Finite size effects in a stochastic condensation model

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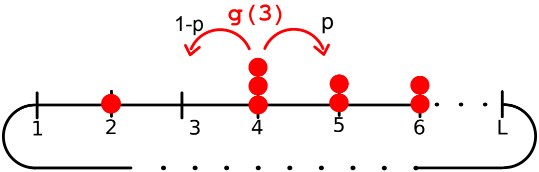
Model: The zero-range process

A continuous-time Markov Chain. Driven diffusive system.



[Spitzer (1970), Andjel (1982)]

Lattice: $\Lambda_L = \{1, \ldots, L\}$ with pbc

Configuration: $\boldsymbol{\eta} = (\eta_x)_{x \in \Lambda_L}$ with $\eta_x \in \mathbb{N}_0$

State space: $X_L = \mathbb{N}_0^{\Lambda_L}$

Jump rates: $g : \mathbb{N}_0 \to [0, \infty)$

$g(n) = 0 \iff n = 0$

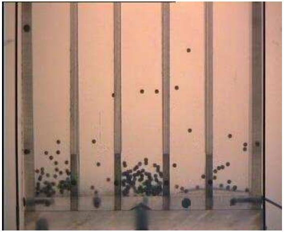
Motivation

$g(k) \searrow \quad \Rightarrow \quad$ effective attraction, condensation possible

$$g(k) \simeq 1 + \frac{b}{k^\gamma}, \qquad b > 0, \ \gamma \in (0, 1)$$

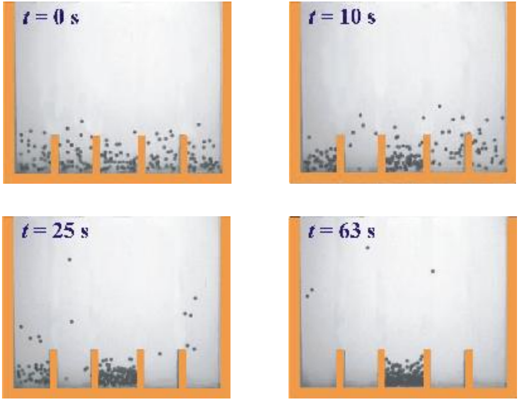
[Evans (2000)]

- Many applications. [Evans and Hanney (2005)]
- Including to granular media and traffic flow.



[van der Meer, van der Weele, Lohse, Mikkelsen, Versluis (2001-02)]

Motivation: Granular media

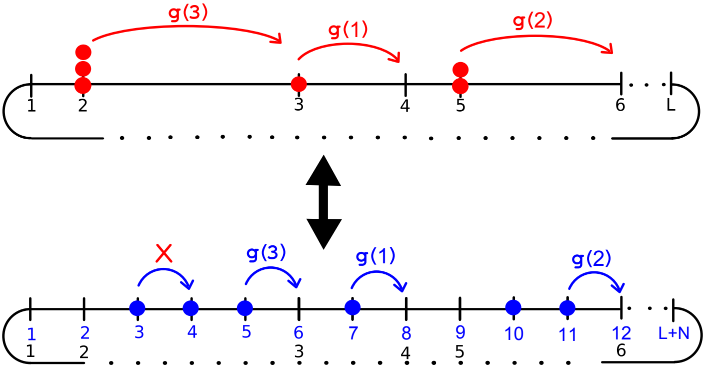


[van der Meer, van der Weele, Lohse, Mikkelsen, Versluis (2001-02)]
stilton.tnw.utwente.nl/people/rene/clustering.html

- Typical sizes $L \sim 10 - 100$, $N \sim 1000$.

Motivation: Traffic

Mapping ZRP to ASEP:



- ASEP can be considered as a simple traffic model

  [Kaupuzs, Mahnke, Harris (2008)]
- Typical sizes $L \sim 1000$, $N \sim 200$.

Outline

Stationary measures: Grand canonical ensemble

There exist stationary product measures.

Stationary weights: $w(n) = \prod_{k=1}^{n} g(k)^{-1}$

Grand canonical ensemble

$$\nu_\phi^L(\boldsymbol{\eta}) = \frac{1}{z(\phi)^L} \prod_{x\in\Lambda_L} w(\eta_x)\, \phi^{\eta_x} \quad \text{where} \quad z(\phi) = \sum_{n=0}^{\infty} w(n)\, \phi^n$$

Fugacity: $\phi \in [0, \phi_c)$

($\nearrow$) **Density:** $\langle \eta_x \rangle_{\nu_\phi} = R(\phi) = \phi \partial_\phi \log z(\phi), \qquad \rho_c = \lim_{\phi\to\phi_c} R(\phi) \in (0, \infty]$

Current: $\langle g(\eta_x) \rangle_{\nu_\phi} = \phi$

Stationary measures: Canonical ensemble

Dynamics conserve the total particle number:

$$\Sigma_L(\boldsymbol{\eta}) := \sum_{x \in \Lambda_L} \eta_x = N$$

Fixed L and N: system is irreducible and finite state so **ergodic.**

Unique stationary measure:

Canonical Ensemble

$$\pi_{L,N}(\boldsymbol{\eta}) := \frac{1}{Z(L,N)} \prod_{x \in \Lambda_L} w(\eta_x) \delta\Big(\sum_x \eta_x - N\Big)$$

Density: $\langle \eta_x \rangle_{\pi_{L,N}} = \rho = N/L$

Thermodynamic limit

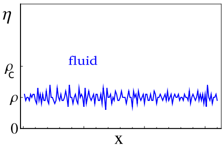
Equivalence of ensembles:

