Generalised extreme value statistics and sum of correlated variables.

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Outline

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global quantities

Definition: global quantity

Measurable quantity,
sum of microscopic or local variables.

Example

- Magnetization of a system of N spins : $M = \frac{1}{N} \sum_{k=1}^{N} \sigma_k$.
- Total dissipated power in a fluid : $P = \sum_k p(r_k)$

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Probleme

Quantity of interest

Probability density function (PDF) of global quantities

Example

Gaussian distribution $\Leftarrow$ central limit theorem (CLT)
Poisson distribution
and a full zoo of other possibilities...

Question :

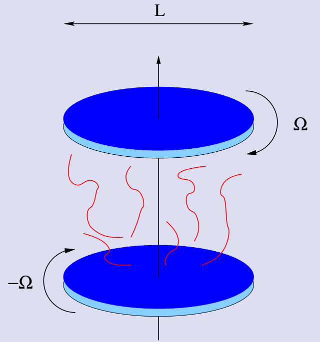
What could we say in general of PDF of global quantities ?

Outline

1. Introduction
   - Global quantities
   - Surprising experimental fact
2. Generalised extreme value statistics
   - Extremes and sums
   - Generalization
   - Limit distribution for correlated sums
3. 1d confined granular gas
   - Model
   - Volume fluctuations
   - Physical interpretation

Injected power fluctuations in a turbulent fluid

[Labbe, Pinton, Fauve, J. Phys II (Paris), **6**, 10099 (1996)]



Experience

- Contra-rotating disks
- Angular speed Ω constant
- Reynolds number
  $\mathrm{Re} = L^2\Omega/\nu \simeq 5 \times 10^5$:
  Turbulent regime.

$\Rightarrow$ Injected power fluctuations

Injected power fluctuations in a turbulent fluid

[Pinton, Holdsworth, Labbé, Phys. Rev. E, **60** R2452 (1999)]

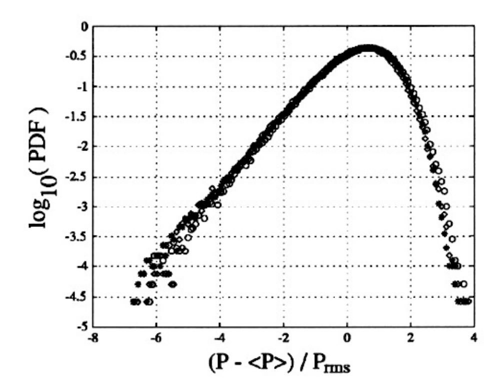


Figure: Measured injected power PDF for various Reynolds numbers.

Magnetisation fluctuations in the 2d XY model

[P. Archambault *et al.*, J. Appl. Phys. 83, 7234 (1998)]

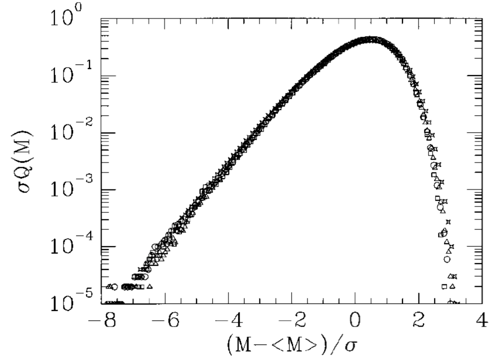


Figure: PDF obtained by Monte Carlo simulations.

A surprising similarity

[Bramwell, Holdsworth, Pinton, Nature (1998)]

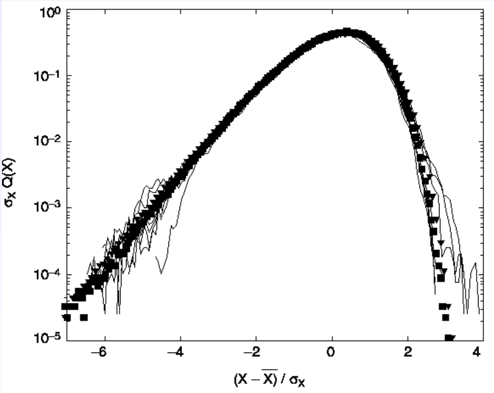


Figure: The two previous results.

