A Metric Rendition of Subadditive Probability

David Miller
Department of Philosophy
University of Warwick
COVENTRY CV4 7AL UK
e-mail: `dwmiller57@yahoo.com`

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Abstract

It is known that the usual axioms of probability on a Boolean algebra are interderivable with a set of axioms for a pseudometric operation on that algebra. A set of axioms for subadditive probability is here demonstrated to be interderivable with a suitably weakened set of pseudometric axioms.

0 Introduction

Throughout this paper, $a, b, c, \ldots$ are elements of the domain B of an arbitrary Boolean lattice $\mathfrak{B} = \langle \mathsf{B}, \leq \rangle$ with zero $\bot$, unit $\top$, meet $\cdot$, join $+$, and complement $'$. The operation $\triangle$ of *symmetric difference* (exclusive disjunction) may be defined on the elements of B by either of the identities $a \triangle c = a \cdot c' + c \cdot a'$ and $a \triangle c = (a + c) \cdot (c' + a')$. Its simpler properties, and the other Boolean identities and inequalities called on, may be checked by truth tables.

An (unnormalized) *measure* or *valuation* on $\mathfrak{B}$ is a real-valued function $\mathfrak{p}$ obeying the axioms

$$(0) \qquad 0 = \mathfrak{p}(\bot) \leq \mathfrak{p}(b)$$

$$(1) \qquad \mathfrak{p}(a + c) + \mathfrak{p}(a \cdot c) = \mathfrak{p}(a) + \mathfrak{p}(c).$$

It is well known that the axiom system for the elementary theory of probability introduced by Kolmogorov (1933), Chapter I, § 1, is obtained by adding to these axioms the further requirement that $\mathfrak{p}(\top) = 1$, but this normalization will not be relevant to anything that follows here.

It is also reasonably well known, and was established in detail in Miller (1984), that the extended set $\mathbb{A}$ that follows, which contains also a definition ($\mathbb{A}0$) of the operator $\mathfrak{d}$,

$$(\mathbb{A}0) \qquad \mathfrak{d}(a, c) =_{\mathrm{Df}} \mathfrak{p}(a \triangle c)$$

$$(\mathbb{A}1) \qquad 0 = \mathfrak{p}(\bot) \leq \mathfrak{p}(b)$$

$$(\mathbb{A}2) \qquad \mathfrak{p}(a + c) + \mathfrak{p}(a \cdot c) = \mathfrak{p}(a) + \mathfrak{p}(c),$$

Most of the results of this paper date from around 1979–1981.

is logically equivalent to the following set $\mathbb{C}$ of axioms for a pseudometric operation $\mathfrak{d}$:

$$(\mathbb{C}0) \qquad \mathfrak{p}(b) =_{\mathrm{Df}} \mathfrak{d}(b, \bot)$$

$$(\mathbb{C}1) \qquad \mathfrak{d}(a, c) \leq \mathfrak{d}(b, a) + \mathfrak{d}(b, c)$$

$$(\mathbb{C}2) \qquad \mathfrak{d}(b + a, b + c) + \mathfrak{d}(b \cdot a, b \cdot c) \leq \mathfrak{d}(a, c),$$

which contains a corresponding definition ($\mathbb{C}0$) of the operator $\mathfrak{p}$ in terms of $\mathfrak{d}$.

Three easily derived consequences, of the system $\mathbb{A}$ state the *monotony* of the measure $\mathfrak{p}$ (with respect to the ordering $\leq$ on $\mathfrak{B}$), a special type of *subadditivity*, and its *complementarity*:

$$(\mathbb{A}3) \qquad \mathfrak{p}(a) \leq \mathfrak{p}(a + c)$$

$$(\mathbb{A}4) \qquad \mathfrak{p}(a + c) \leq \mathfrak{p}(a) + \mathfrak{p}(c)$$

$$(\mathbb{A}5) \qquad \mathfrak{p}(a) + \mathfrak{p}(a') = \mathfrak{p}(\top).$$

Proof The required proofs are familiar and undemanding. Since $a \cdot (c \cdot a') = \bot$ and $a + (c \cdot a') = a + c$ for all $a, c \in \mathsf{B}$, it follows from ($\mathbb{A}2$) and the first part of ($\mathbb{A}1$) that $\mathfrak{p}(a + c) = \mathfrak{p}(a) + \mathfrak{p}(c \cdot a')$, from which ($\mathbb{A}3$) follows by use of the second part of ($\mathbb{A}1$), and ($\mathbb{A}5$) follows by putting $c = \top$. What is more, $c \cdot a' \leq c \cdot a' + c \cdot a = c$, and so by ($\mathbb{A}3$) and what has been proved, $\mathfrak{p}(a + c) = \mathfrak{p}(a) + \mathfrak{p}(c \cdot a') \leq \mathfrak{p}(a) + \mathfrak{p}(c)$, which yields ($\mathbb{A}4$). ■

