \beta_1)c$, and firm 2's marginal cost is equal to $(1 - \beta_2)c$.

b. Show that at the (unique) Nash equilibrium firm 1 produces a higher output than firm 2.

Consider now the following simplified version the Cournot duopoly. Firm 1 and 2 can only produce two levels of output: a high output, q_h, and a low output, q_l. Firms' payoffs are given by the matrix below,

	q_h	q_l
q_h	(Π^N, Π^N)	(Π^D, Π^L)
q_l	(Π^L, Π^D)	(Π^C, Π^C)

where $\Pi^D > \Pi^C > \Pi^N > \Pi^L$. Assume that the two firms interact repeatedly over an infinite horizon and have the same discount factor $\delta, 0 < \delta < 1$.

c. Find the condition on the discount factor under which the strategy pair in which each firm uses the *grim-trigger* strategy is a subgame perfect equilibrium of the infinitely repeated game.

2. Consider the following *procurement* auction. A government agency is considering buying a good from one of two firms, 1 and 2. The agency's valuation of the object, that is, its willingness to pay for the object, is equal to 1. The two firms submit simultaneously sealed-bid offers. The firm offering the lower price is awarded the contract at that price. That is, if firm 1 submits b_1 and firm 2 submits b_2 and $b_1 < b_2$, then firm 1 produces the good for the buyer and receives a payment equal to b_1. In the event of a tie, the contract is awarded by flipping a coin. The production cost is zero for both firms. However, each firm has to pay a *fixed* cost in order to submit its bid. Upon submitting its offer, each firm does not know whether or not its competitor is taking part in the auction.

a. Show that there exists no Nash equilibrium in pure strategy in which firms decide whether to enter the competition or to stay out.

b. Show that, if firms decide to submit an offer, the optimal pricing strategy involves an equilibrium in mixed strategies.

c. Find the equilibrium probability of entering the competition and the equilibrium distribution function of the firms' bids.

3. Consider the following **all-pay** auction. Two people submit sealed bids for an object worth 2 to each of them. Each person's bid may be any nonnegative number up to 2. The winner is the person whose bid is higher; in the event of a tie each person receives half of the object, which she values at 1. Each person pays her bid, *regardless of whether she wins*, and has preferences represented by the expected amount of money she receives.

a. Show that the all-pay auction does not admit a Nash equilibrium in pure strategies.

b. Show that there exists an equilibrium in mixed strategies in which both players randomize according to a uniform distribution on $[0, 2]$.

4. Two people take turns removing stones from a pile of n stones. Each person may, on each of her turns, remove either one stone or two stones. The person who takes the last stone is the winner; she gets \$1 from her opponent. Find the subgame perfect equilibria of the games that model this situation for $n = 1$ and $n = 2$. Find the winner in each subgame perfect equilibrium for $n = 3$, using the fact that the subgame following player 1's removal of one stone is the game for $n = 2$ in which player 2 is the first-mover, and the subgame following player 1's removal of two stones is the game for $n = 1$ in which player 2 is the first mover. Use the same technique to find the winner in each subgame perfect equilibrium for $n = 4$, and, if you can, for an arbitrary value of n.

5. A seller imposes a reserve price r in a second-price sealed-bid auction where there are two bidders. If both bids are below r, then the good is not sold; if one bid is above r and the other one is not, then the winner pays the reserve price r, and if both bids are above the reserve price, then the winner pays the second price. Each bidder i's valuation v_i is distributed uniformly on $[0, 1]$.

a. Prove or disprove that bidding one's own valuation is a Nash Equilibrium.

b. Prove or disprove that bidding one's own valuation is weakly dominant.

c. Find the reserve price that maximizes the expected selling price, given that the bidders play dominant strategies.

Answers

1a. Firm i's objective function can be rewritten as $q_i(a - q_i - q_j - (1 - \beta_i)c)$. Thus it is as if firm i were maximizing profit by producing at a marginal cost of $(1 - \beta_i)c < c$.

1b. The computation of the unique Nash equilibrium is a standard procedure that yields $(q_1^*, q_2^*) = (\frac{a - c(1 + \beta_2 - 2\beta_1)}{3}, \frac{a - c(1 + \beta_1 - 2\beta_2)}{3})$. Since $\beta_1 > \beta_2$, then $q_1^* > q_2^*$.

