The Disposition of Complete Theories

David Miller
Department of Philosophy
University of Warwick, COVENTRY CV4 7AL, UK
&
Instituto de Estudos Avançados
Universidade de São Paulo
Cidade Universitária, 05508 SAO PAULO SP, Brasil

0 Summary

The purpose of this paper is to give a purely logical proof of a result of Mostowski [1937] concerning the complete theories of a calculus based on classical propositional logic; and then modestly to generalize it. Mostowski's result is announced by Tarski on p. 370 of *Logic, Semantics, Metamathematics* [1956]. (All references to Tarski's work here are to this book.) Tarski himself provides only a fragment of a proof, and the proof published by Mostowski makes extensive use of topological methods and results. The proof offered here is undoubtedly longer than Mostowski's and not by any means independent of it. But it should not be beyond the powers of anyone who has followed assiduously a couple of courses in propositional logic and knows a little set theory. The axiom of choice is assumed, but not the continuum hypothesis. a

A *calculus* is what Tarski called a deductive system (originally a deductive theory): the set of sentences of a language plus an operation **Cn** of logical consequence based on that of classical propositional logic. A *theory* is a set of sentences closed under **Cn**. An *axiomatizable theory* is **Cn**(X) for some finite set X of sentences. A *complete theory* is a theory that has no consistent proper extension. The *characteristic pair* of a calculus is the ordered pair consisting of the number of its axiomatizable complete theories and the number of its unaxiomatizable complete theories. If we add the pair $\langle 0, 0\rangle$, which is unaccountably | omitted both by Mostowski and b c | 2

Published in *Coleção Documentos, Série Lógica e Teoria da Ciência*, 10, agosto de 1992, Instituto de Estudos Avançados, Universidade de São Paulo. Digitally remastered February 2009. The notes **a–x** on pp. 25–28 suggest some improvements and clarifications. Most of them are minor, but one (note **n**) is more important.

by Tarski, and correct an obvious slip in the statement of his Theorem 8, Mostowski's result is the following.

Theorem 0: The characteristic pair of a finite or denumerable calculus must take one of the following values (m is a natural number, and n is a positive natural number):

$$\langle m, 0\rangle, \langle m, 2^{\aleph_0}\rangle, \langle \aleph_0, n\rangle, \langle \aleph_0, \aleph_0\rangle, \langle \aleph_0, 2^{\aleph_0}\rangle.$$

We shall prove this Theorem, and also show how to construct examples of calculi of all the different characteristic pairs.

1 Examples of Finite Calculi

We start with a presentation of examples. For the sake of uniformity, they are all of calculi based on propositional languages, even though in some cases there exist more intuitive illustrations amongst predicate languages. Elementary logic with identity $\equiv$ and no other predicates, for example, has characteristic pair $\langle \aleph_0, 1\rangle$ (Tarski, p. 378); whilst augmenting this language with a single monadic predicate P produces a calculus with characteristic pair $\langle \aleph_0, \aleph_0\rangle$. The symbols $\wedge, \vee, \neg, \bot, \rightarrow$ will be used for conjunction, disjunction, negation, the absurdity, and the conditional respectively.

The presentation in this section and the next will be informal. We shall, for example, say that the theory **X** *assigns a truth value* to a letter $\mathtt{p}$, or *asserts the truth or falsehood* of $\mathtt{p}$, or simply *asserts* $\mathtt{p}$ or $\neg\mathtt{p}$, meaning only that either **X** implies $\mathtt{p}$ or that it implies $\neg\mathtt{p}$. It is understood that a complete theory is one that assigns one and only one truth value to each propositional letter in the language.

The easiest cases are the characteristic pairs $\langle 2^j, 0\rangle$. Here it suffices to take a propositional language with j distinct letters $\mathtt{p}_0, \dots, \mathtt{p}_{j-1}$, and adopt for **Cn** the consequence operation of classical propositional logic. Each assignment of truth values to the j letters yields a complete theory, and there accordingly exist 2^j axiomatizable complete theories. Since all theories in the calculus are axiomatizable, there are no other complete theories.

| 3 For the case of the characteristic pair $\langle m, 0\rangle$, where $2^{j-1} < m <$ | 2^j, we

take the same language, but enrich the consequence operation **Cn** by adding as extra logical postulates the negations of any $2^j - m$ of the complete theories. This leaves m complete theories. For instance, if $m = 3$ we could start with a language whose letters are p and q, and then assert $\mathsf{p} \to \mathsf{q}$ as a logical truth. The three complete theories of the calculus are axiomatized by the sentences $\mathsf{p} \wedge \mathsf{q}$, $\neg\mathsf{p} \wedge \mathsf{q}$, and $\neg\mathsf{p} \wedge \neg\mathsf{q}$. d

At the finite level, there remains only the case of $\langle 0, 0\rangle$. In the light of Lindenbaum's theorem (which states that every consistent theory has a complete extension), this characteristic pair may seem not to be possible. But, of course, it is not excluded if there can exist a calculus in which no theory is consistent. There are, in fact, two ways to manage this. One is to add to the laws of logic a rule that allows the absurdity $\perp$ to be derived from the empty set. In this calculus all theories are logically equivalent to $\perp$. The other possibility (not available if logic is formulated in terms of $\perp$ or some other constant) is to take a language with no sentences at all. (Tarski admits this possibility on pp. 31, 63; but on p. 344, at the start of his definitive exposition of the calculus of deductive theories, he rules it out.)

2 Examples of Infinite Calculi

The next easiest cases are the characteristic pairs $\langle m, 2^{\aleph_0}\rangle$. For $m = 0$ we require a propositional language with a denumerable set of letters $\mathsf{p}_0, \mathsf{p}_1, \ldots$. When **Cn** is the ordinary classical consequence operation we have a calculus whose characteristic pair is $\langle 0, 2^{\aleph_0}\rangle$, as is obvious. Each complete theory must, for every natural number i, assign a truth value to p_i, that is, it must asserts either p_i or $\neg\mathsf{p}_i$. No such infinite conjunction of independent conjuncts can be axiomatizable.

For the remaining pairs of the form $\langle m, 2^{\aleph_0}\rangle$ we strengthen the operation **Cn** with various sets of logical postulates. For example, let I be the set of all propositions of the form $\mathsf{p}_0 \to \mathsf{p}_k$ for $k > 0$; and let **Cn** be fortified by the addition of I as a set of axioms. Then p_0 — or, if you like, **Cn**(p_0) — is a complete theory, and is axiomatizable. All other complete theories assert $\neg\mathsf{p}_0$, and otherwise, as before, have denumerably many things to say, and so are nor axiomatizable. The characteristic pair is therefore $\langle 1, 2^{\aleph_0}\rangle$. To

4 obtain the characteristic pair $\langle 2, 2^{\aleph_0} \rangle$ we further strengthen **Cn** with the postulates $\{\mathfrak{p}_1 \to \mathfrak{p}_k \mid k > 1\}$. In this calculus $\mathfrak{p}_0$ and $\neg\mathfrak{p}_0 \wedge \mathfrak{p}_1$ axiomatize complete theories; while all other complete theories assert $\neg\mathfrak{p}_1$ and denumerably many other things, and are not axiomatizable. The general case of a characteristic pair $\langle m, 2^{\aleph_0} \rangle$ is similar. For each i, let I_i be the set $\{\mathfrak{p}_i \to \mathfrak{p}_k \mid k > i\}$, and let **Cn** be enriched with the set $\mathrm{I}_0 \cup \cdots \cup \mathrm{I}_{m-1}$. Then $\mathfrak{p}_0$ and $\neg\mathfrak{p}_0 \wedge \mathfrak{p}_1$ and ...and $\neg\mathfrak{p}_0 \wedge \neg\mathfrak{p}_1 \wedge \cdots \wedge \mathfrak{p}_{m-1}$ all axiomatize complete theories. But there are remain continuum many complete theories — all those that assert $\neg\mathfrak{p}_{m-1}$ — that are unaxiomatizable.

We may construct a calculus with characteristic pair $\langle \aleph_0, 1 \rangle$ by taking the elements of all the I_i as new postulates. For any theory whose truth value assignments ever change from true to false — that is, that asserts both $\mathfrak{p}_i$ and $\neg\mathfrak{p}_k$ for k later than i — is inconsistent. Thus the only complete theories are those whose assignments are truth preserving with increasing index: for each k, there is an axiomatizable one that asserts $\neg\mathfrak{p}_i$ for all $i < k$, and also asserts $\mathfrak{p}_k$; and in addition to these, there is an unaxiomatizable one, which asserts $\neg\mathfrak{p}_i$ for all i.

A slight variation yields the characteristic pair $\langle \aleph_0, 2 \rangle$. We take not all the I_i as new sets of postulates, but only those where $i > 0$. The same argument applies, except that both $\mathfrak{p}_0$ and $\neg\mathfrak{p}_0$ are consistent with all other truth value assignments; in particular, they are both consistent with the theory that asserts $\neg\mathfrak{p}_i$ for every positive i. This produces two unaxiomatizable complete theories. To generate a calculus with characteristic pair $\langle \aleph_0, n \rangle$ for any other positive n is now quite straightforward. First find j such that $2^j < n < 2^{j+1}$, and take as new postulates all the elements of all the I_i for $i > j$. This produces (in the same way as above) 2^{j+1} unaxiomatizable complete theories. By adding further postulates we may disqualify $2^{j+1} - n$ of them as logically false, without reducing to finitude the set of axiomatizable complete theories. (This follows from Theorem 0. If there were only finitely many complete theories left, they would all have to be axiomatizable.)

To obtain a calculus with characteristic pair $\langle \aleph_0, 2^{\aleph_0} \rangle$ we must strengthen **Cn** neither with a finite family nor with a co-finite family of I_i as new postulates, but with a family that is both infinite and co-infinite; for example,
5 all the elements of all the I_{2i}. This certainly yields a denumerable set of axiomatizable complete theories; $\mathfrak{p}_0$, for example, is complete; so too are

both $\neg\mathfrak{p}_0 \wedge \mathfrak{p}_1 \wedge \mathfrak{p}_2$ and $\neg\mathfrak{p}_0 \wedge \neg\mathfrak{p}_1 \wedge \mathfrak{p}_2$; so is $\neg\mathfrak{p}_0 \wedge \mathfrak{p}_1 \wedge \neg\mathfrak{p}_2 \wedge \mathfrak{p}_3 \wedge \mathfrak{p}_4$, and so on. But there are also continuum many unaxiomatizable complete theories. If $\mathbf{X}$ is the theory that asserts $\mathfrak{p}_{2i}$ for every i, then every assignment of truth values to all the odd-numbered letters $\mathfrak{p}_{2i+1}$ is consistent with $\mathbf{X}$, and conjoined with it yields a complete theory.

