Complexity Summer School

Warwick 2013

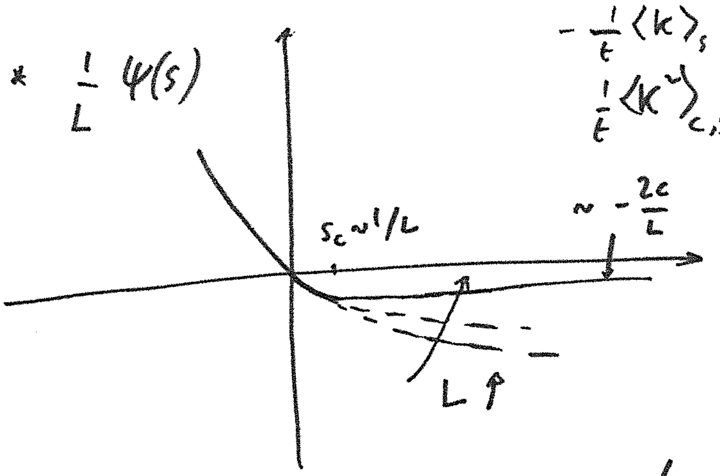
Part III. Applications – Dynamical Heterogeneities in KCMs Transport Models

Outline:

1. Phase Coexistence in 1D & ∞D KCMs
2. 2nd order phase transitions in transport models

1. Phase Coexistence in 1D & ∞D KCMs

1-a. Results for the dynamical free-energy of the 1DFA:

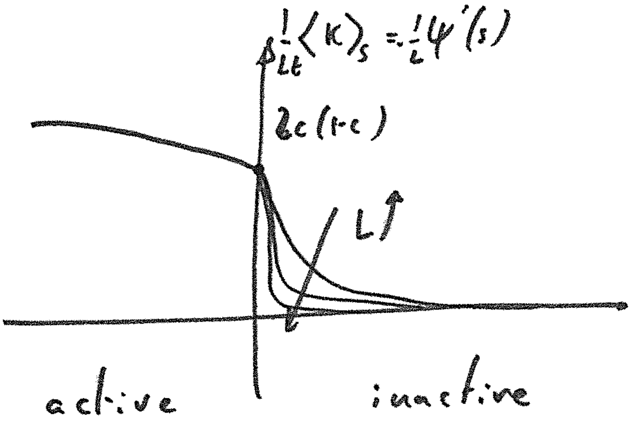


$$\left.\begin{aligned} -\frac{1}{t}\langle K\rangle_s &= \psi'(s) < 0 \\ \frac{1}{t}\langle K^2\rangle_{c,s} &= \psi''(s) > 0 \end{aligned}\right\}: \quad \psi(s) \text{ is decreasing and convex}$$

$$\big(\mathbb{W}(s)\big)_{\mathcal{C}'\mathcal{C}} = e^{-s}\,W(\mathcal{C}\to\mathcal{C}') - \delta_{\mathcal{C}\mathcal{C}'}\,r(\mathcal{C})$$

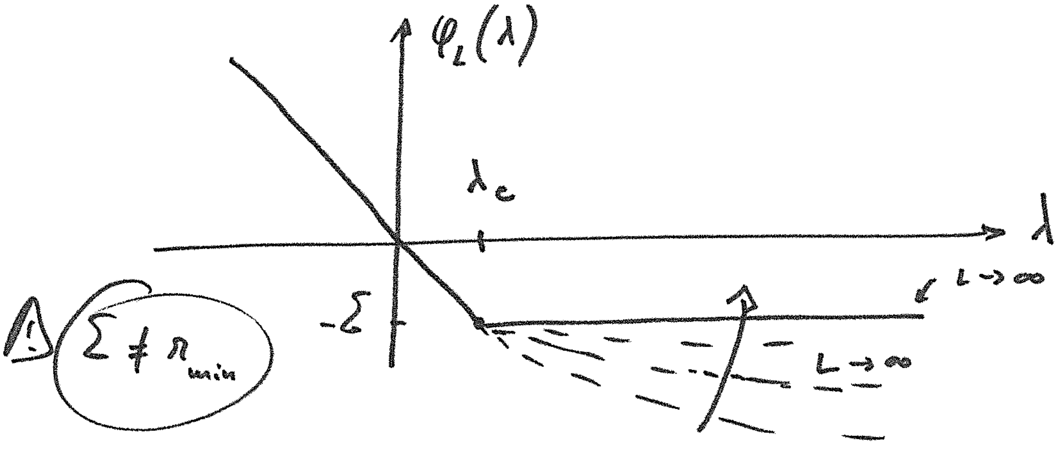
$$\lim_{s\to\infty}\big(\text{—}\big) = -\delta_{\mathcal{C}\mathcal{C}'}\,r(\mathcal{C}) \quad \text{[diagonal operator]}$$

