SOME ACCOUNTING FOR TASTES

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How many different outcomes are possible when n runners race (given that all finish)? How many preference [quasi]-orderings exist on a set of n options? These questions are formally identical, and are non-trivial because of the existence of ties and of indifferent but distinguishable alternatives. Some years ago Don Locke asked me for the answers for some very small n. These I supplied, but no general formula. Here I return to the problem, generalize it, and provide solutions.

A two-horse race may end in three ways (a deadheat, two outright wins), a three-horse race in 13 ways. Let there be $K(n)$ possible results with n horses. For a formula that calculates $K(n)$ from previous values of K, note that there are $\binom{n}{n-k}$ possible $(n-k)$-way ties for 1st place, and $K(k)$ ways for the k others to trail in. Thus

$$(1) \qquad K(n) = \sum_{k=0}^{n-1} \binom{n}{n-k} K(k) = \sum_{k=0}^{n-1} \binom{n}{k} K(k).$$

Hence $K(4) = 75$, $k(5) = 541$, and $K(10) = 102247563$. Mike Paterson has shown me a simple explicit formula for $K(n)$ involving Stirling numbers of the second kind; indeed, it follows immediately from Comtet, *Advanced Combinatorics*, pp. 204f., that

$$(2) \qquad K(n) \quad = \quad \sum_{k=1}^{n} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n.$$

Since Don's original question concerned preference orderings, a natural generalization takes preferences to comprise only a semi-order; an ordering in which indifference may be intransitive. (You may prefer iguanas to skinks, yet confess no preference either between iguanas and geckos or between geckos and skinks.) Place in division undominated options, in division 2 options to which only options in division 1 are preferred, and so on. The *chameleons* of a division are those options not discriminated from some (but not necessarily from all) options in the following division. Let there be $J(n)$ possible semi-orders on n options. Then $n-j$ options may be chosen from division 1 in $\binom{n}{j}$ ways; and (unless $j = 0$) any of the $2^{n-j} - 1$ proper subsets of them may constitute the division 1 chameleons. The remaining j options themselves generate $J(j)$ semi-orders. Thus

$$(3) \qquad J(n) = 1 + \sum_{j=1}^{n-1} \binom{n}{j} (2^{n-j} - 1) J(j).$$

Hence $J(3) = 19$, $J(4) = 159$, $J(5) = 1651$, and $J(10) = 1934655063$. More directly: a total of $n-j$ chameleons overall may be picked in $\binom{n}{j}$ ways, and the other j options ordered (in one of $K(j)$ ways) into k disjoint divisions for some positive $k < j$; then the chameleons may be dispersed in $(k-1)^{n-j}$ ways among the $k-1$ divisions preceding division k. By (2),

$$(4) \qquad J(n) \quad = \quad \sum_{j=0}^{n} \binom{n}{j} \sum_{k=1}^{j} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^j (k-1)^{n-j}.$$

Doubtless (4) can be simplified. There exists a proof that (3) and (4) define the same function.

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