This leaves only the characteristic pair $\langle \aleph_0, \aleph_0 \rangle$ — for which, as noted in section 1, there is a simple example in predicate logic. For an example in propositional logic, we add more postulates to the calculus described in the previous paragraph. Let K_{2i+1} be defined in somewhat the same way as I_i, but with a restriction to odd-numbered letters: that is, K_{2i+1} is defined to be the set $\{\mathfrak{p}_{2i+1} \to \mathfrak{p}_{2k+1} \mid k > i\}$. Then the logical postulates added to the present calculus are the elements of all the I_{2i} and the elements of all the K_{2i+1}. The only consistent assignments of truth values to the odd-numbered $\mathfrak{p}_{2i+1}$ will, in the same way as before, be those that are truth preserving with advancing i. Not all the axiomatizable complete theories will be retained, since (for instance) $\neg\mathfrak{p}_0 \wedge \mathfrak{p}_1 \wedge \mathfrak{p}_2 \wedge \neg\mathfrak{p}_3$ is no longer consistent; but $\mathfrak{p}_0$ survives, as do $\neg\mathfrak{p}_0 \wedge \mathfrak{p}_1 \wedge \neg\mathfrak{p}_2$ and $\neg\mathfrak{p}_0 \wedge \mathfrak{p}_1 \wedge \mathfrak{p}_2 \wedge \mathfrak{p}_3 \wedge \mathfrak{p}_4$, and all others with no negated odd-numbered letters. That is, the axiomatizable complete theories are still denumerable in number. As for the unaxiomatizable ones, it is clear that each of these asserts $\neg\mathfrak{p}_{2i}$ for every i, and in addition either $\mathfrak{p}_1$ or $\neg\mathfrak{p}_1 \wedge \mathfrak{p}_3$, or $\neg\mathfrak{p}_1 \wedge \neg\mathfrak{p}_3 \wedge \mathfrak{p}_5$, or ..., or, finally, $\neg\mathfrak{p}_{2i+1}$ for every i. All these latter theories are unaxiomatizable, and they are clearly denumerably many.

3 Basics

Let S, the set of all sentences of a language, be either finite or denumerably infinite. An operation $\mathbf{Cn} : \wp(S) \mapsto \wp(S)$ is a *consequence operation* if the following three conditions hold (Tarski, pp. 31, 63f.).

(1) if $X \subseteq S$ then $X \subseteq \mathbf{Cn}(X) \subseteq S$

(2) if $X \subseteq S$ then $\mathbf{Cn}(\mathbf{Cn}(X)) = \mathbf{Cn}(X)$

(3) if $X \subseteq S$ then $\mathbf{Cn}(X) = \bigcup\{\mathbf{Cn}(Y) \mid Y \subseteq X \text{ and } |Y| < \aleph_0\}$. | | 6

Condition (3) is often called *compactness*, though as formulated here it is a rather stronger constraint. If sentences A, B in S are such that e

$A \in \mathbf{Cn}(\{B\})$ and $B \in \mathbf{Cn}(\{A\})$, we call them *logically equivalent* under $\mathbf{Cn}$. Henceforth, logically equivalent sentences will, for convenience, be identified, and will be referred to as *propositions*. (Nothing important hangs on this identification.) We shall write $\mathbf{S}$ for the set of all propositions, and in future use $\mathbf{Cn}$ for the corresponding consequence operation on subsets of $\mathbf{S}$. Like S, $\mathbf{S}$ is at most denumerable. The pair $\langle \mathbf{S}, \mathbf{Cn} \rangle$ will be called a [deductive] *calculus*. It will be assumed that the operation $\mathbf{Cn}$ is based on the consequence operation of classical propositional logic; that is, not simply that if X implies x classically then x is an element of $\mathbf{Cn}(X)$; but also that all the standard classical theorems, including the deduction theorem, are satisfied by $\mathbf{Cn}$.

A subset $\mathbf{X} \subseteq \mathbf{S}$ is called a [deductive] *theory* if $\mathbf{X} = \mathbf{Cn}(\mathbf{X})$. If $\mathbf{X} = \mathbf{Cn}(X)$ for some finite set $X \subseteq \mathbf{S}$, we call $\mathbf{X}$ [finitely] *axiomatizable*. Note that, because $\mathbf{Cn}$ incorporates the standard rules for conjunction, and
f $\mathbf{Cn}(\varnothing) = \mathbf{Cn}(\{t\})$ for any tautological t, any finite set of propositions
is equivalent to a single proposition. We usually abbreviate $\mathbf{Cn}(\{x\})$ by $\mathbf{Cn}(x)$, or simply by x. The connectives $\vee$ and $\wedge$ may be straightforwardly extended from propositions to theories.

$$\mathbf{X} \vee \mathbf{Z} = \mathbf{X} \cap \mathbf{Z} \tag{4}$$
$$\mathbf{X} \wedge \mathbf{Z} = \mathbf{Cn}(\mathbf{X} \cup \mathbf{Z}). \tag{5}$$

It is easily checked that $\vee$ and $\wedge$ so defined are extensions of their proposi-
g tional ancestors. Expressions such as $x \wedge \mathbf{X}$ will be used in the obvious way.
We define analogously the disjunction and conjunction of arbitrary classes of theories.

$$\bigvee \mathfrak{K} = \bigcap \mathfrak{K} \tag{6}$$
$$\bigwedge \mathfrak{K} = \mathbf{Cn}(\bigcup \mathfrak{K}). \tag{7}$$

Many distributive laws fail for infinite disjunctions and conjunctions. One
h that holds, $x \wedge \bigvee \mathfrak{K} = \bigvee\{x \wedge \mathbf{X} \mid \mathbf{X} \in \mathfrak{K}\}$, is left as an exercise. It will used
below in Lemma 22 and Theorem 25.

| 7 Taken as a set of propositions, the largest theory is $\mathbf{S}$, the | self-contradictory theory, whilst the smallest is $\mathbf{T} = \mathbf{Cn}(\varnothing)$, the tautological theory. Each is axiomatizable; $\mathbf{S}$ by the absurdity $\perp$, and $\mathbf{T}$ by the empty set $\varnothing$. Any theory distinct from $\mathbf{S}$ is *consistent*. If $\mathbf{X} \subseteq \mathbf{Z}$ we say that $\mathbf{Z}$ is an

extension of $\mathbf{X}$, or that $\mathbf{Z}$ *implies* $\mathbf{X}$, and write $\mathbf{Z} \vdash \mathbf{X}$. If $\mathbf{Z}$ is an extension of $\mathbf{X}$ and distinct from $\mathbf{X}$, then it is a *proper extension* of $\mathbf{X}$.

A theory is called *complete* if its only proper extension is $\mathbf{S}$. We shall use $\mathbf{\Omega}$ and $\mathbf{\Psi}$ for complete theories, and ω and ψ when they are axiomatizable. The class of complete theories of a calculus is denoted by $\mathfrak{S}$. Suppose that $\mathbf{\Omega}$ is complete and does not imply x. Then $\mathbf{\Omega} \wedge \neg x$ cannot be $\mathbf{S}$, and so (by the definition of completeness) it is $\mathbf{\Omega}$; hence $\mathbf{\Omega} \vdash \neg x$. This is the property of negation completeness. Lindenbaum's theorem (mentioned above) is another central result concerning complete theories, but its proof is so well known that we do not pause to repeat it here. (The proof of Lemma 6 is variation on it.) We shall however prove the following refinement of Lindenbaum's theorem, due to Tarski.

THEOREM 1: For any theory $\mathbf{X}$ in a calculus that is based on classical propositional logic,

$$\mathbf{X} \;=\; \bigvee\{\mathbf{\Omega} \mid \mathbf{\Omega} \vdash \mathbf{X}\}.$$

PROOF: Let $\mathfrak{K}$ be the class of complete theories that imply $\mathbf{X}$. It is clear that $\bigvee \mathfrak{K}$ implies $\mathbf{X}$. If $\mathbf{X} = \mathbf{S}$ or if $\bigvee \mathfrak{K}$ is empty (as it will be only if $\mathbf{S}$ is) then $\mathbf{X}$ implies $\bigvee \mathfrak{K}$ and there is nothing more to be proved. Otherwise, suppose that z is a consequence of $\bigvee \mathfrak{K}$ that is not also a consequence of $\mathbf{X}$. Then $\mathbf{X} \cup \{\neg z\}$ is consistent, so may be extended to a complete $\mathbf{\Omega}$. Obviously $\mathbf{\Omega}$ implies $\neg z$. But since z is implied by $\bigvee \mathfrak{K}$, it is implied by every complete theory that extends $\mathbf{X}$, and hence $\mathbf{\Omega} \vdash z$. This makes $\mathbf{\Omega}$ inconsistent, which is impossible. ∎

By the *range* $\mathcal{R}[\mathbf{X}]$ of a theory $\mathbf{X}$ is meant the class of all complete theories that extend $\mathbf{X}$. Tarski's theorem tells us that $\mathbf{X} = \bigvee \mathcal{R}[\mathbf{X}]$. If $\mathfrak{J}$ is a class of complete theories then $\mathfrak{J} \subseteq \mathcal{R}[\bigvee \mathfrak{J}]$, but the converse is not true in general. Let $\mathbf{\Omega}$ be unaxiomatizable, for instance, and let $\mathfrak{J} = \mathfrak{S} \setminus \{\mathbf{\Omega}\}$. Then if $\bigvee \mathfrak{J} \vdash z$, we have $\bigvee \mathfrak{S} = \bigvee \mathfrak{J} \vee \mathbf{\Omega} \vdash z \vee \mathbf{\Omega}$. By Theorem 1, $\bigvee \mathfrak{S} = \mathbf{T}$. Hence $\neg z \vdash \mathbf{\Omega}$, which means that z is in $\mathbf{T}$. Hence $\bigvee \mathfrak{J} = \mathbf{T}$, and $\mathcal{R}[\bigvee \mathfrak{J}]$ is not included in $\mathfrak{J}$. | | 8

THEOREM 2: Let $\mathfrak{J}$ and $\mathfrak{K}$ be the classes respectively of the axiomatizable and unaxiomatizable complete theories of a calculus. Then $\bigvee \mathfrak{K} = \bigwedge\{\neg\omega \mid \omega \in \mathfrak{J}\}$ and $\mathcal{R}[\bigvee \mathfrak{K}] = \mathfrak{K}$. i

Proof: If $\boldsymbol{\Omega} \in \mathfrak{K}$ and $\omega \in \mathfrak{I}$, then $\boldsymbol{\Omega}$ cannot imply ω, and so it implies $\neg\omega$. Thus $\boldsymbol{\Omega} \vdash \bigwedge\{\neg\omega \mid \omega \in \mathfrak{I}\}$, which we call $\mathbf{X}$. If $\mathbf{X}$ had a complete extension not in $\mathfrak{K}$, that extension would be axiomatizable; that is, some ω would imply $\mathbf{X}$, and thus imply $\neg\omega$. This is impossible. In other words, $\mathcal{R}[\mathbf{X}] = \mathfrak{K}$, so by Theorem 1, $\mathbf{X} = \bigvee \mathfrak{K}$. ∎

4 Simple Results

The number of axiomatizable complete theories in a calculus will be denoted by α, the number of unaxiomatizable complete theories by β. As has already been noted, the axiom of choice is assumed throughout, but the continuum hypothesis is not assumed. (For a result related to the present inquiry that can be proved without the axiom of choice, see Tarski, pp. 82f.) In this section we prove some of the simpler constraints on the pair $\langle \alpha, \beta \rangle$.

Theorem 3: For every calculus that is based on classical propositional logic,

(8) $$0 \leq \alpha \leq \aleph_0,$$

(9) $$0 \leq \beta \leq 2^{\aleph_0}.$$

Proof: There are at most denumerably many distinct propositions, so at most denumerably many distinct axiomatizable theories. On the other hand, every theory is a subset of $\mathbf{S}$, so there are at most $2^{\aleph_0}$ distinct theories. ∎

Theorem 4: For every calculus that is based on classical propositional logic,

(10) $$\text{if} \quad \alpha + \beta < \aleph_0, \quad \text{then} \quad \beta = 0.$$

Proof: By Theorem 1, every theory is the disjunction of all the complete theories that extend it. Since finitely many theories can be disjoined in only finitely many ways, there are only finitely many theories; and therefore, since each proposition x generates its own axiomatizable theory $\mathbf{Cn}(x)$, only finitely many propositions. But then no theory can fail to be finitely
| 9 axiomatizable, and $\beta = 0$. ∎|

Theorem 5: For every calculus that is based on classical propositional logic,

$$\text{if} \quad \beta = 0, \quad \text{then} \quad \alpha < \aleph_0. \tag{11}$$ j

Proof: Let $\mathfrak{J}$ be the class of all the axiomatizable complete theories, and ψ, ω be distinct elements of $\mathfrak{J}$. Since ω has no consistent proper extension, ψ does not imply ω; hence ψ, being negation complete, implies $\neg\omega$.