Density fluctuation in a tokamak

[B. Ph. van Milligen et al., Phys. Plasmas **12**, 052507 (2005)]

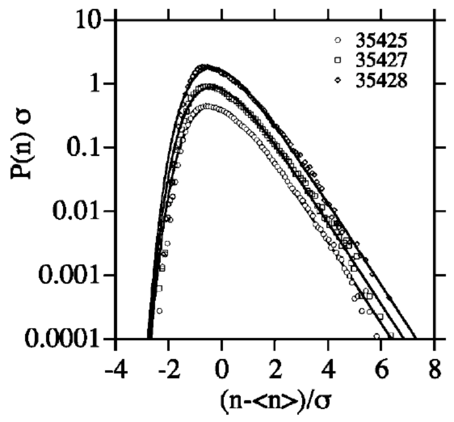
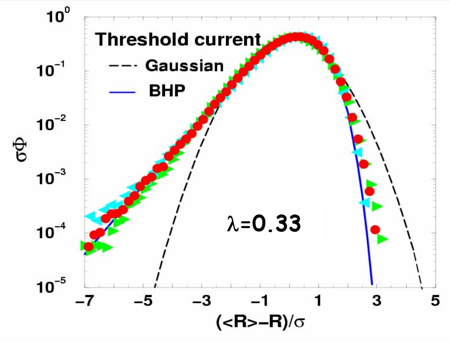


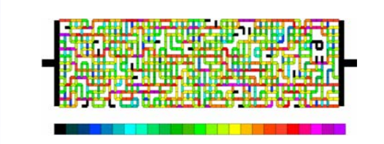
Figure: Plasma density fluctuations in a tokamak.

Resistance of a disordered conductor

[C. Pennetta et al., SPIE Proc. 5471 (2004)]



- 2d resistor network
- In a steady state
- Monte Carlo simulations
- Fluctuations closed to the threshold.



Probleme

Experimental fact

Similar distributions
in various complex systems
Turbulent flows, magnetic systems, SOC models, electroconvections, Freederick transitions, granular materials, spin glass, universe sheets ...

Question :

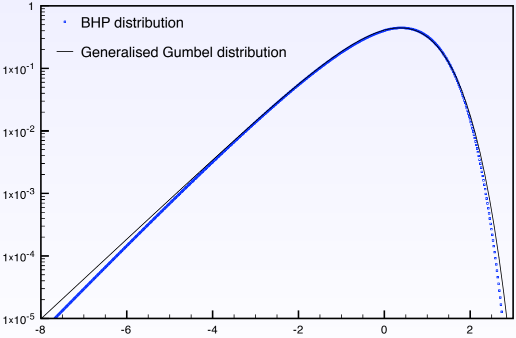
Origine for this similarity ?

- ”Limit theorem” for complex systems ?
- Similar physics behind these distributions ?

BHP distribution and extreme value statistics

Interesting observation :

$$G_a(\mu) \propto \exp\left[ab_a(\mu - s_a) - ae^{b_a(\mu - s_a)}\right], \boldsymbol{a} = \pi/2$$



Extreme value statistics

Extreme value statistics

- For a **integer**, G_a is **Gumbel distribution**,
  an aymptotic distribution for **extremes**

Example

Let $\mathfrak{X}_n = \{x_1, ..., x_n\}$ be a set of n random variables independent and identically distributed.

Let $z_n = \mathrm{Max}^{(k)}(\mathfrak{X}_n)$ be the k^{th} largest element of $\mathfrak{X}_n$.

$$P_n(z_n) \underset{n\to\infty}{\longrightarrow} G_k(\tilde{z})$$

Note that here k is an integer **by construction**

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A phenomenology based on extremes ?

Question :

What is the **meaning**
of a **generalized** Gumbel distribution ($a \in \mathbb{R}$) ?

- Physics controled by a hidden complex process ?

$$X = \sum_{k=1}^{n} x_k \simeq \mathrm{Max}[f(x_1 ... x_n)] \text{ ?}$$

- **Interpretation of generalised extreme value statistics ?**

A phenomenology based on extremes ?