The set $\mathbb{A}^\star = \{(\mathbb{A}0), (\mathbb{A}3), (\mathbb{A}4), (\mathbb{A}5)\}$ is strictly weaker than the set $\mathbb{A}$, for it admits interpretations in which $\mathfrak{p}(a + c) = \mathfrak{p}(a) + \mathfrak{p}(c)$ for all $a, c \in \mathsf{B}$. It should be noted that, like $\mathbb{A}$, it admits interpretations in which $\mathfrak{p}$ is identically 0.

From the system $\mathbb{C}$ we can also derive with ease the following two inequalities concerning $\mathfrak{d}$:

$$(\mathbb{C}3) \qquad \mathfrak{d}(b + a, b + c) \leq \mathfrak{d}(a, c)$$

$$(\mathbb{C}4) \qquad \mathfrak{d}(a, c) + \mathfrak{d}(a, c') \leq \mathfrak{d}(c, c').$$

Proof It follows from suitable substitutions in ($\mathbb{C}1$) and ($\mathbb{C}2$) that $\mathfrak{d}(b, b) \leq 2\mathfrak{d}(b, b) \leq \mathfrak{d}(b, b)$, and hence that $\mathfrak{d}(b, b) = 0$; and then, from two further applications of ($\mathbb{C}1$), that $0 \leq \mathfrak{d}(c, a) = \mathfrak{d}(a, c)$. ($\mathbb{C}3$) now follows from ($\mathbb{C}2$). By ($\mathbb{C}1$), ($\mathbb{C}2$), and ($\mathbb{C}1$) again,

$$\begin{aligned} \mathfrak{d}(\top, \bot) \leq \mathfrak{d}(c, \top) + \mathfrak{d}(c, \bot) &= \mathfrak{d}(c + c, c + c') + \mathfrak{d}(c \cdot c, c \cdot c') \\ &\leq \mathfrak{d}(c, c') \leq \mathfrak{d}(a, c) + \mathfrak{d}(a, c') \leq \mathfrak{d}(\top, \bot), \end{aligned}$$

by writing $\top$ for a and $\bot$ for c. The identity of the two sides of ($\mathbb{C}4$) now follows. ■

To obtain the set $\mathbb{C}^\star$ we add to ($\mathbb{C}3$) and ($\mathbb{C}4$) the definition ($\mathbb{C}0$) of $\mathfrak{p}$ in terms of $\mathfrak{d}$ and the triangle inequality ($\mathbb{C}1$).

In the next two sections we shall show that the sets $\mathbb{A}^\star$ and $\mathbb{C}^\star$ are logically equivalent.

1 Proof of the pseudometric axioms $\mathbb{C}^\star$ from the measure axioms $\mathbb{A}^\star$

In the present section the four axioms ($\mathbb{A}0$), ($\mathbb{A}3$), ($\mathbb{A}4$), ($\mathbb{A}5$), of $\mathbb{A}^\star$ are assumed.

Lemma 1.0 $\mathfrak{p}(b) = \mathfrak{d}(b, \bot)$.

Proof This follows immediately from ($\mathbb{A}0$), given that $b \bigtriangleup \bot = b$. ■

LEMMA 1.1 $\mathfrak{d}(a,c) \leq \mathfrak{d}(b,a) + \mathfrak{d}(b,c)$.

PROOF Since $(b \vartriangle a) + (b \vartriangle c) = (a \vartriangle c) + ((b \vartriangle a) + (b \vartriangle c))$, we have from ($\mathbb{A}3$) and ($\mathbb{A}4$) that $\mathfrak{p}(a \vartriangle c) \leq \mathfrak{p}((b \vartriangle a) + (b \vartriangle c)) \leq \mathfrak{p}(b \vartriangle a) + \mathfrak{p}(b \vartriangle c)$. By ($\mathbb{A}0$), $\mathfrak{d}(a,c) \leq \mathfrak{d}(b,a) + \mathfrak{d}(b,c)$. ■

LEMMA 1.2 $\mathfrak{d}(b+a, b+c) \leq \mathfrak{d}(a,c)$.