Assume that $\mathfrak{J}$ is infinite. We shall show that there is a complete theory $\boldsymbol{\Omega}$ outside $\mathfrak{J}$.

Put $\mathfrak{J}^\star = \{\neg\omega \mid \omega \in \mathfrak{J}\}$, the set of the negations of elements of $\mathfrak{J}$. Let Y be any finite subset of $\mathfrak{J}^\star$, and ψ an element of $\mathfrak{J}$ whose negation $\neg\psi$ is not in Y. Since ψ implies all the elements of Y, Y is consistent. This holds for any finite $Y \subseteq \mathfrak{J}^\star$, so by compactness $\mathfrak{J}^\star$ is consistent. By Lindenbaum's theorem there is a complete theory $\boldsymbol{\Omega}$ that implies every element of $\mathfrak{J}^\star$. It is clear that $\boldsymbol{\Omega}$ cannot be an axiomatizable theory ω, for then $\neg\omega$ would be in $\mathfrak{J}^\star$; this would mean that $\boldsymbol{\Omega}$ implied both ω and $\neg\omega$, and was not consistent. ∎

5 Incomplete Finite Completability

Theorem 4 assures us that if α and β are both finite then $\beta = 0$. In response to the question of what is possible if α is finite and β is infinite we shall show in Theorem 7 that β can only be $2^{\aleph_0}$. Remember that the continuum hypothesis is not being assumed, so that more needs to shown than simply that β exceeds $\aleph_0$. To start thinking along the right lines for the proof, consider an informal proof, based on Gödel's incompleteness theorem, that Peano arithmetic $\mathbf{P}$ (which is assumed to be consistent) can be completed in continuum many ways. Gödel's theorem shows that for any effectively presented extension $\mathbf{X}$ of $\mathbf{P}$ (including of course $\mathbf{P}$ itself) there is an effectively specifiable sentence G (the gödelsentence of $\mathbf{X}$) such that both $\mathbf{X} \wedge G$ and $\mathbf{X} \wedge \neg G$ are consistent. Let G_1 be the gödelsentence for $\mathbf{P}$, and G_0 its negation. Likewise, let G_{11} be the gödelsentence for $\mathbf{P} \wedge G_1$, and G_{10} its negation; and G_{01} be the gödelsentence for $\mathbf{P} \wedge G_0$, and G_{00} its negation; and so on. Continuing to eternity, we obtain as many consistent extensions of $\mathbf{P}$ as there are denumerable sequences of 0s and 1s; that is, there exist (at least) $2^{\aleph_0}$ extensions of the theory | $\mathbf{P}$. Since by Lindenbaum's | 10 theorem, each has a complete extension, $\mathbf{P}$ has at least (in fact, exactly) $2^{\aleph_0}$ complete extensions.

This proof is hardly watertight — to take the most glaring gap, there is no explicit argument to the effect that each two extensions are different — or even that G_{01} is different from G_{10} —, but it is not irremediable. We redeem it below it Lemma 6.

We call a theory $\mathbf{Z}$ a *finite extension* of the theory $\mathbf{X}$ if there is some proposition x for which $\mathbf{Z} = x \wedge \mathbf{X}$. More specifically, $\mathbf{X}$ is *finitely completable* if there is some complete theory that finitely extends it. Finite completability has little to do with logical strength — what we might think of informally as distance from a complete theory. $\mathbf{T}$ is finitely completable whenever $\alpha > 0$, but many of its extensions may not be. In general, if $\mathbf{X}$ has a finitely axiomatizable complete extension ω, then it is finitely completable: $\omega = \omega \wedge \mathbf{X}$. But the converse is often false; for, since $\mathbf{\Omega} = x \wedge \mathbf{\Omega}$ for any consequence x of $\mathbf{\Omega}$, any complete theory, axiomatizable or not, is finitely completable.

For our present purposes the crucial distinction is between those calculi in which all consistent theories are finitely completable, and those in which there is at least one consistent theory that is not. In calculi of the former kind, β is at most $\aleph_0$, as we shall show in the next section; in those of the latter kind, $\beta = 2^{\aleph_0}$, as will be shown here.

LEMMA 6: If $\mathbf{X}$ is consistent and not finitely completable, then it has $2^{\aleph_0}$ complete extensions.

k PROOF: Let $z_0 = \bot$, and let $\{z_i \mid i > 0\}$ be an enumeration without repetition of the consistent propositions of the calculus. For each deductive theory $\mathbf{Z}$ define

$$\begin{aligned} \mathbf{Z1} &= \mathbf{Z} \wedge z_k \\ \mathbf{Z0} &= \mathbf{Z} \wedge \neg z_k \end{aligned}$$

where $k = k[\mathbf{Z}]$ is determined in the following manner:

(A) if $\mathbf{Z}$ is consistent but not complete, then $k[\mathbf{Z}]$ is the least i for
| 11 which neither z_i nor $\neg z_i$ is implied by $\mathbf{Z}$; |

(B) if $\mathbf{Z}$ is [consistent and] complete, then $k[\mathbf{Z}] = 0$;

(C) if $\mathbf{Z}$ is inconsistent, then $k[\mathbf{Z}] = 0$;

For any ω-sequence (progression) σ of 0s and 1s we define $\mathbf{Z}\boldsymbol{\sigma}$ as $\bigwedge\{\mathbf{Z}\boldsymbol{\tau} | \tau \text{ is an initial segment of } \sigma\}$.

We shall show that (i) if $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are distinct sequences of 0s and 1s, $\mathbf{Z}\boldsymbol{\sigma}$ and $\mathbf{Z}\boldsymbol{\tau}$ are distinct provided that they are not both $\mathbf{S}$; and (ii) if $\boldsymbol{\sigma}$ is infinite and $\mathbf{Z}\boldsymbol{\sigma}$ is consistent, then $\mathbf{Z}\boldsymbol{\sigma}$ is complete. It is clear that if $\mathbf{X}$ is not finitely completable then for every sequence $\boldsymbol{\sigma}$ the theory $\mathbf{X}\boldsymbol{\sigma}$ is consistent. It follows that $\mathbf{X}$ has as many complete extensions as there are progressions of 0s and 1s: that is, it has $2^{\aleph_0}$ extensions.

Let σ and τ be sequences of 0s and 1s. It follows from the definitions that if τ is an initial segment of σ, then $\mathbf{X}\boldsymbol{\sigma}$ implies $\mathbf{X}\boldsymbol{\tau}$. We can therefore visualize the theories $\mathbf{X}\boldsymbol{\sigma}$ as branches of a tree rooted at $\mathbf{X}$, where at each level the branch $\mathbf{Z}$ that has developed to date either bifurcates, sends out a single twig to the next level, which then dries up, or dries up at once. Bifurcation necessarily occurs if $k[\mathbf{Z}]$ is determined by (A); for then both $\mathbf{Z1}$ and $\mathbf{Z0}$ are strictly stronger than $\mathbf{Z}$ is. In case (B), $\mathbf{Z1}$ is inconsistent, and so stronger than $\mathbf{Z}$ is, while $\mathbf{Z0} = \mathbf{Z}$; hence there is a single final extension of the branch. In case (C), $\mathbf{Z} = \mathbf{Z1} = \mathbf{Z0}$, and the branch dies. It is apparent that every infinite branch bifurcates at every level. Whether or not the branch it represents is infinite, $\mathbf{X}\boldsymbol{\sigma}$ is defined for every infinite sequence σ of 0s and 1s.

(i) Let σ and τ be distinct infinite sequences of 0s and 1s. If one of $\mathbf{X}\boldsymbol{\sigma}$ and $\mathbf{X}\boldsymbol{\tau}$ is consistent, and the other is not, the certainly they are not identical. So we may suppose that both are consistent, which means that $\mathbf{X}$ is consistent and not complete, and (A) applies. Let π be the longest initial segment — which is finite, and perhaps empty — that σ and τ have in common (that is, they differ for the first time immediately after the segment π). Then each of $\mathbf{X}\boldsymbol{\sigma}$ and $\mathbf{X}\boldsymbol{\tau}$ implies $\mathbf{X}\boldsymbol{\pi}$; and indeed, one of them implies $\mathbf{X}\boldsymbol{\pi}\mathbf{1}$ and the other implies $\mathbf{X}\boldsymbol{\pi}\mathbf{0}$. But $\mathbf{X}\boldsymbol{\pi}\mathbf{1}$ and $\mathbf{X}\boldsymbol{\pi}\mathbf{0}$ are incompatible; and so, since neither $\mathbf{X}\boldsymbol{\sigma}$ nor $\mathbf{X}\boldsymbol{\tau}$ is $\mathbf{S}$, they cannot be identical.

(ii) Suppose that $\mathbf{X}\boldsymbol{\sigma}$ is consistent, but not complete. Let i be | the least | 12
number for which neither z_i nor $\neg z_i$ is implied by $\mathbf{X}\boldsymbol{\sigma}$, and let τ be the sequence consisting of the first $i + 1$ elements of $\mathbf{X}\boldsymbol{\sigma}$. The $\mathbf{X}\boldsymbol{\tau}$ must have been obtained from $\mathbf{X}$ by $i + 1$ applications of (A), and so $\mathbf{X}$ must have been enriched with $i + 1$ new propositions. By the stage of $\mathbf{X}\boldsymbol{\tau}$, that is, the

proposition z_i must have been taken account of, and so $\mathbf{X}\sigma$ (which implies $\mathbf{X}\tau$) implies either z_i or $\neg z_i$, contrary to the specification of i. Hence there is no such i, and $\mathbf{X}\sigma$ is complete. ∎

THEOREM 7: For every calculus that is based on classical propositional logic,

$$\text{if } \alpha < \aleph_0 \leq \beta, \quad \text{then} \quad \beta = 2^{\aleph_0}. \tag{12}$$

PROOF: Let $\mathfrak{K}$ be the class of unaxiomatizable complete theories. Since α is finite, we may deduce from Theorem 2 that $\bigvee \mathfrak{K}$ is axiomatizable and that its range is $\mathfrak{K}$. Thus it is consistent. But it is not finitely completable, for an unaxiomatizable theory cannot finitely extend an axiomatizable theory.
l ory. We apply the Lemma with $\mathbf{X} = \bigvee \mathfrak{K}$. ∎

If α is infinite, the disjunction of the unaxiomatizable complete theories
m is unaxiomatizable (a simple exercise), and the conclusion of Lemma 6 does not follow unconditionally. The best that we can prove is this.