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Generalised extreme value statistics

[É. Bertin and M. Clusel, J.Phys.A **39** 7607 (2006)]

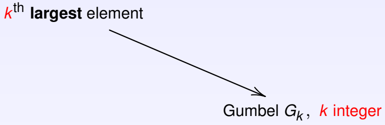
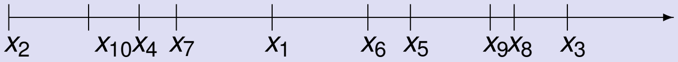


Figure: Principle of the proof

Equivalence: ordering

Ordering

- $\mathfrak{X}_N$ a set of N IID random variables x_k.



Increments

$$u_k \equiv z_k - z_{k+1}, \ \forall\ 1 \leq k \leq N-1, \qquad u_N \equiv z_N.$$

$$z_n \equiv \mathrm{Max}^{(n)}\left[\mathfrak{X}_N\right] = \sum_{k=n}^{N} u_k,$$

Interpretation

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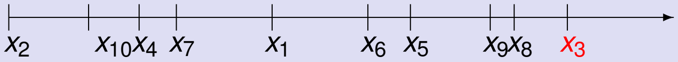

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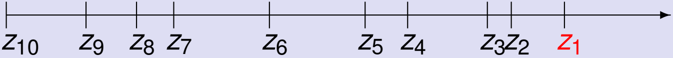

- $z_k = x_{\sigma(k)}$, with σ ordering permutation $z_1 \geq z_2 ... \geq z_N$.

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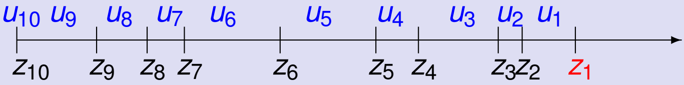

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Interpretation

Equivalence between **sums** and **extremes.**

Equivalence: correlation

Induced correlations

Ordering process $\Rightarrow$ *a priori* **non-IID**.

Joint probability

$$\tilde{J}_{k,N}(u_k, ..., u_N) = N! \int_0^{\infty} dz_N P(z_N) ... \int_{z_2}^{\infty} dz_1 P(z_1) \times \delta(u_N - z_N) \prod_{n=k}^{N-1} \delta(u_n - z_n + z_{n+1})$$

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Equivalence: correlation

Integration

- Successive integration

$$\int_u^\infty \mathrm{d}z_{k-1} P(z_{k-1}) \ldots \int_{z_2}^\infty \mathrm{d}z_1 P(z_1) = \frac{1}{(k-1)!} F(u)^{k-1},$$

- $F(u) \equiv \int_u^\infty \mathrm{d}z \; P(z)$
- Shift of indices.

Final result

$$\tilde{J}_{k,N}(u_k, \ldots, u_N) = \frac{N!}{(k-1)!} F\left(\sum_{i=k}^{N} u_i\right)^{k-1} \prod_{n=k}^{N} P\left(\sum_{i=n}^{N} u_i\right).$$

Equivalence: correlation

Final result

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General case:

- Not factorised : impossible to write

$$J_{N'}(u_1, \dots, u_{N'}) \neq \prod_{n=1}^{N'} \pi_n(u_n),$$

- Random variables U_n are **not IID.**

Generalised extreme value statistics

[É. Bertin and M. Clusel, J.Phys.A **39** 7607 (2006)]

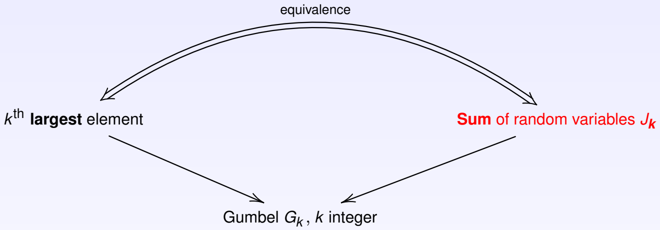

Figure: Principle of the proof

Outline

Generalisation: extended joint probability

Previous results

$$\tilde{J}_{k,N}(u_k, \ldots, u_N) = \frac{N!}{(k-1)!} F\left(\sum_{i=k}^{N} u_i\right)^{k-1} \prod_{n=k}^{N} P\left(\sum_{i=n}^{N} u_i\right).$$