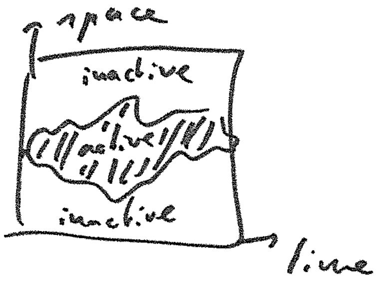
$$\hookrightarrow \psi(s\to\infty) = -\min_{\mathcal{C}} r(\mathcal{C}) = -2c$$



} 1st order jump in the (dynamical) order parameter

* <u>Finite-size effects</u>. $s = \frac{\lambda}{L}$ $\quad \varphi_L(\lambda) = \Psi_L\left(\frac{\lambda}{L}\right)$





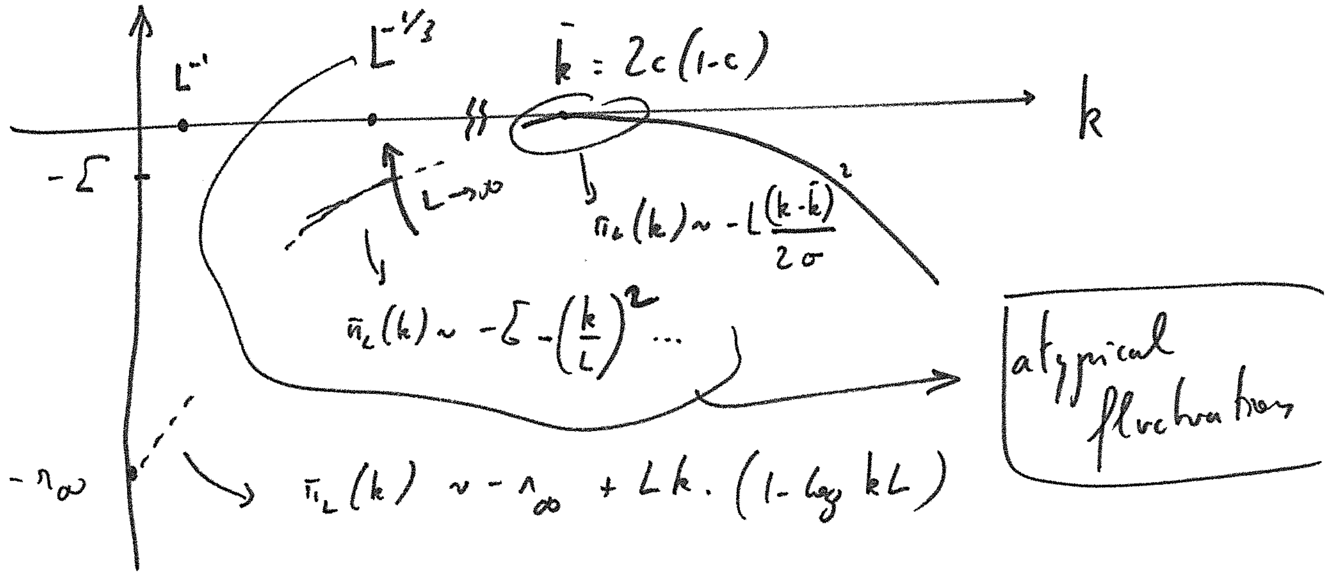
$\underline{\lambda > \lambda_c}: \varphi(\lambda) = -\Sigma + \left(\frac{\lambda}{L}\right)^{2/3} + \cdots$

$-\Sigma$ → optimal configuration is a bubble

$\left(\frac{\lambda}{L}\right)^{2/3}$ → Finite-size scaling. The boundary of the bubble is a random walk

↳ Finite-size scaling of Ψ tells us about the nature of the interface between active & inactive regions

* <u>Finite-size scaling in 'direct space'</u>: $P(K = kt) \sim e^{t\,\pi_L(k)}$



1-6- A "mean-field" picture: FA on a complete graph.