THEOREM 8: For every calculus that is based on classical propositional logic and contains at least one consistent theory that is not finitely completable

$$\text{if } \alpha = \aleph_0 \quad \text{then} \quad \beta = 2^{\aleph_0}. \tag{13}$$

PROOF: Immediate. ∎

n 6 Complete Finite Completability

We shall show in this section that if every consistent theory of a calculus is only a proposition away from completeness, then the complete theories can be listed, in disjoint non-empty blocks $\mathfrak{A}_\nu$ that are at most denumerable, in this way: start with the complete theories that are finite completions of $\mathbf{T}$ (that is, those that are axiomatizable); continue with the finite completions of the disjunction of those that remain; and so on, until all complete theories
| 13 are accounted for. It will be established that no proposition y | can be involved in more than one of these acts of completion; and thus that there is not more than a denumerable number of complete theories.

To be more exact: (14) defines for any calculus whatever two sequences $\mathfrak{S}_\nu$ and $\mathfrak{A}_\nu$ of classes of complete theories. The definition is by simultaneous recursion. The $\mathfrak{S}$-sequence, we shall show in (16), is contracting, while the elements of the $\mathfrak{A}$-sequence are pairwise disjoint (this is (18)). With the help of the axiom of choice we may show that from some ordinal o onwards every element of the $\mathfrak{A}$-sequence is empty.

$$
\begin{array}{lllcl}
(14) & \text{(a)} & \mathfrak{S}_0 & = & \mathfrak{S} \\
 & \text{(b)} & \text{for all } \nu,\ \mathfrak{A}_\nu & = & \{\mathbf{\Omega} \mid \mathbf{\Omega} \text{ finitely extends } \bigvee \mathfrak{S}_\nu\} \\
 & \text{(c)} & \mathfrak{S}_{\nu+1} & = & \mathfrak{S}_\nu \setminus \mathfrak{A}_\nu \\
 & \text{(d)} & \text{for limit } \lambda,\ \mathfrak{S}_\nu & = & \bigcap\{\mathfrak{S}_\nu \mid \nu < \lambda\}.
\end{array}
$$

It is a trivial consequence of Theorem 1 that $\bigvee \mathfrak{S} = \mathbf{T}$. Thus $\mathfrak{A}_0$ is the class of all the axiomatizable complete theories. Accordingly, $\mathfrak{S}_1$ is the class of unaxiomatizable complete theories. The characteristic pair $\langle \alpha, \beta \rangle$ of a calculus is the pair of cardinals $\langle |\mathfrak{A}_0|, |\mathfrak{S}_1| \rangle$.

Lemma 9: For every calculus that is based on classical propositional logic,

$$
\begin{array}{ll}
(15) & \mathfrak{A}_\nu \text{ is at most denumerable;} \\
(16) & \text{if } \mu < \nu \text{ then } \mathfrak{S}_\nu \subseteq \mathfrak{S}_\mu; \\
(17) & \mathfrak{A}_\nu \subseteq \mathfrak{S}_\nu; \\
(18) & \text{if } \mu \neq \nu \text{ then } \mathfrak{A}_\mu \cap \mathfrak{A}_\nu = \varnothing.
\end{array}
$$

Proof: The proof of (15) is an immediate consequence of (14b); the theory $\bigvee \mathfrak{S}_\nu$ can be finitely extended in at most a denumerable number of ways. (16) follows at once from (14c) and (14d).

As for (17), suppose that $\mathbf{\Psi}$ belongs to $\mathfrak{A}_\nu$. Then $\mathbf{\Psi} = y \wedge \bigvee \mathfrak{S}_\nu$ for some proposition y. Clearly $\bigvee \mathfrak{S}_\nu$, which is consistent, cannot imply $\neg y$, and so some $\mathbf{\Omega}$ in $\mathfrak{S}_\nu$ must imply y. Any such $\mathbf{\Omega}$ consequently implies $\mathbf{\Psi}$, which is the conjunction of y and $\bigvee \mathfrak{S}_\nu$. Since no complete theory can imply any other complete theory, $\mathbf{\Omega} = \mathbf{\Psi}$. Hence $\mathbf{\Psi}$ belongs to $\mathfrak{S}_\nu$.

To prove (18), suppose that $\mu < \nu$ and that $\mathbf{\Omega}$ belongs to $\mathfrak{A}_\mu$. By | (17) | 14
$\mathbf{\Omega}$ belongs to $\mathfrak{S}_\mu$, and so by (14c) $\mathbf{\Omega}$ does not belong to $\mathfrak{S}_{\mu+1}$. But then, by (16), $\mathbf{\Omega}$ is not in $\mathfrak{S}_\nu$. So by (17) again, $\mathbf{\Omega}$ is not in $\mathfrak{A}_\nu$. ∎

LEMMA 10: For every ordinal ν o

$$\mathcal{R}[\bigvee \mathfrak{S}_\nu] \quad = \quad \mathfrak{S}_\nu. \tag{19}$$

PROOF: We need prove only that $\mathcal{R}[\bigvee \mathfrak{S}_\nu] \subseteq \mathfrak{S}_\nu$, since the converse is obvious. For $\nu = 0$, the result is trivial, since $\mathfrak{S}_0 = \mathfrak{S}$. Suppose that (19) holds for ν, and that $\mathbf{\Psi} \vdash \bigvee \mathfrak{S}_{\nu+1}$. By (16), $\mathbf{\Psi} \vdash \bigvee \mathfrak{S}_\nu$, and so by the induction hypothesis $\mathbf{\Psi} \in \mathfrak{S}_\nu$. By (14c), either $\mathbf{\Psi} \in \mathfrak{S}_{\nu+1}$ or $\mathbf{\Psi} \in \mathfrak{A}_\nu$. We show that the latter is not possible.

If $\mathbf{\Psi} \in \mathfrak{A}_\nu$, then $\mathbf{\Psi} = y \wedge \bigvee \mathfrak{S}_\nu$ for some y, and so $y \wedge \bigvee \mathfrak{S}_\nu \vdash \bigvee \mathfrak{S}_{\nu+1} \vdash \bigvee \mathfrak{S}_\nu$. Thus $\mathbf{\Psi} = y \wedge \bigvee \mathfrak{S}_\nu = y \wedge \bigvee \mathfrak{S}_{\nu+1}$. It follows that $\mathbf{\Psi}$ is in both $\mathfrak{A}_\nu$
p and $\mathfrak{A}_{\nu+1}$, in contradiction to (18). ∎

LEMMA 11: For every limit ordinal λ

$$\bigvee \mathfrak{S}_\lambda \quad = \quad \bigwedge\{\mathfrak{S}_\nu \mid \nu < \lambda\} \tag{20}$$

PROOF: By (19), $\mathbf{\Omega} \vdash \bigvee \mathfrak{S}_\lambda$ if and only if $\mathbf{\Omega} \in \mathfrak{S}_\lambda$; that is, by (14d), if and only if $\mathbf{\Omega} \in \mathfrak{S}_\nu$ for every $\nu < \lambda$. By (19) again, together with (7), this holds if and only if $\mathbf{\Omega} \vdash \bigwedge\{\bigvee \mathfrak{S}_\nu \mid \nu < \lambda\}$. But by Theorem 1, distinct theories cannot have the same range. ∎

LEMMA 12: Let $\mathbf{\Psi} \in \mathfrak{A}_\mu$ and $\mathbf{\Omega} \in \mathfrak{A}_\nu$ be distinct complete theories. Then there is no proposition y for which both $\mathbf{\Psi} = y \wedge \bigvee \mathfrak{S}_\mu$ and $\mathbf{\Omega} = y \wedge \bigvee \mathfrak{S}_\nu$.

PROOF: Suppose for convenience that $\mu \leq \nu$. Then $\mathfrak{S}_\nu \subseteq \mathfrak{S}_\mu$ by (16), and so $\bigvee \mathfrak{S}_\nu \vdash \bigvee \mathfrak{S}_\mu$. It follows that $\mathbf{\Omega} = y \wedge \bigvee \mathfrak{S}_\nu \vdash y \wedge \bigvee \mathfrak{S}_\mu = \mathbf{\Psi}$. Being complete theories, $\mathbf{\Omega}$ and $\mathbf{\Psi}$ are therefore identical, contrary to hypothesis. ∎

LEMMA 13: There exists an ordinal o for which $\mathfrak{A}_o = \varnothing$.

PROOF: By the axiom of choice, there is a least ordinal λ with cardinality greater than $2^{\aleph_0}$. Since there are just $2^{\aleph_0}$ complete theories, the elements of $\mathfrak{S}$ cannot be well ordered in a sequence of length λ. But by (14c) whenever
| 15 $\mathfrak{A}_\nu$ is not empty $\mathfrak{S}_{\nu+1}$ is a proper subset of $\mathfrak{S}_\nu$. It | follows that $\mathfrak{A}_0$ becomes empty for some $o < \lambda$. ∎

Theorem 14: For every calculus that is based on classical propositional logic and contains no consistent theory that is not finitely completable

$$(21) \qquad \beta \leq \aleph_0.$$

Proof: Let o be the least ordinal for which $\mathfrak{A}_o$ is empty. Since every consistent theory is finitely completable, it follows from (14b) that $\bigvee \mathfrak{S}_o = \mathbf{S}$, and therefore that o is the least ordinal for which $\mathfrak{S}_o$ is empty. Thus by (16) and (18), every element of $\mathfrak{S}$ belongs to exactly one $\mathfrak{A}_\nu$ for $\nu < o$. Now each element of each $\mathfrak{A}_\nu$ is obtained by conjoining some proposition y to the appropriate $\bigvee \mathfrak{S}_\nu$, and by Lemma 12 no proposition y is in this way associated with more than one complete theory. It follows that $\mathfrak{S}$ is at most denumerable, and thus that there are at most $\aleph_0$ unaxiomatizable complete theories. ∎

Theorem 15: For every calculus that is based on classical propositional logic,

$$(22) \qquad \text{if} \quad \alpha = \aleph_0 \leq \beta < 2^{\aleph_0} \quad \text{then} \quad \beta = \aleph_0.$$

Proof: By Theorems 8 and 14. ∎

7 Further Results

Theorem 0, stated in section 0, may be obtained by consolidating formulas (8)–(12) and (22). As already noted, Theorem 0 was proved by Mostowski by topological methods, exploiting a topology first defined by Stone on the class $\mathfrak{S}$. The crucial connection (which will not be proved here) is stated in Theorem 16.

Theorem 16: Let $\mathfrak{K}$ be a class of complete theories in a calculus that is based on classical propositional logic, and $\mathbf{\Omega}$ be any complete theory. Then $\mathbf{\Omega}$ is an accumulation point of $\mathfrak{K}$ in the Stone topology if and only if $\mathbf{\Omega} \vdash \bigvee \mathfrak{K}$ but is not a finite extension of $\bigvee \mathfrak{K}$.

Proof: Omitted. ∎

In topological terms, $\mathfrak{S}_{\nu+1}$ is the derived set of $\mathfrak{S}_\nu$, the set of its accumulation points. In logical terms, $\mathfrak{A}_\nu$ is the set of complete | theories that | 16 may be finitely axiomatized given that each element of each earlier $\mathfrak{A}_\mu$ is rejected.

In section 3 of [1937] Mostowski laid the foundations for a more detailed investigation of the structure of $\mathfrak{S}$, in which calculi are characterized not by $\langle \alpha, \beta \rangle$, the pair of cardinalities of $\mathfrak{A}_0$ and $\mathfrak{S}_1$, but by the values assumed by the cardinality $|\mathfrak{A}_\nu|$ for all ν, together with the cardinality of the set of complete theories not in any $\mathfrak{A}_\nu$. (Alternatively, as Theorem 20 shows, we may use $|\mathfrak{S}|$.) We have seen in (15) that each $\mathfrak{A}_\nu$ is at most denumerable. In the rest of this section, which does not go appreciably beyond Mostowski's work, we shall prove that the sequence of cardinalities $|\mathfrak{A}_\nu|$ is non-increasing, and contains at most one finite non-zero element between the denumerable elements (if any) and the zero ones. We first strengthen Lemma 13.