Generalisation

$$J_N(u_1, \ldots, u_N) = \frac{\Gamma(N)}{Z_N} \Omega\left[F\left(\sum_{n=1}^{N} u_n\right)\right] \prod_{n=1}^{N} P\left(\sum_{i=n}^{N} u_i\right)$$

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Functions

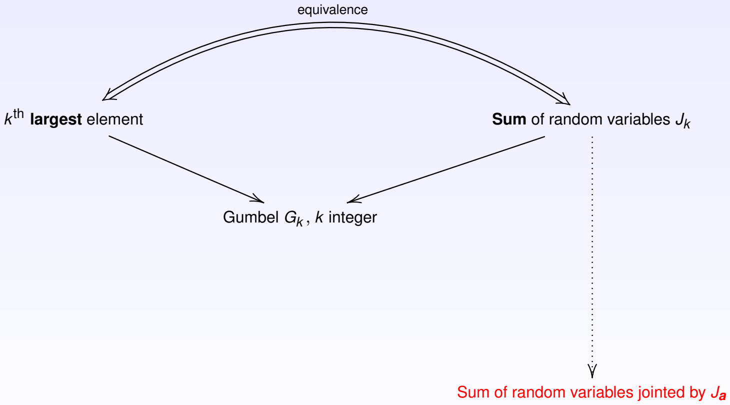
- $\Omega(F)$, (arbitrary) positive function of F

$$\Omega(F) \underset{F \to 0}{\sim} F^{a-1}, \mathbf{a} \in \mathbb{R}$$

- $Z_N = \int_0^1 \mathrm{d}v \, \Omega(v) \, (1-v)^{N-1}$.

Generalised extreme value statistics

[É. Bertin and M. Clusel, J.Phys.A **39** 7607 (2006)]



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Outline

The simple exponential case

Factorisation conditions

- $\forall(x,y),\; P(x+y) = P(x)P(y) \Rightarrow P(x) = \kappa e^{-\kappa x}$
- $\forall(F_1,F_2),\; \Omega(F_1 F_2) = \Omega(F_1)\Omega(F_2) \Rightarrow \Omega(F) = F^{a-1}$

Factorised joint probability

$$J_{N'}(u_1,\ldots,u_{N'}) = \prod_{n=1}^{N'} \pi_n(u_n),$$

$$\pi_n(u_n) = (n+a-1)\kappa\, e^{-(n+a-1)\kappa u_n}.$$

Variables u_n

Independent, but non-identically distributed.

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The simple exponential case

Definitions

- u_n is distributed according to $\pi_n(u_n)$, $a \in \mathbb{R}$.
- $s_N = \sum_{n=1}^{N} u_n$.
- Let Υ_N be the distribution of s_n.

Fourier transform of Υ_N

$$\mathcal{F}\left[\Upsilon_N\right](\omega) = \prod_{n=1}^{N} \mathcal{F}\left[\pi_n\right](\omega) = \prod_{n=1}^{N} \left(1 + \frac{i\omega}{\kappa(n+a-1)}\right)^{-1}.$$

Question :

What is the asymptotic distribution $\lim_{N\to\infty} \Upsilon_N$?

The simple exponential case

First two moments of Υ_N

$$\begin{aligned}\langle S_N \rangle &= \sum_{n=1}^{N} \langle U_n \rangle = \frac{1}{\kappa} \sum_{n=1}^{N} \frac{1}{n+a-1}, \\ \sigma_N^2 &= \sum_{n=1}^{N} \mathrm{Var}(U_n) = \frac{1}{\kappa^2} \sum_{n=1}^{N} \frac{1}{(n+a-1)^2}.\end{aligned}$$

Breakdown of central limit theorem

$$\lim_{N\to\infty} \sigma_N < \infty$$

Lindeberg condition not satisfied.