$1 \leq n \leq L$ = # of active sites

$$\begin{cases} W(n \to n+1) = \underset{\text{activation rate}}{c}\ \underset{\text{\# activable sites}}{(L-n)}\ \underset{\text{\# active neighbors (Kinetic Constraint)}}{n} \\ W(n \to n-1) = \underset{\text{inactiva. rate}}{(1-c)}\ \underset{\text{\# sites that can be desactivated}}{n}\ \underset{\text{\# active neighbors}}{(n-1)} \end{cases} \quad r(n) = W(n \to n+1) + W(n \to n-1)$$

(the factor n in $W(n \to n+1)$: Important to avoid absorbing state $n=0$)

Rk: this is an example of one-step process; see Tobia's lecture

Exercises:

- $\boxed{P_{eq}(n) = \binom{L}{n} c^n (1-c)^{L-n}}$ is the steady-state (equilibrium distribution)

[Show that $W(n \to n+1)\, P_{eq}(n) = W(n+1 \to n)\, P_{eq}(n+1)$]

- $\boxed{\mathbb{W}^{sym}(s) = P_{eq}^{-1/2}(\hat{n})\ \mathbb{W}(s)\ P_{eq}^{1/2}(\hat{n}) \text{ is a symmetric matrix}}$

of elements $\left(\mathbb{W}^{sym}(s)\right)_{nn'} = e^{-s}\left\{\sqrt{W(n \to n+1)W(n+1 \to n)}\ \delta_{n\,n'+1} + \sqrt{W(n \to n-1)W(n-1 \to n)}\ \delta_{n'\,n-1}\right\}$

$- r(n)\, \delta_{nn'}$

__Use the following theorem__: for a symmetric matrix $\mathbb{W}^{sym}$

$$\psi(s) = \max \mathrm{Sp}\, \mathbb{W}^{sym} = \max_{|P\rangle \neq 0} \frac{\langle P | \mathbb{W}^{sym} | P \rangle}{\langle P | P \rangle} \quad (*)$$

with $|P\rangle = \sum_{1 \leq n \leq L} P(n)\, |n\rangle$ and $\underbrace{P(n) = e^{L f(n/L)}}_{\text{large deviation scaling}}$

substitute this form of $P(n)$ into $(*)$ to show that

$$\frac{1}{L}\psi(s) \underset{L\to\infty}{\sim} \max_f \frac{\sum_n \left[\sqrt{w^+w^-}\left(e^{L\,g(\rho)} + e^{-L\,g(\rho)}\right) - r\right] e^{2L\,f(\rho)}}{e^{2L\,f(\rho)}}$$

$\rho = \frac{k}{L}$

$0<\rho<1$

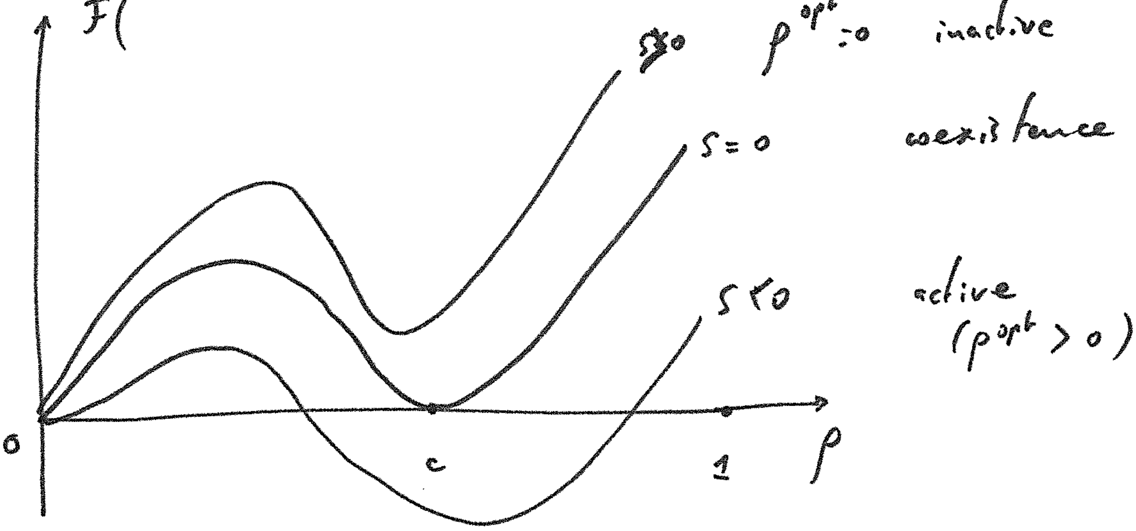
in the large L limit, extremal values of f dominate, & verify $f'(\rho) = 0$