Lemma 17: There exists a denumerable ordinal o for which $\mathfrak{A}_o = \varnothing$.

Proof: Let o be the least ordinal for which $\mathfrak{A}_o$ is empty. By (14b), this is also the least ordinal for which $\bigvee \mathfrak{S}_o$ is not finitely completable. The elements of each $\mathfrak{A}_\nu$ for $\nu < o$ are thus all finite extensions of $\mathfrak{S}_\nu$; and, as in the proof of Theorem 14, there cannot be more than denumerable number of them. Hence o is denumerable. ∎

Lemma 18: For every calculus that is based on classical propositional logic and contains no consistent theory that is not finitely completable, the least ordinal o such that $\mathfrak{A}_o = \varnothing$ is a successor ordinal.

Proof: This is a simple compactness argument. By (16) the $\mathfrak{S}_\nu$ form a contracting sequence, so the theories $\bigvee \mathfrak{S}_\nu$ become stronger with increasing ν. Let λ be some limit ordinal. Then the conjunction $\bigwedge\{\bigvee \mathfrak{S}_\nu \mid \nu < \lambda\} = \mathbf{S}$ only if $\bigvee \mathfrak{S}_\nu = \mathbf{S}$ for some $\nu < \lambda$. But by (20), this conjunction is identical with $\bigvee \mathfrak{S}_\lambda$. Thus λ cannot be the least o for which $\mathfrak{S}_o$ is empty. ∎

Lemma 19: For every calculus that is based on classical propositional
| 17 logic, |

$$\text{if} \quad |\mathfrak{A}_\mu| = \aleph_0 \quad \text{then} \quad |\mathfrak{A}_{\mu+1}| > 0. \tag{23}$$

Proof: This is proved in the same way as Theorem 5. ∎

THEOREM 20: For every calculus that is based on classical propositional logic,

(24) if $|\mathfrak{S}| \leq \aleph_0$, then $|\mathfrak{S} \setminus \mathfrak{A}| = 0$;

(25) if $|\mathfrak{S}| = 2^{\aleph_0}$, then $|\mathfrak{S} \setminus \mathfrak{A}| = 2^{\aleph_0}$,

where o is the least ordinal for which $\mathfrak{A}_o$ is empty, and $\mathfrak{A} = \bigcup\{\mathfrak{A}_\nu \mid \nu < o\}$.

PROOF: We have seen already in Theorems 8 and 14 that $|\mathfrak{S}| = 2^{\aleph_0}$ if there is a consistent theory that is not finitely completable, and that $|\mathfrak{S}| \leq \aleph_0$ otherwise. The proof of Lemma 13 makes plain that in the latter case all complete theories belong to some $\mathfrak{A}_\nu$ for $\nu < o$. The proofs of Lemmas 6 and 17 make plain that in the former case $\bigvee \mathfrak{S}_o$ has $2^{\aleph_0}$ unaxiomatizable complete extensions, none of which belongs either to $\mathfrak{A}_o$ or to any earlier $\mathfrak{A}_\nu$. ∎

Mostowski *op. cit.*, section 3, p. 13, called the *characteristic* of a calculus in which $|\mathfrak{S}| = \aleph_0$ the pair $\langle \nu, n\rangle$, where ν is the greatest ordinal for which $\mathfrak{S}_\nu$ is not empty, and $|\mathfrak{S}_\nu| = n$. Theorem 14 and Lemma 18 ensure that there always is such an ordinal ν; as there is when $\mathfrak{S}$ is finite but not empty. It suits our purposes to transform this definition into one concerning the elements of the $\mathfrak{A}$-sequence.

LEMMA 21: For every calculus that is based on classical propositional logic,

(26) if $|\mathfrak{S}_\mu| < \aleph_0$, then for all $\nu \geq \mu$, $\mathfrak{A}_\nu = \mathfrak{S}_\nu$ and $\mathfrak{S}_{\nu+1} = \varnothing$.

PROOF: Suppose that $\mathfrak{S}_\mu = \{\boldsymbol{\Omega}_i \mid i < k\}$. It is easy to show that there q
must exist a set $Y = \{y_i \mid i < k\}$ of pairwise incompatible propositions such that for each $i < k$, $\boldsymbol{\Omega}_i \vdash y_i$. It follows that, for each $i < k$, $y_i \wedge \bigvee \mathfrak{S}_\mu = \bigvee\{y_i \wedge \boldsymbol{\Omega}_j \mid j < k\} = \boldsymbol{\Omega}_i$. Thus each element $\boldsymbol{\Omega}_i$ of $\mathfrak{S}_\mu$ is a finite completion of $\bigvee \mathfrak{S}_\mu$. Hence $\mathfrak{S}_\mu \subseteq \mathfrak{A}_\mu$. By (17), $\mathfrak{A}_\mu = \mathfrak{S}_\mu$. By (14c), $\mathfrak{S}_{\mu+1} = \varnothing$. Thus $\mathfrak{A}_{\mu+1} = \varnothing$ by (14b) or (17). The extension to $\nu > \mu + 1$ is immediate. ∎| | 18

LEMMA 22: For every calculus that is based on classical propositional logic,

(27) if $|\mathfrak{A}_\mu| < \aleph_0$, then for all $\nu > \mu$, $\mathfrak{A}_\nu = \mathfrak{S}_{\mu+1}$ and $\mathfrak{A}_\nu = \varnothing$.

PROOF: Suppose that $\mathfrak{A}_\mu = \{\mathbf{\Omega}_i \mid i < k\} = \{y_i \wedge \bigvee \mathfrak{S}_\mu \mid i < k\}$. Let $u = y_0 \vee \cdots \vee y_{k-1}$. If $\mathbf{\Psi} \in \mathfrak{S}_\mu$ and $\mathbf{\Psi} \vdash u$, then $\mathbf{\Psi} \vdash u \wedge \bigvee \mathfrak{S}_\mu$; that is, $\mathbf{\Psi} \vdash \bigvee\{\mathbf{\Omega}_i \mid i < k\}$. So $\mathbf{\Psi} \in \mathfrak{A}_\mu$. The converse is immediate.

Otherwise, if $\mathbf{\Psi}$ is not in $\mathfrak{A}_\mu$, then $\mathbf{\Psi} \vdash \neg u$, and so $\neg u \wedge \mathbf{\Psi} = \mathbf{\Psi}$. Hence $\neg u \wedge \bigvee \mathfrak{S}_\mu$, which by the infinite distributive law recorded in section 3, is the same as $\bigvee\{\neg u \wedge \mathbf{\Psi} \mid \mathbf{\Psi} \in \mathfrak{S}_\mu\}$, is identical also with $\bigvee\{\mathbf{\Psi} \mid \mathbf{\Psi} \in \mathfrak{S}_\mu \setminus \mathfrak{A}_\mu\}$; that is, with $\bigvee \mathfrak{S}_{\mu+1}$.

Thus $\bigvee \mathfrak{S}_{\mu+1}$ is a finite extension of $\bigvee \mathfrak{S}_\mu$. Hence any finite completion of $\bigvee \mathfrak{S}_{\mu+1}$ is a finite completion of $\bigvee \mathfrak{S}_\mu$. That is, $\mathfrak{A}_{\mu+1} \subseteq \mathfrak{A}_\mu$. By (18) these two sets are disjoint, so it follows that $\mathfrak{A}_{\mu+1} = \varnothing$. Thus by (14c), $\mathfrak{S}_{\mu+2} = \mathfrak{S}_{\mu+1}$. The extension to $\nu > \mu + 1$ is immediate. ∎

THEOREM 23: Let $\langle \nu, n \rangle$ be the characteristic of a calculus based on classical propositional logic in which $0 < |\mathfrak{S}| \leq \aleph_0$. Then ν is the least ordinal for which $\mathfrak{A}_\nu$ is finite and $n = |\mathfrak{A}_\nu|$.

PROOF: Immediate. ∎

The advantage of this formulation is that the characteristic may be defined in the same way for a calculus with $2^{\aleph_0}$ complete theories (and also for one with no complete theories). The only difference is that now n may be 0, though it may not be. We may call the *characteristic triple* of a calculus the triple $\langle \nu, n, \theta \rangle$ where ν is the least ordinal for which $\mathfrak{A}_\nu$ is finite, $n = |\mathfrak{A}_\nu|$, and $\theta = |\mathfrak{S}|$.

THEOREM 24: The characteristic triple of a finite or denumerable calculus must take one of the following values (m is a natural number, n is a natural number greater than 0, μ is a finite or denumerable ordinal and ν is a finite or denumerable ordinal greater than 0):

$$\langle 0, m, n \rangle, \langle \nu, n, \aleph_0 \rangle, \langle \mu, m, 2^{\aleph_0} \rangle.$$

19 PROOF: By Theorems 0 and 14, and Lemmas 17 and 19. ∎

8 Further Examples

It is not difficult to exhibit examples, similar to those given in sections 1 and 2, for all the possibilities not excluded by Theorem 24. Indeed, the examples can be generated in a uniform manner, depending only on which type of characteristic triple is involved. For the triple $\langle 0, m, m\rangle$, which is possessed only by calculi that are essentially finite, nothing more need be said: we may proceed exactly as in section 1.

For the other cases, we must base our calculi on a propositional language with denumerably many letters, and the trick is to arrange these letters in an appropriate transfinite sequence. For the triple $\langle \nu, n, \aleph_0\rangle$, the letters are arranged in a sequence of length $\omega^\nu \cdot n$; and for the triple $\langle \mu, m, 2^{\aleph_0}\rangle$, they are arranged in a sequence of length $\omega^\mu \cdot (m+1) + \omega$. When $\mu = 0$ this boils down to a sequence of length ω, and the construction given below reduces to that described in section 2.

Although there is not a great deal of difference between the two types of triple, $\langle \nu, n, \aleph_0\rangle$ and $\langle \mu, m, 2^{\aleph_0}\rangle$, it is easiest to take them one at a time. Let ν and n be respectively a fixed positive denumerable ordinal and fixed positive natural number; and $\{\mathsf{p}_\kappa \mid 0 < \kappa < \omega^\nu \cdot n\}$ be an ordering of the letters of some denumerable propositional language. The consequence operation will be that of classical propositional logic enriched with the logical postulates $\{\mathsf{p}_\iota \to \mathsf{p}_\kappa \mid 0 < \iota < \kappa < \omega^\nu \cdot n\}$. For each ξ satisfying $0 < \xi \leq \omega^\nu \cdot n$, we write

(28) $$\boldsymbol{\Omega}_\xi \quad = \quad \bigwedge\{\neg \mathsf{p}_\kappa \mid 0 < \kappa < \xi\} \wedge \bigwedge\{\mathsf{p}_\kappa \mid \xi \leq \kappa < \omega^\nu \cdot n\};$$

from which we derive as easy consequences

(29) (a) if $\xi < \omega^\nu \cdot n$ then $\boldsymbol{\Omega}_\xi \vdash \mathsf{p}_\xi$,

(b) $$\boldsymbol{\Omega}_{\omega^\nu \cdot n} \quad = \quad \bigwedge\{\neg \mathsf{p}_\kappa \mid 0 < \kappa < \omega^\nu \cdot n\}.$$

It is clear that the $\boldsymbol{\Omega}_\xi$ are all complete, and clear also that they are all the complete theories of the calculus. Conversely, this information, along with the fact that the underlying logic is classical, suffices to identify the logical postulates of the calculus; for if $0 < \iota < \kappa < \omega^\nu \cdot n$ then every $\boldsymbol{\Omega}_\xi$ that implies p_ι also implies p_κ; which by Theorem 1 amounts to asserting that p_ι implies p_κ. (Note that $\boldsymbol{\Omega}_\xi$ is a function of | ν and n as well as of ξ; and in | 20
the same way, $\mathfrak{S}_\eta$ and $\mathfrak{A}_\eta$ below are functions of ν and n, as well as of η. But since these arguments are here fixed, we gain notational perspicuity, and lose nothing important, by suppressing reference to them.)