The simple exponential case

First two moments of Υ_N

$$\langle S_N \rangle = \sum_{n=1}^{N} \langle U_n \rangle = \frac{1}{\kappa} \sum_{n=1}^{N} \frac{1}{n+a-1},$$

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The simple exponential case: asymptotic distribution

Rescaled variable

$$\mu = \frac{s - \langle S_N \rangle}{\sigma}, \text{ with } \sigma = \lim_{N \to \infty} \sigma_N.$$

$$\Phi_N(\mu) = \sigma \Upsilon_N(\sigma\mu + \langle S_N \rangle).$$

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Large N limit: $\Phi = \lim_{N\to\infty} \Phi_N$

$$\mathcal{F}\left[\Phi_\infty\right](\omega) = \prod_{n=1}^{\infty} \left(1 + \frac{i\omega}{\sigma\kappa(n+a-1)}\right)^{-1} \exp\left(\frac{i\omega}{\sigma\kappa(n+a-1)}\right).$$

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Large N limit: $\Phi = \lim_{N\to\infty} \Phi_N$

$$\Phi_\infty(\mu) = G_a(\mu),$$

G_a, Gumbel distribution of parameter $a \in \mathbb{R}$.

Generalised extreme value statistics

[É. Bertin and M. Clusel, J.Phys.A **39** 7607 (2006)]

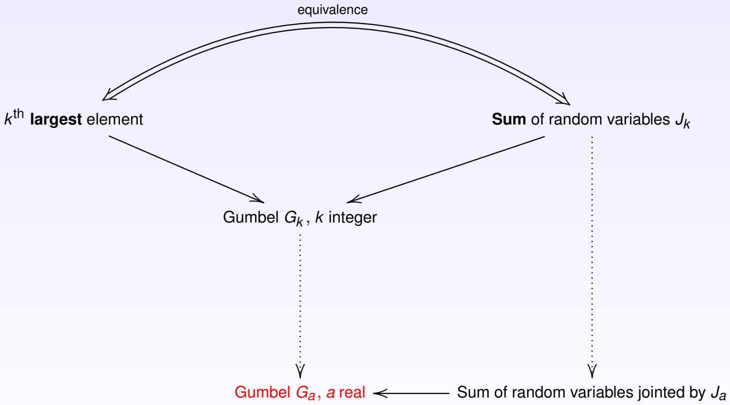

Figure: Principle of the proof

Physical interpretation

$1/f$ noise with cut-off

- 1D lattice model, continuous variable ϕ_x on each site.
- $\hat{\phi}_q = \frac{1}{\sqrt{L}} \sum_{x=1}^{L} \phi_x e^{iqx}$.
- Total energy : $E = \sum_q |\hat{\phi}_q|^2 = \sum_q u_q$,
- $P(u_n) = \kappa e^{-\kappa u_n}$ and $\Omega(F) = F^{a-1}$.

$$\Rightarrow P(E) = G_a(E).$$

Correlation

- $\langle |\hat{\phi}_q|^2 \rangle \propto \frac{1}{|q|+m}, \quad m = \frac{2\pi(a-1)}{L}$
- Correlation length $\xi = \frac{L}{2\pi(a-1)}$

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Limit cases

- $\xi \to \infty$: pure $1/f$ noise $\Rightarrow$ **Gumbel** $a = 1$
- $\xi \to 0$: uncorrelated white noise $\Rightarrow$ **Gaussian** $a \to \infty$

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Interpretation of generalised extreme value statistics

Summary

- Extreme value statistics $\Longleftrightarrow$ sums of correlated random variables
- Generalization of the correlated sums,
  and not of the extreme problem.

Conclusion

Observation of generalised Gumbel distribution
↪Correlation and not necessarly extremes

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➥**Correlation** and not necessarly extremes

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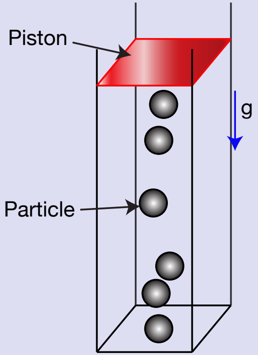
A simple model for 1d confined granular gas

[É. Bertin et al., J.Stat.Mech P07019 (2008)]



Model

- N point particles
- Quasi 1d cylinder
- Equilibrium at T
- Reflective wall at $z = 0$
- Piston at z_p

Potentials

- Piston: $U_\mathrm{p}(z_\mathrm{p})$
- Particle: $U(z)$

A simple model for 1d confined granular gas

Joint probability

$$P_N(z_1, \ldots, z_N, z_\mathrm{p}) = \frac{1}{Z}\, e^{-\beta U_\mathrm{p}(z_\mathrm{p})} \prod_{i=1}^{N} e^{-\beta U(z_i)}\, \Theta(z_\mathrm{p} - z_i),$$

Volume fluctuations

Integration on particles positions:

$$P(z_\mathrm{p}) = \frac{1}{Z}\, e^{-\beta U_\mathrm{p}(z_\mathrm{p})} \left(\int_0^{z_\mathrm{p}} dz\, e^{-\beta U(z)} \right)^N,$$

Outline

Two simple cases

Free particles

$$U(z) = 0$$

$$P(z_{\rm p}) = \frac{1}{Z} z_{\rm p}^N \, e^{-\beta U_{\rm p}(z_{\rm p})}$$

Gaussian fluctuations in the large N limit.