↳ this "optimizal" principle is thus independent of f and writes

$$\frac{1}{L}\psi(s) \underset{L\to\infty}{=} -\min_\rho \mathcal{F}(\rho, s)$$

$\mathcal{F}(\rho,s)$ ↳ "Landau free energy" (dynamical)

$$\mathcal{F}(\rho, s) = \rho\cdot\left(c(1-\rho) + \rho(1-c)\right) - 2e^{-s}\rho\sqrt{c(1-c)\rho(1-\rho)}$$

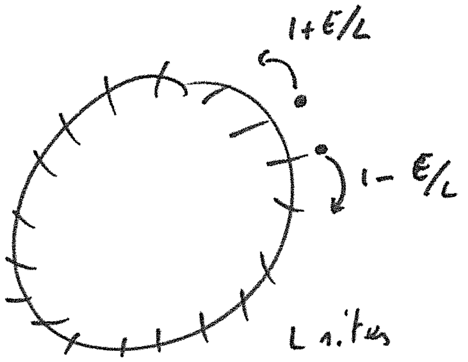


Akin to static 1st order phase transition

2 – 2ND ORDER PHASE TRANSITIONS IN TRANSPORT MODEL

Weakly Asymmetric SEP → WASEP

2-a- Current fluctuations in a periodic WASEP



slight asymmetry ~ 1/L b/w ↻ & ↺

E is a "field"

$0 < x < 1$ spatial coord.

↓

Macroscopic description with a field $\rho(x,t)$ = local density field

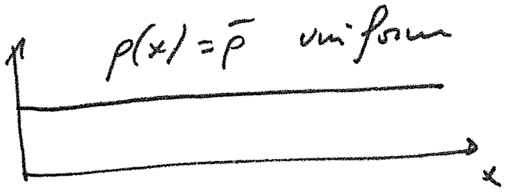
$$\partial_t \rho = -\nabla_x J \quad , \quad J = -D\nabla_x \rho - E\rho(1-\rho) + \frac{1}{\sqrt{L}}\sqrt{\rho(1-\rho)}\,\eta$$

"small" multiplicative white noise; η ← white noise

Representation of this Langevin evolution with an action

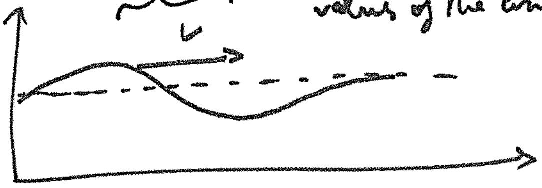
$$\langle e^{-sQ}\rangle \sim \underbrace{\int \mathcal{D}\rho\; e^{-L\,\overbrace{S[\rho;s]}^{\text{action}}}}_{\text{dominated by the saddle point}}$$

Q ← total current on $[0,t]$



$|s-E| < E_c$

optimal profile is flat, uniform, stationary



$|s-E| > E_c$

optimal profile is non-uniform, moving at constant velocity v

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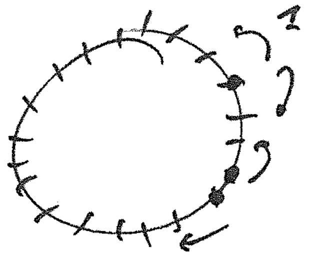
2.b. Activity fluctuations in a periodic SSEP



fixed # particles, mean density ρ_0

K = # events on histories $[0,t]$

in the same way: $\langle e^{-sK} \rangle \sim \int \mathcal{D}\rho(x,t)\, e^{-L \int_0^t \{-\mathcal{H}(\rho(x,t))\} d\tau}$ (action density)

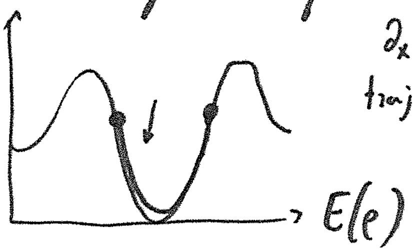
Optimal path obtained by saddle-point evaluation

Path are stationary $\rho(x)$.

Saddle point equation take the form

$\partial_x \rho = -E'(\rho)$

trajectories $\rho(x)$ with x being a time



microscopical view: to decrease activity, particles gather in clusters.