The intuitive idea is that $\mathfrak{A}_\eta$ is the set of all those $\boldsymbol{\Omega}_\xi$ for which ξ is a multiple of ω^η but not of $\omega^{\eta+1}$. Hence $\mathfrak{A}_0$ consists of all $\boldsymbol{\Omega}_\xi$ for which ξ is not a limit ordinal (since there is no $\boldsymbol{\Omega}_0$, this is the same as saying that ξ is a successor). $\mathfrak{S}_1$ therefore contains all $\boldsymbol{\Omega}_\xi$ for which ξ is a positive power of ω; and $\mathfrak{A}_1$ contains all of these that are not powers of ω^2. In general, $\mathfrak{S}_{\eta+1}$ selects every ωth element from the set $\mathfrak{S}_\eta$; in brief, its limit points (see the remark immediately following Theorem 16 above). Theorem 25 does little more than spell this idea out in detail. When $\nu \geq \eta$ we use the expression
s $\nu - \eta$ for the unique ordinal ζ for which $\nu = \eta + \zeta$.

Theorem 25: In a classical calculus whose complete theories are given in (28) above, for every $\eta \leq \nu$,

(30) $$\mathfrak{S}_\eta = \{\boldsymbol{\Omega}_{\omega^\eta\cdot\xi} \mid 0 < \xi \leq \omega^{\nu-\eta}\cdot n\},$$

(31) $$\mathfrak{A}_\eta = \{\boldsymbol{\Omega}_{\omega^\eta\cdot(\xi+1)} \mid 0 \leq \xi < \omega^{\nu-\eta}\cdot n\}.$$

Proof: The proof is really a proof of (30) by induction on η, in the course of which it is shown also that if (30) hold for η then so does (31). The proof is conveniently split into five stages.

- (i) the base case: (30) holds when $\eta = 0$;
- (ii) if (30) holds for $\eta \leq \nu$, and $0 \leq \xi < \omega^{\nu-\eta}\cdot n$, then $\boldsymbol{\Omega}_{\omega^\eta\cdot(\xi+1)}$ finitely extends $\bigvee \mathfrak{S}_\eta$;
- (iii) if (30) holds for $\eta < \nu$, and $0 \leq \xi < \omega^{\nu-\eta-1}\cdot n$, then $\boldsymbol{\Omega}_{\omega^{\eta+1}\cdot\xi}$ does not finitely extend $\bigvee \mathfrak{S}_\eta$;
- (iv) [from (ii) and (iii)] if (30) holds for $\eta \leq \nu$, then (31) holds for η; and if $\eta < \nu$ then (30) holds for $\eta + 1$;
- (v) if λ is a limit ordinal $\leq \nu$, and (30) holds for every $\nu < \lambda$, then it holds also for $\nu = \lambda$.

(i) The case of $\eta = 0$ is entirely trivial, since $\mathfrak{S}_0 = \mathfrak{S} = \{\boldsymbol{\Omega}_\xi \mid 0 < \xi \leq$
| 21 $\omega^\nu \cdot n\}$. |

(ii) Suppose next that (30) holds for some $\eta \leq \nu$. By the distributive
t law cited in section 3 above,

$$\begin{aligned} \mathfrak{p}_{\omega^\eta} \wedge \bigvee \mathfrak{S}_\eta &= \bigvee\{\mathfrak{p}_{\omega^\eta} \wedge \boldsymbol{\Omega}_{\omega^\eta\cdot\xi} \mid 0 < \xi \leq \omega^{\nu-\eta}\cdot n\} \\ &= (\mathfrak{p}_{\omega^\eta} \wedge \boldsymbol{\Omega}_{\omega^\eta}) \vee \bigvee\{\mathfrak{p}_{\omega^\eta} \wedge \boldsymbol{\Omega}_{\omega^\eta\cdot\xi} \mid 1 < \xi \leq \omega^{\nu-\eta}\cdot n\}, \end{aligned}$$

which by (28), (29a), and (29b) is simply $\boldsymbol{\Omega}_{\omega^\eta}$. Hence $\boldsymbol{\Omega}_{\omega^\eta}$ is a finite extension of $\bigvee \mathfrak{S}_\eta$. In other words,

$$\text{(32)} \qquad \text{if} \quad \xi = 0, \quad \text{then} \quad \boldsymbol{\Omega}_{\omega^\eta\cdot(\xi+1)} \in \mathfrak{A}_\eta.$$

If $\eta = \nu$ and $n = 1$ then nothing remains to be proved under (ii). Otherwise it is possible to choose some ξ in the range $0 < \xi < \omega^{\nu-\eta}\cdot n$. Since $\xi + 1 \leq \omega^{\nu-\eta}\cdot n$, $\boldsymbol{\Omega}_{\omega^\eta\cdot(\xi+1)}$ belongs to $\mathfrak{S}_\eta$; and by (28), $\boldsymbol{\Omega}_{\omega^\eta\cdot(\xi+1)} \vdash \neg\mathfrak{p}_{\omega^\eta\cdot\xi} \wedge \mathfrak{p}_{\omega^\eta\cdot(\xi+1)}$. Thus $\boldsymbol{\Omega}_{\omega^\eta\cdot(\xi+1)} \vdash (\neg\mathfrak{p}_{\omega^\eta\cdot\xi} \wedge \mathfrak{p}_{\omega^\eta\cdot(\xi+1)}) \wedge \bigvee \mathfrak{S}_\eta$. The converse implication also holds. For by (29a) and (28), we have both

$$\text{(33)} \qquad \text{if} \quad 0 < \kappa < \xi + 1, \quad \text{then} \quad \boldsymbol{\Omega}_{\omega^\eta\cdot\kappa} \vdash \mathfrak{p}_{\omega^\eta\cdot\xi},$$

$$\text{(34)} \qquad \text{if} \quad \xi + 1 \leq \kappa \leq \omega^{\nu-\eta}\cdot n, \quad \text{then} \quad \boldsymbol{\Omega}_{\omega^\eta\cdot\kappa} \vdash \neg\mathfrak{p}_{\omega^\eta\cdot(\xi+1)}.$$ u

Hence almost vacuously,

$$\text{(35)} \quad \text{if} \quad 0 < \kappa \leq \omega^{\nu-\eta}\cdot n, \ \text{then} \ (\neg\mathfrak{p}_{\omega^\eta\cdot\xi} \wedge \mathfrak{p}_{\omega^\eta\cdot(\xi+1)}) \wedge \boldsymbol{\Omega}_{\omega^\eta\cdot\kappa} \vdash \boldsymbol{\Omega}_{\omega^\eta\cdot(\xi+1)};$$

whence by (30) and distributivity once more, $(\neg\mathfrak{p}_{\omega^\eta\cdot\xi} \wedge \mathfrak{p}_{\omega^\eta\cdot(\xi+1)}) \wedge \bigvee \mathfrak{S}_\eta \vdash \boldsymbol{\Omega}_{\omega^\eta\cdot(\xi+1)}$.

We may conclude that $\boldsymbol{\Omega}_{\omega^\eta\cdot(\xi+1)}$ and $(\neg\mathfrak{p}_{\omega^\eta\cdot\xi} \wedge \mathfrak{p}_{\omega^\eta\cdot(\xi+1)}) \wedge \bigvee \mathfrak{S}_\eta$ are identical; that is, that $\boldsymbol{\Omega}_{\omega^\eta\cdot(\xi+1)}$ is an element of $\mathfrak{A}_\eta$. This result holds for any ξ greater than 0 and less that $\omega^{\nu-\eta}\cdot n$, and may be combined with (32) to yield

$$\text{(36)} \qquad \text{if} \quad 0 \leq \xi < \omega^{\nu-\eta}\cdot n, \quad \text{then} \quad \boldsymbol{\Omega}_{\omega^\eta\cdot(\xi+1)} \in \mathfrak{A}_\eta.$$

This establishes what was required under (ii): the set named on the right of (31) is included in that named on the left.

(iii) We continue to suppose that (30) holds. Suppose $\eta < \nu$. Let ξ satisfy the inequality $0 < \xi < \omega^{\nu-\eta-1}\cdot n$. Then $0 < \omega\cdot\xi \leq \omega^{\nu-\eta}\cdot n$, and so by (30) $\boldsymbol{\Omega}_{\omega^{\eta+1}\cdot\xi} \in \mathfrak{S}_\eta$. Using again the definition (28) we may therefore conclude that $\boldsymbol{\Omega}_{\omega^{\eta+1}\cdot\xi} \vdash \mathfrak{p}_{\omega^{\eta+1}\cdot\xi} \wedge \bigwedge\{\neg\mathfrak{p}_{\omega^\eta\cdot\kappa} \mid 0 < \kappa < \omega\cdot\xi\} \wedge \bigvee \mathfrak{S}_\eta$. The converse implication also holds, since $\bigvee \mathfrak{S}_\eta \vdash \bigwedge\{\neg\mathfrak{p}_\iota \mid \iota < \omega^\eta\}$. This ensures that, in the presence of the logical postulates of the calculus, | | 22
the conjunction $\bigwedge\{\neg\mathfrak{p}_{\omega^\eta\cdot\kappa} \mid 0 < \kappa < \omega\cdot\xi\} \wedge \bigvee \mathfrak{S}_\eta$ negates all letters $\mathfrak{p}_\iota$ for $\iota < \omega^{\eta+1}\cdot\xi$, and $\mathfrak{p}_{\omega^{\eta+1}\cdot\xi}$ affirms all the others. Thus $\boldsymbol{\Omega}_{\omega^{\eta+1}\cdot\xi} = \mathfrak{p}_{\omega^{\eta+1}\cdot\xi} \wedge \bigwedge\{\neg\mathfrak{p}_{\omega^\eta\cdot\kappa} \mid 0 < \kappa < \omega\cdot\xi\} \wedge \bigvee \mathfrak{S}_\eta$.

Since ξ is not 0, it is plain that $\mathbf{\Omega}_{\omega^{\eta+1}\cdot\xi}$ is the conjunction of a sequence $\{\mathbf{X}_\kappa \mid 0 < \kappa < \omega\cdot\xi\}$ of theories each of which properly implies all earlier ones: $\mathbf{X}_\kappa = \mathfrak{p}_{\omega^{\eta+1}\cdot\xi} \wedge \bigwedge\{\neg\mathfrak{p}_{\omega^\eta\cdot\iota} \mid 0 < \iota < \kappa\} \wedge \bigvee \mathfrak{S}_\eta$. The sequence having no last element, a standard compactness argument shows that $\mathbf{\Omega}_{\omega^{\eta+1}\cdot\xi}$ is not a finite extension of $\bigvee \mathfrak{S}_\eta$.