1c. The payoff structure is that of the *Prisoner's Dilemma.* Suppose that the 'row' firm adopts the grim-trigger strategy. If the 'column' firm chooses q_h in one period, it gets Π^D in that period and Π^N from the next period onwards. This yields an average discounted payoff of $\Pi^D(1 - \delta) + \delta\Pi^N$. The strategy of choosing q_l yields the 'column' firm an average discounted payoff of Π^C. Then (q_l, q_l) is a Nash equilibrium of the infinitely repeated game IFF

$$\Pi^C \geq \Pi^D(1 - \delta) + \delta\Pi^N \Leftrightarrow \delta \geq \frac{\Pi^D - \Pi^C}{\Pi^D - \Pi^N}.$$

2a. Suppose that both firms enter with probability one. Then they will optimally submit zero, and get negative expected profit (because of the entry cost). Then either firm has an incentive to deviate and stay out. Suppose that they stay out with probability one. Then either firm has an incentive to enter, submit one and get profit of $1 - c$ instead of zero (by staying out). Finally, suppose that firm 1 stays out and firm 2 enters. The latter can get $1 - c$ in profit by submitting a bid equal to one. Then firm 1 has an incentive to deviate, that is, to enter and to slightly undercut firm 2. This would yield firm 1 strictly positive profit instead of zero (by staying out).

2b. Suppose that, conditional on entering, both firms submit bids (b_1, b_2) with probability one. Notice that bids cannot be lower than c. If $b_1 = b_2 = c$, either firm would profitably deviate by staying out. If $1 \geq b_1 > b_2 > c$, then firm 1 would profitably undercut firm 2.

2c. At a mixed strategy equilibrium, firms have to be indifferent between entering and staying out. Then the equilibrium expected payoff is zero. Consider now firm 1's problem. It has to optimally choose the probability of staying out, q, and the distribution function over bids, $Q(\cdot)$, so as to make *firm 2* indifferent between staying out and entering. Thus, firm 1 chooses the couple $(q, Q(\cdot))$ such that

$$qx + (1 - q)[x(1 - Q(x)] - c = 0,$$

where x is firm 2's bid. At $x = 1$, $Q(1) = 1$ which implies $q = c$. Thus firm 2's expected payoff by bidding x writes

$$cx + (1-c)[x(1-Q(x))] - c = 0,\ x \in [c, 1].$$

Then,

$$Q(x) = 1 - \frac{c(1-x)}{(1-c)x},$$

which is indeed a continuous and strictly increasing function on $[c, 1]$ satisfying $Q(c) = 0$ and $Q(1) = 1$.

3a. Let b_1 and b_2 be bidder 1's and bidder 2's offers respectively. If $b_1 = b_2 < 2$, then either player has an incentive to increase her bid. If $b_1 = b_2 = 2$, then either player can increase her payoff by bidding zero. If $b_1 \neq b_2$, then the bidder who has submitted the lower offer can increase her payoff by bidding zero.

3b. Suppose that bidder 2 bids according to a uniform distribution on [0,2]. By bidding $x \in [0, 2]$, bidder 1's expected payoff writes

$$-x \ \Pr(b_2 > x) + (2-x) \ \Pr(b_2 \leq x) = -x(1 - \frac{x}{2}) + (2-x)\frac{x}{2} = 0.$$

If bidder 1 submits $x > 2$, she wins with probability 1 and gets a strictly negative payoff.

3c. This is a standard textbook exercise (see Osborne (2003), p. 430).

4. For $n = 1$ the game has a unique subgame perfect equilibrium, in which player 1 takes one stone. The outcome is that player 1 wins. For $n = 2$ the game has a unique subgame perfect equilibrium in which

- player 1 takes two stones
- after a history in which player 1 takes one stone, player 2 takes one stone.

The outcome is that player 1 wins.

For n = 3, the subgame following the history in which player 1 takes one stone is the game for n = 2 in which player 2 is the first mover, so player 2 wins. The subgame following the history in which player 1 takes two stones is the game for n = 1 in which player 2 is the first mover, so player 2 wins. Thus there is a subgame perfect equilibrium in which player 1 takes one stone initially, and one in which she takes two stones initially. In both subgame perfect equilibria player 2 wins.