PROOF It follows from ($\mathbb{A}3$) that $\mathfrak{p}(a \cdot c) \leq \mathfrak{p}(a)$. Since $(b+a) \vartriangle (b+c) = b' \cdot (a \vartriangle c)$, we may conclude by ($\mathbb{A}0$) that $\mathfrak{d}(b+a, b+c) \leq \mathfrak{d}(a,c)$. ■

LEMMA 1.3 $\mathfrak{d}(a,c) + \mathfrak{d}(a,c') \leq \mathfrak{d}(c,c')$.

PROOF By ($\mathbb{A}5$), $\mathfrak{p}(a \vartriangle c) + \mathfrak{p}(a \vartriangle c') = \mathfrak{p}(a \vartriangle c) + \mathfrak{p}((a \vartriangle c)') = \mathfrak{p}(\top) = \mathfrak{p}(c \vartriangle c')$. The result follows by ($\mathbb{A}0$). ■

It may be noted that ($\mathbb{A}5$) is needed only for the proof of ($\mathbb{C}4$) in Lemma 1.3. Matters are quite different in the converse direction, where ($\mathbb{C}4$) plays a central role throughout.

2 Proof of the measure axioms $\mathbb{A}^\star$ from the pseudometric axioms $\mathbb{C}^\star$

In the present section the four axioms ($\mathbb{C}0$), ($\mathbb{C}1$), ($\mathbb{C}3$), ($\mathbb{C}4$), of $\mathbb{C}^\star$ are assumed.

LEMMA 2.0 $0 = \mathfrak{d}(b,b) \leq \mathfrak{d}(a,c) = \mathfrak{d}(c,a) = \mathfrak{d}(a',c') \leq \mathfrak{d}(b,b')$.

PROOF It follows immediately from ($\mathbb{C}1$) that $\mathfrak{d}(b,b) \leq 2\mathfrak{d}(b,b)$, and hence that $0 \leq \mathfrak{d}(b,b)$ for any $b \in \mathsf{B}$. Writing b for both a and c in ($\mathbb{C}4$), we see that $\mathfrak{d}(b,b) \leq 0$. This proves that $\mathfrak{d}(b,b) = 0$. It follows from ($\mathbb{C}1$) that $\mathfrak{d}(a,c) \leq \mathfrak{d}(c,a)$, and therefore that $\mathfrak{d}(a,c) = \mathfrak{d}(c,a)$ for every $a, c \in \mathsf{B}$. It follows also from ($\mathbb{C}1$) that $\mathfrak{d}(c,c) \leq 2\mathfrak{d}(a,c)$, and hence that $\mathfrak{d}(b,b) \leq \mathfrak{d}(a,c)$.

To conclude we note first that ($\mathbb{C}1$) allows ($\mathbb{C}4$) to be strengthened to an identity, and so

$$2\mathfrak{d}(c,c') = \mathfrak{d}(a,c) + \mathfrak{d}(a,c') + \mathfrak{d}(a',c) + \mathfrak{d}(a',c') =$$
$$\mathfrak{d}(a,c) + \mathfrak{d}(a',c) + \mathfrak{d}(a,c') + \mathfrak{d}(a',c') = 2\mathfrak{d}(a,a'),$$

by the commutative property of $\mathfrak{d}$ already proved. We obtain $\mathfrak{d}(c,c') = \mathfrak{d}(a,a')$ for every $a, c \in \mathsf{B}$. A further application of ($\mathbb{C}4$) completes the proof that $\mathfrak{d}(a,c) \leq \mathfrak{d}(b,b')$. It now follows from ($\mathbb{C}4$) that $\mathfrak{d}(a,c) + \mathfrak{d}(a,c') = \mathfrak{d}(a,c') + \mathfrak{d}(a',c')$, and accordingly that $\mathfrak{d}(a,c) = \mathfrak{d}(a',c')$. ■

COROLLARY 2.0.0 $\mathfrak{d}(a,c) + \mathfrak{d}(a,c') = \mathfrak{d}(b,b') = \mathfrak{d}(a,d) + \mathfrak{d}(a',d)$.

PROOF Immediate. ■

LEMMA 2.1 $\mathfrak{d}(a,c) = \mathfrak{p}(a \vartriangle c)$.