(iv) Assume that $\eta \leq \nu$. Each ordinal ξ in the interval $0 < \xi \leq \omega^{\nu-\eta}\cdot n$ is either a successor ordinal $\iota + 1$ where $0 \leq \iota < \omega^{\nu-\eta}\cdot n$, or a limit ordinal $\omega\cdot\kappa$ where $0 < \kappa \leq \omega^{\nu-\eta-1}\cdot n$. In the first case $\mathbf{\Omega}_{\omega^\eta\cdot\xi}$ belongs to $\mathfrak{A}_\eta$, as shown in (ii). In the second case (which is not possible at all if $\omega^{\nu-\eta}\cdot n$ is finite; that is, if $\eta = \nu$), $\mathbf{\Omega}_{\omega^\eta\cdot\xi}$ does not belong to $\mathfrak{A}_\eta$, as shown in (iii). Given that (30) holds for η, we may conclude via (17) that (31) does too; and that provided $\eta < \nu$, (30) holds also when η is replaced by $\eta + 1$.

(v) Let λ be a limit ordinal not greater than ν, and suppose that (30) holds for all $\eta < \lambda$. Choose $0 < \iota < \omega^\nu\cdot n$. Then $\mathbf{\Omega}_\iota$ belongs to $\mathfrak{S}_\lambda$, which is defined by (14d) to be $\bigcap\{\mathfrak{S}_\eta \mid \eta < \lambda\}$, if and only if ι is of the form $\omega^\eta\cdot\xi$ for every $\eta < \lambda$. It is immediate that $\iota = \omega^\lambda\cdot\xi$ for some ξ; for it is certainly a right multiple of a power of ω, but there is no $\eta < \lambda$ such that $\iota = \omega^\eta\cdot j$ for a finite j. But if $\iota < \omega^\nu\cdot n$, then $\iota = \omega^\lambda\cdot\xi$ if and only if $\xi \leq \omega^{\nu-\lambda}\cdot n$. We conclude that (30) holds when $\eta = \lambda$. ∎

Corollary 26: Every calculus satisfying the conditions of the Theorem has characteristic triple $\langle\nu, n, \aleph_0\rangle$.

Proof: It follows at once from (31) that if $\eta < \nu$ then $\mathfrak{A}_\eta$ is denumerably infinite. It is finite for the first time when $\eta = \nu$, and $\mathfrak{A}_\nu$ contains exactly n complete theories. ∎

We turn now to the characteristic triple $\langle\mu, m, 2^{\aleph_0}\rangle$, where μ is any denumerable ordinal and m any natural number. The intuitive idea here is much the same as before, except that to the end of the sequence $\{\mathfrak{p}_\kappa\}$ of propositional letters, each one of which is implied by all its predeces-
| 23 sors, we now add denumerably many more letters q_k, each implied by |
each $\mathfrak{p}_\kappa$, but themselves completely independent. In short, the letters are the elements of the disjoint sets $\{\mathfrak{p}_\kappa \mid 0 < \kappa < \omega^\mu\cdot(m+1)\}$ and $\{\mathsf{q}_k \mid 0 < k < \omega\}$, and the consequence operation is that of classical propositional logic with $\{\mathfrak{p}_\iota \to \mathfrak{p}_\kappa \mid 0 < \iota < \kappa < \omega^\mu\cdot(m+1)\}$ and $\{\mathfrak{p}_\iota \to \mathsf{q}_k \mid 0 < \iota < \omega^\mu\cdot(m+1); 0 < k < \omega\}$ as additional postulates. Since

the q_k constitute a completely independent set, and for no k is either q_k or $\neg\mathsf{q}_k$ implied by the theory $\bigwedge\{\neg\mathsf{p}_\kappa \mid 0 < \kappa < \omega^\mu \cdot (m+1)\}$, this latter theory, which turns out to be identical with $\bigvee \mathfrak{S}_{\mu+1}$, has $2^{\aleph_0}$ unaxiomatizable complete extensions and no axiomatizable ones.

In order to be able to give a simple name to each complete theory of this calculus, we adopt the convention that $\neg^i$ is to be the negation sign if $i = 0$ and to be the empty string if $i = 1$ (for $i > 1$ it is undefined). Each complete assignment of truth values to the letters q_k may thus be represented by the theory $\mathbf{Q}(\sigma) = \bigwedge\{\neg^{\sigma(k)}\mathsf{q}_k \mid 0 < k < \omega\}$, where σ is a function from the positive natural numbers to $\{0, 1\}$. We shall (rather lazily) write 2^ω for the set of all such functions, and υ for that element of 2^ω that has constant value 1. For each ξ satisfying $0 < \xi < \omega^\mu \cdot (m+1)$, there is, in parallel to (28), a complete theory

$$\text{(37)} \qquad \mathbf{\Omega}_\xi = \bigwedge\{\neg\mathsf{p}_\kappa \mid 0 < \kappa < \xi\} \wedge \wedge \bigwedge\{\mathsf{p}_\kappa \mid \xi \leq \kappa < \omega^\mu \cdot (m+1)\} \wedge \mathbf{Q}(\upsilon);$$

and for each σ in 2^ω there is a complete theory

$$\text{(38)} \qquad \mathbf{\Omega}_\sigma = \bigwedge\{\neg\mathsf{p}_\kappa \mid 0 < \kappa < \omega^\mu \cdot (m+1)\} \wedge \mathbf{Q}(\sigma).$$

It is clear that the $\mathbf{\Omega}_\xi$ and the $\mathbf{\Omega}_\sigma$ are all complete theories of the calculus, since each of them settles the truth value of each p_κ and of each q_k. It is clear also that they are all the complete theories, and that there are $2^{\aleph_0}$ of them. Their disposition among the $\mathfrak{S}_\eta$ and $\mathfrak{A}_\eta$ is stated in Theorem 27. (Note that $\omega^{\mu-\eta} \cdot (m+1) - 1 = \omega^{\mu-\eta} \cdot (m+1)$ when $\eta < \mu$.)

Theorem 27: In a classical calculus whose complete theories are given in (37) and (38) above, for every $\eta \leq \mu$,

$$\text{(39)} \qquad \mathfrak{S}_\eta = \{\mathbf{\Omega}_{\omega^\eta \cdot \xi} \mid 0 < \xi \leq \omega^{\mu-\eta} \cdot (m+1) - 1\} \cup \{\mathbf{\Omega}_\sigma \mid \sigma \in 2^\omega\},$$

$$\text{(40)} \qquad \mathfrak{A}_\eta = \{\mathbf{\Omega}_{\omega^\eta \cdot (\xi+1)} \mid 0 \leq \xi < \omega^{\mu-\eta} \cdot (m+1) - 1\};$$

while for every $\eta > \mu$ | 24

$$\text{(41)} \qquad \mathfrak{S}_\eta = \{\mathbf{\Omega}_\sigma \mid \sigma \in 2^\omega\} \quad \text{and} \quad \mathfrak{A}_\eta = \varnothing.$$

Proof: The proof is very similar to the proof of Theorem 25. The details are omitted. ∎

COROLLARY 28: Every calculus satisfying the conditions of the Theorem has characteristic triple $\langle \mu, m, 2^{\aleph_0} \rangle$.

PROOF: It follows at once from (40) that if $\eta < \mu$ then $\mathfrak{A}_\eta$ is denumerably infinite. It is finite for the first time when $\eta = \mu$, and $\mathfrak{A}_\mu$ contains exactly m complete theories. ∎

Mostowski has shown (*op. cit.*, Corollary 12) that any two calculi with the same characteristic triple $\langle \nu, n, \aleph_0 \rangle$ are isomorphic with respect to the relation $\vdash$; that is, they are of the same structural type (Tarski, p. 370). Matters are less neat when $|\mathfrak{S}| = 2^{\aleph_0}$; here sameness of structural type is assured for the characteristic triples $\langle 0, m, 2^{\aleph_0} \rangle$, but not otherwise (*op. cit.*, Theorem 14). The failure of isomorphism is illustrated by our final example: a calculus with triple $\langle 1, 0, 2^{\aleph_0} \rangle$ that is distinct from the one whose complete theories are obtained by setting $\mu = 1$ and $m = 0$ in (37) and (38) above. The new calculus has the same letters $\{\mathsf{p}_k \mid k < \omega\} \cup \{\mathsf{q}_k \mid k < \omega\}$, but its postulates are only the conditionals $\{\mathsf{p}_i \to \mathsf{p}_k \mid 0 < i < k < \omega\}$ and $\{\mathsf{p}_k \to \mathsf{q}_k \mid 0 < k < \omega\}$. It may be shown that in the old calculus the disjunction $\bigvee \mathfrak{A}_0$ is identical with $\bigvee \{\mathsf{p}_k \mid k < \omega\}$, while in the new one $\bigvee \mathfrak{A}_0 = \mathsf{T}$. Indeed, in the first case $\bigvee \mathfrak{A}_0$ implies each q_k, while in the second (in which, for instance, both $\neg \mathsf{p}_0 \wedge \mathsf{p}_1 \wedge \mathsf{q}_0$ and $\neg \mathsf{p}_0 \wedge \mathsf{p}_1 \wedge \neg \mathsf{q}_0$ are complete), $\bigvee \mathfrak{A}_0$ implies no q_k. Hence the calculi cannot be of the same structural type.

References

MOSTOWSKI, Andrzej [1937]. 'Abzählbare Boolesche Körper und ihre Anwendung auf die allgemeine Metamathematik', *Fundamenta Mathematicae* **29**, pp. 34–53. English translation: *Foundational Studies. Selected Works*, Volume II, North-Holland Publishing Company, Amsterdam, and Polish Scientific Publishers, Warsaw, 1979, pp. 1–17.

TARSKI, Alfred [1956]. *Logic, Semantics, Metamathematics*, Clarendon Press, Oxford.

Notes (2009)

a The results of this paper are in effect translations into logical terminology of abstract theorems about the ultrafilters of a Boolean algebra. I hope that the logical treatment given here will be attractive to philosophers whose acquaintance with abstract algebra and point-set topology is, like mine, somewhat limited.

b I now prefer to refer to a set $\langle S, \mathbf{Cn}\rangle$ as a *logic*, not as a *calculus*, a term that suggests some machinery of calculation. Tarski's original term *deductive theory* is best used, as it is in this paper, for a set of expressions closed under a consequence operation $\mathbf{Cn}$, and his later term *deductive system* for a particular formulation of a logic.

c I now prefer the term *maximal theory* to *complete theory*.

d Or, more briefly, by the three sentences $\mathfrak{p}$, $\neg\mathfrak{p} \wedge \mathfrak{q}$, and $\neg\mathfrak{q}$.

e The condition that if $y \in \mathbf{Cn}(X)$ then $y \in \mathbf{Cn}(Y)$ for some finite subset $Y \subseteq X$, in this paper called *compactness*, is better called *finitariness*. In line with traditional usage, $\mathbf{Cn}$ may be called *compact* if (in the language of the present section) the inconsistent theory $\mathbf{S}$ is finitely axiomatizable. It is not hard to give examples of compact logics that are not finitary, and of finitary logics that are not compact.

f The expression 'any tautological t' is misleading. For although there are infinitely many tautological sentences, there is only one tautological proposition.

g That is to say, $x \wedge \mathbf{X}$ is identified with $\mathbf{Cn}(x) \wedge \mathbf{X}$.