A particle as piston

$$U_{\rm p}(z_{\rm p}) = U(z_{\rm p})$$

$$P(z_{\rm p}) = \frac{d}{dz_{\rm p}} G(z_{\rm p})^{N+1}$$

Standard extreme value statistics.

Two simple cases

Free particles

$$U(z) = 0$$

$$P(z_{\rm p}) = \frac{1}{Z} z_{\rm p}^N \, e^{-\beta U_{\rm p}(z_{\rm p})}$$

Gaussian fluctuations in the large N limit.

A particle as piston

$$U_{\rm p}(z_{\rm p}) = U(z_{\rm p})$$

$$P(z_{\rm p}) = \frac{d}{dz_{\rm p}} G(z_{\rm p})^{N+1}$$

Standard extreme value statistics.

Case $U(z) = U_0 \, z^\alpha, \qquad U_\mathrm{p}(z_\mathrm{p}) = U'_0 \, z_\mathrm{p}^\gamma$

Ordering

- $z_1, \ldots, z_N$ satisfying $0 < z_i < z_\mathrm{p}$
- permutation: $\sigma : z_{\sigma(1)} \leq z_{\sigma(2)} \leq \ldots \leq z_{\sigma(N)}$
- space interval: h_i $h_i = z_{\sigma(i)} - z_{\sigma(i-1)}, \qquad i = 2, \ldots, N.$
  $h_i = u_{N+2-i}$

Equilibrium probability distribution

$$\tilde{J}_N(h_1, \ldots, h_{N+1}) = K \, \Omega \left[F \left(\sum_{i=1}^{N+1} h_i \right) \right] \prod_{i=1}^{N+1} P \left(\sum_{j=1}^{i} h_i \right),$$

Case $U(z) = U_0\, z^{\alpha}, \qquad U_{\rm p}(z_{\rm p}) = U_0'\, z_{\rm p}^{\gamma}$

Equilibrium probability distribution

$$\tilde{J}_N(h_1, ..., h_{N+1}) = K\, \Omega \left[F\left(\sum_{i=1}^{N+1} h_i \right) \right] \prod_{i=1}^{N+1} P\left(\sum_{j=1}^{i} h_i \right),$$

Functions

$$P(z) = \lambda\, e^{-\beta U(z)},\; F(z) = \int_z^{\infty} P(z')dz'.$$

$$\Omega(y) = \frac{1}{\lambda} \exp\left[\beta U(F^{-1}(y)) - \beta U_{\rm p}(F^{-1}(y)) \right], \forall y \in]0,1[.$$

Case $U(z) = U_0\, z^{\alpha}$, $\quad U_\mathrm{p}(z_\mathrm{p}) = U_0'\, z_\mathrm{p}^{\gamma}$

Equilibrium probability distribution

$$\tilde{J}_N(h_1, \ldots, h_{N+1}) = K\,\Omega\left[F\left(\sum_{i=1}^{N+1} h_i\right)\right]\prod_{i=1}^{N+1} P\left(\sum_{j=1}^{i} h_i\right),$$

Limit $N \to \infty$: case $\alpha = \gamma$

$$\Omega(y) \approx \frac{\Gamma\left(\frac{1}{\alpha}\right)^a}{\alpha(\beta U_0)^{1/\alpha}}\, y^{a-1}\left(\ln\frac{1}{y}\right)^{(a-1)(1-\frac{1}{\alpha})} \qquad (y \to 0)$$

with $a = U_0'/U_0$

➥generalised Gumbel distribution of parameter a

Case $U(z) = U_0 \, z^\alpha, \qquad U_p(z_p) = U_0' \, z_p^\gamma$

Equilibrium probability distribution

$$\tilde{J}_N(h_1, \dots, h_{N+1}) = K \, \Omega \left[F\left(\sum_{i=1}^{N+1} h_i \right) \right] \prod_{i=1}^{N+1} P\left(\sum_{j=1}^{i} h_i \right),$$