PROOF By ($\mathbb{C}3$), $\mathfrak{d}(b'+a', b'+c') \leq \mathfrak{d}(a',c')$, and hence by Lemma 2.0,

$$(\mathbb{C}5) \qquad \mathfrak{d}(b \cdot a, b \cdot c) \;\leq\; \mathfrak{d}(a,c).$$

Since $(a \cdot c) + (a \vartriangle c) = a + c$, it follows from ($\mathbb{C}3$) again that $\mathfrak{d}(a+c, a \cdot c) = \mathfrak{d}(a \cdot c + (a \vartriangle c), ac + \bot) \leq \mathfrak{d}(a \vartriangle c, \bot)$. Since $(a \vartriangle c) \cdot (a+c) = a \vartriangle c$, and $(a \vartriangle c) \cdot (a \cdot c) = \bot$, it follows from ($\mathbb{C}5$) also that $\mathfrak{d}(a \vartriangle c, \bot) = \mathfrak{d}((a \vartriangle c) \cdot (a+c), (a \vartriangle c) \cdot (a \cdot c)) \leq \mathfrak{d}(a+c, a \cdot c)$. We may conclude that $\mathfrak{d}(a \vartriangle c, \bot) = \mathfrak{d}(a+c, a \cdot c)$.

By Corollary 2.0.0, $\mathfrak{d}(a \vartriangle c, \bot) + \mathfrak{d}(a' \vartriangle c, \bot) = \mathfrak{d}(a+c, a \cdot c) + \mathfrak{d}(a' \cdot c', a \cdot c)$, from which it follows that $\mathfrak{d}(a' \vartriangle c, \bot) = \mathfrak{d}(a' \cdot c', a \cdot c)$. By writing a' for a in this equation, we may conclude that $\mathfrak{d}(a \vartriangle c, \bot) = \mathfrak{d}(a \cdot c', a' \cdot c)$. It now follows from ($\mathbb{C}3$), ($\mathbb{C}5$), and some Boolean identities that $\mathfrak{d}(a,c) = \mathfrak{d}((a \cdot c) + (a \cdot c'), (a \cdot c) + (a' \cdot c)) \leq \mathfrak{d}(a \cdot c', a' \cdot c) = \mathfrak{d}((a \vartriangle c) \cdot a, (a \vartriangle c) \cdot c) \leq \mathfrak{d}(a,c)$. This completes the proof that $\mathfrak{d}(a,c) = \mathfrak{d}(a \vartriangle c, \bot)$, and hence, by ($\mathbb{C}0$), that ($\mathbb{A}0$) holds. ■

COROLLARY 2.1.0 $\mathfrak{d}(a, c) = \mathfrak{d}(a + c, a \cdot c)$.

PROOF Immediate. ∎

LEMMA 2.2 $\mathfrak{p}(a) \leq \mathfrak{p}(a + c)$.

PROOF Suppose that $c \leq b \leq a$. It follows at once from ($\mathbb{C}5$) that $\mathfrak{d}(b, c) \leq \mathfrak{d}(a, c)$. Since $\bot \leq a \leq a + c$, we may conclude that $\mathfrak{d}(a, \bot) \leq \mathfrak{d}(a + c, \bot)$. The result follows by ($\mathbb{C}0$). ∎

LEMMA 2.3 $\mathfrak{p}(a + c) \leq \mathfrak{p}(a) + \mathfrak{p}(c)$.

PROOF By ($\mathbb{C}1$) and ($\mathbb{C}3$), $\mathfrak{d}(a + c, \bot) \leq \mathfrak{d}(a, a + c) + \mathfrak{d}(a, \bot) = \mathfrak{d}(a + \bot, a + c) + \mathfrak{d}(a, \bot) \leq \mathfrak{d}(\bot, c) + \mathfrak{d}(a, \bot)$. The result follows from the commutativity of $\mathfrak{d}$ (Lemma 2.0) and ($\mathbb{C}0$). ∎

LEMMA 2.4 $\mathfrak{p}(a) + \mathfrak{p}(a') = \mathfrak{p}(\top)$.

PROOF By Corollary 2.0.0, $\mathfrak{d}(a, \bot) + \mathfrak{d}(a', \bot) = \mathfrak{d}(\top, \bot)$. The result follows by ($\mathbb{C}0$). ∎

3 Conclusion

Axiom ($\mathbb{A}4$) is only one way of formulating the subadditivity of a measure $\mathfrak{p}$, and by no means obviously the best. Other formulations are

$$\mathfrak{p}(a + c) + \mathfrak{p}(a \cdot c) \leq \mathfrak{p}(a) + \mathfrak{p}(c) \tag{2}$$

$$a \cdot c = \bot \rightarrow \mathfrak{p}(a + c) \leq \mathfrak{p}(a) + \mathfrak{p}(c). \tag{3}$$

It is clear that (2) is at least as strong as ($\mathbb{A}4$), which itself is at least as strong as (3). It is not known whether there are attractive sets of pseudometric axioms that are equivalent to $\{(\mathbb{A}0), (\mathbb{A}3), (2), (\mathbb{A}5)\}$ or to $\{(\mathbb{A}0), (\mathbb{A}3), (3), (\mathbb{A}5)\}$.

References

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