h A proof of $x \wedge \bigvee \mathfrak{K} = \bigvee\{x \wedge \mathbf{X} \mid \mathbf{X} \in \mathfrak{K}\}$ is available when the logic contains a conditional operation $\rightarrow$. Suppose first that $x \wedge \bigvee \mathfrak{K} \vdash z$. Then by the deduction theorem, $\bigvee \mathfrak{K} \vdash x \rightarrow z$, and hence $\mathbf{X} \vdash x \rightarrow z$ for each $\mathbf{X} \in \mathfrak{K}$. This implies (by modus ponens) that $x \wedge \mathbf{X} \vdash z$ for each $\mathbf{X} \in \mathfrak{K}$, and therefore that $\bigvee\{x \wedge \mathbf{X} \mid \mathbf{X} \in \mathfrak{K}\} \vdash z$. The proof of the converse goes in reverse.

i The original text had the word 'language' here instead of 'calculus'.

j (11) can of course be equivalently expressed as the converse of (10).

k The original text used 's', rather than '$\perp$', but this is evidently an irregularity.

l Lemma 6.

m Let $\mathfrak{J}, \mathfrak{K}$ be the classes of axiomatizable and unaxiomatizable complete theories respectively, and let $\alpha = |\mathfrak{J}| = \aleph_0$. If $\bigvee \mathfrak{K} = \mathsf{Cn}(y)$ for some proposition y, then by Theorem 2, $\mathsf{Cn}(y) = \bigwedge\{\neg\omega \mid \omega \in \mathfrak{J}\}$, so that by (3), there is some finite subset $\{\omega_0, \ldots, \omega_{i-1}\} \subseteq \mathfrak{J}$ for which $\neg\omega_0 \wedge \cdots \wedge \neg\omega_{i-1} \vdash y$; that is, for which $\neg\omega_0 \wedge \cdots \wedge \neg\omega_{i-1} \vdash \neg\omega_i$, where ω_i is distinct from each element of $\{\omega_0, \ldots, \omega_{i-1}\}$. But since distinct complete theories are mutually contradictory, $\omega_i \vdash \neg\omega_j$ for each $j < i$, and hence $\omega_i \vdash \neg\omega_i$, which is absurd.

n Wilfrid Hodges has shown me that the principal result of this section, Theorem 14, can be proved much more simply, avoiding all appeal to the theory of ordinals. Definition (14) will, however, be used in the final two sections of the paper.

Hodges's proof proceeds in four short stages as follows.

LEMMA A: If $\mathfrak{J}$ is a finite set of theories, then $\mathcal{R}[\bigvee \mathfrak{J}] = \bigcup\{\mathcal{R}[Y] \mid Y \in \mathfrak{J}\}$.

PROOF: Since $Y \vdash \bigvee \mathfrak{J}$ for each $Y \in \mathfrak{J}$, whether or not $\mathfrak{J}$ is finite, it is trite that $\bigcup\{\mathcal{R}[Y] \mid Y \in \mathfrak{J}\} \subseteq \mathcal{R}[\bigvee \mathfrak{J}]$. Conversely, if $\mathbf{\Omega} \vdash Y$ for no $Y \in \mathfrak{J}$, then in each $Y \in \mathfrak{J}$ there is a proposition y_Y such that $\mathbf{\Omega} \vdash \neg y_Y$. Let $y = \bigvee\{y_Y \mid Y \in \mathfrak{J}\}$. Then $\bigvee \mathfrak{J} \vdash y$ and $\mathbf{\Omega} \vdash \neg y$, and therefore $\mathbf{\Omega} \nvdash \bigvee \mathfrak{J}$. It may be concluded that $\mathcal{R}[\bigvee \mathfrak{J}] \subseteq \bigcup\{\mathcal{R}[Y] \mid Y \in \mathfrak{J}\}$. ∎

LEMMA B: Let $X = \{x \mid |\mathcal{R}[x]| \leq \aleph_0\}$ and $\mathsf{Z} = \bigwedge\{\neg x \mid x \in X\}$. If $\beta > \aleph_0$, then Z is consistent.

PROOF: If $\mathsf{Z} \vdash \bot$ then by (3) there is some finite subset $Y \subseteq X$ such that $\bigwedge\{\neg y \mid y \in Y\} \vdash \bot$; that is, such that $\bigwedge\{\neg y \mid y \in Y\} = \mathsf{S}$. It follows that if $\bigvee\{y \mid y \in Y\} \vdash u$ then $\neg u \vdash \mathsf{S}$, and hence that $\bigvee\{y \mid y \in Y\} = \mathsf{T}$, whence by Lemma A, $\mathcal{R}[\mathsf{T}] = \bigcup\{\mathcal{R}[y] \mid y \in Y\}$, which is denumerable. This contradicts the assumption that $\beta > \aleph_0$, which implies that $|\mathcal{R}[\mathsf{T}]| = \alpha + \beta > \aleph_0$. ∎

THEOREM C: The theory Z defined in Lemma B has no finite completion.

PROOF: Suppose that $\mathsf{Z} \wedge z = \mathbf{\Omega}$. We shall show that $\mathcal{R}[z]$ is denumerable, which implies that $z \in X$, and hence that z contradicts Z. It follows that Z is not finitely completable.

Let $\mathbf{\Psi}$ be a complete theory that implies z. If $\mathbf{\Psi} \neq \mathbf{\Omega}$ then $\mathbf{\Psi} \nvdash \mathsf{Z}$, and hence there is some $x \in X$ for which $\mathbf{\Psi} \vdash x$. Now for each $x \in X$, the range $\mathcal{R}[x]$ is denumerable, and therefore there can be only denumerably many complete theories $\mathbf{\Psi}$ that imply z. That is, $\mathcal{R}[z]$ is denumerable. ∎

Corollary D: For every calculus that is based on classical propositional logic and contains no consistent theory that is not finitely completable

$$\beta \leq \aleph_0.$$

Proof: Immediate. ∎

I should like to thank Hodges warmly for this proof, and for his critical reading seventeen years ago of sections 0–6 of the paper. Responsibility for errors and misjudgements is strictly reserved.

o This Lemma, and Lemma 11, are better placed in the next section, between Lemmas 17 and 18.

p Formula (19) must be proved also for limit ordinals λ. Again it suffices to prove that $\mathcal{R}[\bigvee \mathfrak{S}_\lambda] \subseteq \mathfrak{S}_\lambda$, so suppose that $\mathbf{\Psi} \vdash \bigvee \mathfrak{S}_\lambda$. By (16), $\mathbf{\Psi} \vdash \bigvee \mathfrak{S}_\nu$ for all $\nu < \lambda$, and hence by the induction hypothesis, $\mathbf{\Psi} \in \mathfrak{S}_\nu$ for all $\nu < \lambda$. It follows by (14d) that $\mathbf{\Psi} \in \mathfrak{S}_\lambda$, which is what was to be proved.

q The original proof of Lemma 21 contains some small errors, and is also in some respects incomplete. In the first sentence 'ι', rather than 'i', was used as a subscript, and at the end of the fourth sentence '$\bigvee \mathfrak{S}_\mu$' appeared without the symbol '$\bigvee$'. These are evidently typographical irregularities. The equation in the third sentence, which depends implicitly on the distributive law, originally read erroneously '$y_i \wedge \bigvee \mathfrak{S}_\mu = \bigvee\{y_j \wedge \mathbf{\Omega}_j \mid j < k\} = \mathbf{\Omega}_i$'.

To prove the existence of a set Y of pairwise incompatible propositions, exactly one of which follows from each of the complete theories $\{\mathbf{\Omega}_i \mid i < k\}$, we note first that if $\mathbf{\Omega} \neq \mathbf{\Psi}$ then there is some proposition y such that $\mathbf{\Omega} \vdash y$ and $\mathbf{\Psi} \nvdash y$; that is (by negation completeness) some proposition y such that $\mathbf{\Omega} \vdash y$ and $\mathbf{\Psi} \vdash \neg y$. For each distinct $i, j < k$, let w_{ij} be a proposition such that $\mathbf{\Omega}_i \vdash w_{ij}$ and $\mathbf{\Omega}_j \vdash \neg w_{ij}$. It is evident that $\mathbf{\Omega}_i$ implies $y_i = \bigwedge\{w_{ij} \mid j \neq i; j < k\} \wedge \bigwedge\{\neg w_{ji} \mid j \neq i; j < k\}$ for each $i < k$, and that the propositions $\{y_i \mid i < k\}$ are pairwise incompatible.

It may be noted that no such set Y of pairwise incompatible consequences need exist for an infinite set $\mathfrak{K}$ of complete theories. This is immediate if $\mathfrak{K}$ is not denumerable. It is also immediate that if $\mathfrak{K}$ is a denumerable set of axiomatizable complete theories then Y may be identified with $\mathfrak{K}$. But the picture is different when unaxiomatizable complete theories are considered, even in the most primitive case of a calculus with characteristic pair $\langle \aleph_0, 1 \rangle$, such as elementary logic with identity and no other predicates (see the first paragraph of section 1 above). The axiomatizable complete theories ω_j of this calculus state, for each positive j, that exactly j distinct objects exist, while the one unaxiomatizable complete theory $\mathbf{\Omega} = \bigwedge\{\neg\omega_j \mid j \in \mathcal{N}\}$ states that there is no finite bound on their number. It is plain that if y is a proposition that is incompatible with each ω_j, then $y \vdash \mathbf{\Omega}$, and hence $y = \bot$. In other words, $\mathbf{\Omega}$ implies no proposition that is incompatible with each ω_j; and a fortiori, $\mathbf{\Omega}$ implies no proposition that is incompatible with at least one consequence of each ω_j. For the set $\mathfrak{S}$ of complete theories of this calculus, no set Y of pairwise incompatible consequences exists.

r Here there is an implicit use of Lemma A in note **n** above.

s That there is a unique such ordinal may be proved by a straightforward induction on η. Note that $\nu - \eta$ may equal ν even if $\eta \neq 0$; a simple example is provided by $\omega - 1$, which equals ω. (See also the parenthetical sentence just before Theorem 27 on p. 23.) The term $\nu - \eta - 1$, used in parts (iii) and (iv) of the proof of Theorem 25, may be identified with either $\nu - (\eta + 1)$ or $(\nu - \eta) - 1$, since these terms are easily proved to be equal. For on the one hand, $\zeta = \nu - (\eta + 1)$ if and only if $\nu = (\eta + 1) + \zeta$. On the other hand, $\zeta = (\nu - \eta) - 1$ if and only if $\nu - \eta = 1 + \zeta$; that is, if and only if $\nu = \eta + (1 + \zeta)$. The result follows from the associativity of ordinal addition.

t See note **f** and text.

u In the original text the antecedent of formula (34) was incorrectly given as $\xi + 1 < \kappa \leq \omega^{\nu-\eta} \cdot n$.

v Y is a completely independent set of propositions if for every $X \subseteq Y$ the set $X \cup \{\neg y \mid y \in Y \setminus X\}$ is consistent. The relation of complete independence, which is central to the theory of elementary propositions of Wittgenstein's *Tractatus Logico-Philosophicus* (1921), originates with E. H. Moore (1910), p. 82. 'Introduction to a Form of General Analysis'. In E. H. Moore, J. Wilczynski, & M. Mason (1910). *The New Haven Mathematical Colloquium*, pp. 1–150. AMS Colloquium Publications, volume 2. It is mentioned on p. 36 of *Logic, Semantics, Metamathematics*, but not further discussed.

w When $\mu = 1$ and $m = 0$ the denumerable sequence $\{\mathfrak{p}_\kappa \mid 0 < \kappa < \omega^\mu \cdot (m + 1)\}$ reduces to a simple progression.

x The translations in *Logic, Semantics, Metamathematics* were prepared by J. H. Woodger. The 2nd edition, edited by J. Corcoran, was published in 1983. Indianapolis: Hackett Publishing Company.