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➥generalised Gumbel distribution of parameter a

Case $U(z) = U_0\, z^\alpha$, $U_p(z_p) = U'_0\, z_p^\gamma$

Equilibrium probability distribution

$$\tilde{J}_N(h_1, \ldots, h_{N+1}) = K\, \Omega \left[F\left(\sum_{i=1}^{N+1} h_i \right) \right] \prod_{i=1}^{N+1} P\left(\sum_{j=1}^{i} h_i \right) ,$$

Limit $N \to \infty$

Depending of the behaviour of Ω

- $\gamma > \alpha$: Gaussian distribution
- $\gamma < \alpha$: Exponential distribution
- $\gamma = \alpha$: Generalised extreme value distribution

Outline

Physical interpretation: gravity case $\alpha = \gamma = 1$

Equation of states

- $\langle z_{\rm p}\rangle = \sum_{j=1}^{N+1} \langle h_j\rangle \simeq \frac{k_{\rm b} T}{mg} \ln\left(1 + \frac{N}{a}\right)$.
- $P_a \langle V\rangle = a k_{\rm b} T \ln\left(1 + \frac{N}{a}\right)$, $P_a = Mg/S = amg/S$.
- $M \gg Nm$: $P\langle V\rangle = N k_{\rm b} T$
- $M \ll Nm$: $P_a\langle V\rangle = a k_{\rm b} T \ln(N/a)$

Compressibility

- $-\frac{\partial\langle V\rangle}{\partial P} = \frac{1}{k_{\rm b} T}\left(\langle V^2\rangle - \langle V\rangle^2\right)$
- $\kappa = -\frac{1}{\langle V\rangle}\frac{\partial\langle V\rangle}{\partial P} \propto \frac{1}{\ln N}$.
- Abnormally small fluctuations, logarithmic decay

Conclusion

Generalised extreme value statistics

- So called “generalised EVS”:
  $\Rightarrow$Associated with sum of correlated random variables
- *A priori* no simple extreme processes at play
- Misleading name !
- Any suggestion ?

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Consider a set of realisations $\{u_n\}$ of N (correlated) random variables U_n, with the joint probability (**??**). We then define as above the random variable $S_N = \sum_{n=1}^{N} U_n$, and let Υ_N be the probability density of S_N. Then Υ_N is given by

$$\begin{aligned}\Upsilon_N(s) &= \int_0^\infty \mathrm{d}u_N ... \mathrm{d}u_1 J_N(u_1, ..., u_N)\, \delta\left(s - \sum_{n=1}^{N} u_n\right). \\ &= \frac{\Gamma(N)}{Z_N} P(s)\, \Omega(F(s))\, I_N(s),\end{aligned}$$

with

$$I_N(s) = \int_0^\infty \mathrm{d}u_N P(u_N) ... \int_0^\infty \mathrm{d}u_1 \delta\left(s - \sum_{n=1}^{N} u_n\right).$$

To evaluate I_N, let us start by integrating over u_1, using

$$\int_0^\infty \mathrm{d}u_1 \delta\left(s - \sum_{n=1}^{N} u_n\right) = \Theta\left(s - \sum_{n=2}^{N} u_n\right),$$

where Θ is the Heaviside distribution. This changes the upper bound of the integral over u_2 by $u_2^{\max} = \max\left(0, s - \sum_{n=2}^{N} u_n\right)$. Then the integration over u_2 leads to

$$\int_0^{u_2^{\max}} \mathrm{d}u_2 P(u_2) = \left[F\left(\sum_{n=3}^{N} u_n\right) - F(s)\right] \Theta\left(s - \sum_{n=3}^{N} u_n\right),$$

By recurrence it is then possible to show that

$$I_N(s) = \frac{1}{\Gamma(N)} \Big(1 - F(s)\Big)^{N-1},$$

finally yielding the following expression for Υ_N:

$$\Upsilon_N(s) = \frac{1}{Z_N} P(s)\, \Omega\big(F(s)\big) \Big(1 - F(s)\Big)^{N-1}. \qquad (1)$$

In the following sections, we assume that $\Omega(F)$ behaves asymptotically as a power law $\Omega(F) \sim \Omega_0 \, F^{a-1}$ when $F \to 0$ ($a > 0$). Under this assumption, we deduce from Eq. (1) the different limit distributions associated with the different classes of asymptotic behaviours of P at large x.