For n = 4, the subgame following the history in which player 1 takes one stone is the game for n = 3 in which player 2 is the first-mover, so player 1 wins. The subgame following the history in which player 1 takes two stones is the game for n = 2 in which player 2 is the first-mover, so player 2 wins. Thus in every subgame perfect equilibrium player 1 takes one stone initially, and wins. Continuing this argument for larger values of n, we see that if n is a multiple of 3 then in every subgame perfect equilibrium player 2 wins, while if n is not a multiple of 3 then in every subgame perfect equilibrium player 1 wins. We can prove this claim by induction on n. The claim is correct for n = 1, 2, and 3, by the arguments above. Now suppose it is correct for all integers through n - 1. I will argue that it is correct for n.

First suppose that n is divisible by 3. The subgames following player 1's removal of one or two stones are the games for n - 1 and n - 2 inwhich player 2 is the first-mover. Neither n - 1 nor n - 2 is divisible by 3, so by hypothesis player 2 is the winner in every subgame perfect equilibrium of both of these subgames. Thus player 2 is the winner in every subgame perfect equilibrium of the whole game. Now suppose that n is not divisible by 3. As before, the subgames following player 1's removal of one or two stones are the games for n - 1 and n - 2 in which player 2 is the first-mover. Either n - 1 or n - 2 is divisible by 3, so in one of these subgames player 1 is the winner in every subgame perfect equilibrium. Thus player 1 is the winner in every subgame perfect equilibrium of the whole game.

5a & 5b. The argument that for each player a bid equal to her valuation weakly dominates all other bids (and hence it is a Nash Equilibrium) is the same as the one in the absence of a reserve price. Consider the following scheme, which compares player i's payoffs to the bid v_i with her payoffs to a bid $b_i < v_i$ (top table), and to a bid $b_i > v_i$ (bottom table), as a function of the highest of the other players' bids, denoted b. In each case, for all bids of the other players, player i's payoffs to v_i are at least as large as her payoffs to the other bid, and for bids of the other players such that b is in the middle column of each table, player i's payoffs to v_i are greater than her payoffs to the other bid. Thus player i's bid v_i weakly dominates all her other bids.

	$b < b_i$ or $b = b_i$ & b_i wins	$b_i < b < v_i$ or $b = b_i$ & b_i loses	$b > v_i$
$b_i < v_i$	$v_i - b$	0	0
$b_i = v_i$	$v_i - b$	$v_i - b$	0

	$b < v_i$	$v_i < b < b_i$ or $b = b_i$ & b_i wins	$b > b_i$ or $b = b_i$ & b_i loses
$b_i = v_i$	$v_i - b$	0	0
$b_i > v_i$	$v_i - b$	$v_i - b$	0

5c. Consider the expected price at which the object is sold when the reserve price is r. Because this is a second prince auction, the dominant strategy for a player of type x is to bid x. Hence the expected revenue for the auctioneer is:

$$ER = r \Pr(x_1 < r, x_2 \geq r) + r \Pr(x_1 \geq r, x_1 < r) + E[\min\{x_1, x_2\} | x_1 > r, x_2 > r] \Pr(x_1 > r, x_2 > r),$$

where x_i is the type of player i.

To calculate ER, note that

$$r \Pr(x_1 < r, x_2 \geq r) = r \Pr(x_1 \geq r, x_1 < r) = r \cdot r(1-r) = r^2 - r^3,$$

and that the density of $\min\{x_1, x_2\}$ is $2(1-x)$,

$$\Pr\{\min\{x_1, x_2\} \leq x\} = 1 - \Pr\{x_1 > x, x_2 > x\} = 1 - (1-x)^2,$$

Thus

$$E[\min\{x_1, x_2\} | x_1 > r, x_2 > r] \Pr(x_1 > r, x_2 > r) = \int_r^1 x \cdot 2(1-x)\, dx = \frac{2}{3}r^3 - r^2 + \frac{1}{3}.$$

Wrapping up, we obtain:

$$ER(r) = 2(r^2 - r^3) + \frac{2}{3}r^3 - r^2 + \frac{1}{3} = -\frac{4}{3}r^3 + r^2 + \frac{1}{3}.$$

The first order condition is:

$$ER'(r) = -2r(2r-1) = 0$$

which yields $r = 0$ and $r = 1/2$ as solutions. The associated second order conditions are: $ER''(0) = 2$ and $ER''(1/2) = -2$. Hence, the optimal reserve price is $r = 1/2$, which yields expected revenue of $5/12$. Whereas a reserve price of 0 yields expected revenue of $1/3$.