$P(x)$ decays faster than any power law at large x.
To that purpose, we define s_N^* by $F(s_N^*) = a/N$. If a is an integer, this is nothing but the typical value of the a^{th} largest value of s in a sample of size N. As P is unbounded we have

$$\lim_{N\to\infty} s_N^* = +\infty.$$

Let us introduce $g(s) = -\ln F(s)$ and, assuming $g'(s_N) \neq 0$, define the rescaled variable v by

$$s = s_N^* + \frac{v}{g'(s_N^*)}. \tag{2}$$

For large N, series expansion of g around s_N^*:

$$g(s) = g(s_N^*) + v + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{g^{(n)}(s_N^*)}{g'(s_N^*)^n} v^n.$$

For P in the Gumbel class, $g^{(n)}(s_N^*)/g'(s_N^*)^n$ is bounded as a function of n so that the series converges. In addition, one has

$$\lim_{N\to\infty} \frac{g^{(n)}(s_N^*)}{g'(s_N^*)^n} = 0, \ \forall n \geq 2,$$

so that $g(s)$ may be written as

$$g(s) = g(s_N^*) + v + \varepsilon_N(v), \quad \text{with} \lim_{N\to\infty} \varepsilon_N(v) = 0. \tag{3}$$

Given that $P(s) = g'(s)\,F(s)$, one gets using Eqs. (1) and (3)

$$\begin{aligned}\Phi_N(v) &= \frac{1}{g'(s_N^*)}\Upsilon_N(s)\\ &= \frac{1}{Z_N}\frac{g'(s)}{g'(s_N^*)}\,F(s)\,\Omega\big(F(s)\big)\Big(1-F(s)\Big)^{N-1},\end{aligned}$$

where s is given by Eq. (2). For P in the Gumbel class, it can be checked that, for fixed v

$$\lim_{N\to\infty}\frac{g'\left(s_N^* + v/g'(s_N^*)\right)}{g'(s_N^*)} = 1.$$

Besides, $F\left(s_N^* + v/g'(s_N^*)\right) \to 0$ when $N \to \infty$, so that one can use the small F expansion of $\Omega(F)$.

Altogether, one finds

$$\Phi_N(v) \sim \frac{\Omega_0}{Z_N}\left(\frac{a}{N}\right)^a e^{-av-a\varepsilon_N(v)}\left[1-\frac{a}{N}e^{-v-\varepsilon_N(v)}\right]^{N-1}$$

Using a simple change of variable in Eq. (**??**), one can show that

$$\lim_{N\to\infty}\frac{N^a Z_N}{\Omega_0} = \Gamma(a)$$

It is then straightforward to take the asymptotic limit $N \to \infty$, leading to

$$\Phi_\infty(v) = \frac{a^a}{\Gamma(a)}\exp\left[-av - ae^{-v}\right].$$

In order to recover the usual expression for the generalised Gumbel distribution, one simply needs to introduce the reduced variable

$$\mu = \frac{v - \langle v \rangle}{\sigma_v},$$

with, Ψ being the digamma function,

$$\langle v \rangle = \ln a - \Psi(a), \quad \sigma_v^2 = \Psi'(a).$$

The variable μ is then distributed according to a generalised Gumbel distribution.

To sum up, if one considers the sum S_N of $N \gg 1$ random variables linked by the joint probability (**??**), then the asymptotic distribution of the reduced variable μ defined by

$$\mu = \frac{s_N - \langle S_N \rangle}{\sigma_N},$$

with

$$\langle S_N \rangle = s_N^* + \frac{\ln a - \Psi(a)}{g'(s_N^*)}, \quad \sigma_N = \frac{\sqrt{\Psi'(a)}}{g'(s_N^*)}.$$

is the generalised Gumbel distribution. back