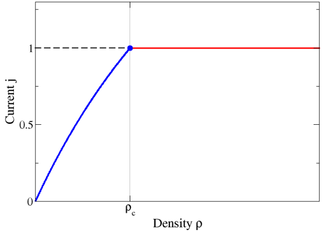
Previous results [Großkinsky, Schütz, Spohn (2003)]

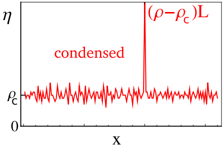
In the thermodynamic limit $L, N \to \infty$, $N/L \to \rho$

$$\pi_{L,N} \xrightarrow{w} \nu_{\Phi(\rho)} \quad \text{where} \quad \Phi(\rho) = \begin{cases} R^{-1}(\rho) & \text{if } \rho < \rho_c \\ \phi_c & \text{if } \rho \geq \rho_c \end{cases}$$

$$g(k) = 1 + \frac{b}{k^\gamma}, \qquad b > 0,\ \gamma \in (0,1) \implies \rho_c < \infty$$







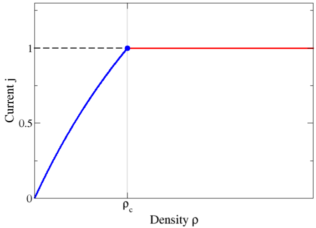
Thermodynamic limit

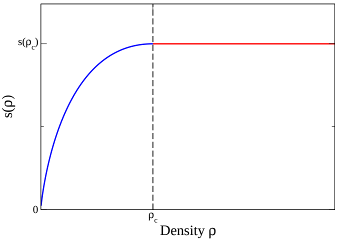
Thermodynamic entropy:

$$s(\rho) = \sup_{\phi \in [0,1)} \left(\log z(\phi) - \rho \log \phi \right)$$

Convergence of canonical entropy:

$$\lim_{L \to \infty} \frac{1}{L} \log Z(L, [\rho L]) = \log z(\Phi(\rho)) - \rho \log \Phi(\rho) = s(\rho)$$





$$\Phi(\rho) = e^{-\partial_\rho s(\rho)}$$

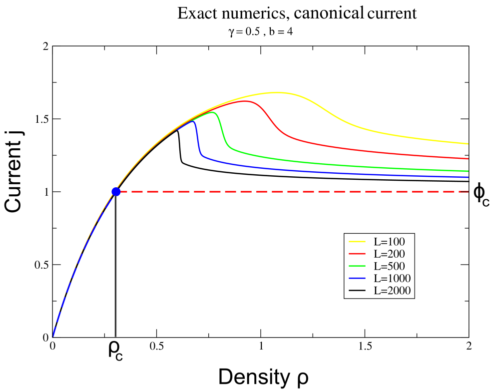
Numerics in the Canonical Ensemble

Recursion relation

$$Z(L,N) = \sum_{k=0}^{N} w(k)\, Z(L-1, N-k)$$

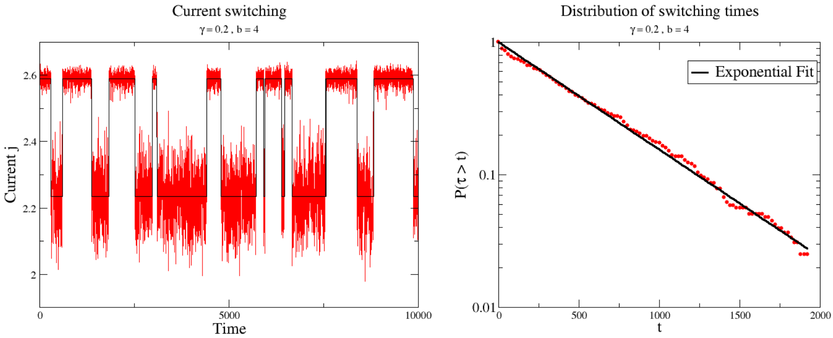
Canonical current

$$j := \langle g(\eta_x) \rangle_{\pi_{L,N}} = \frac{Z(L,N-1)}{Z(L,N)}$$



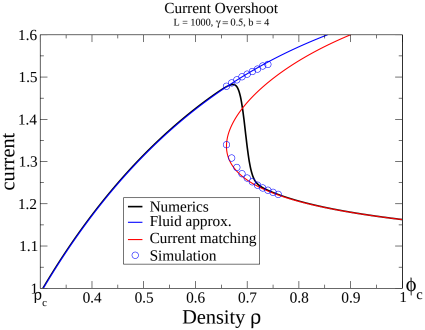
- Large fluid current overshoot.
- Sharp transition to putative condensed phase.

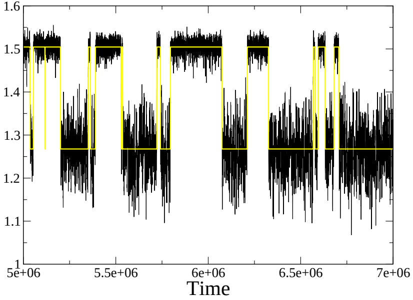
Monte Carlo simulations



- Left: Metastable switching between fluid and condensed currents.
- Right: Exponential distribution of waiting time in the condensed and fluid ‘phases’.

Fluid and Condensed approximations





- Describe where the approximations come from
- How they can be used to find the leading order effect

Rate function

Aim:

- Find the scaling of the overshoot region.
- Understand the apparent metastability close to the maximum current .

Approach:

- Find an effective rate function which describes the metastability in some scaling limit in terms of the background density.

$$\pi_{L,N}\left(\frac{N-\max_{x\in\Lambda}\eta_x}{L-1}\asymp\rho_{\mathrm{bg}}\right)\sim\exp\left(-L^{\beta}I_{\rho}(\rho_{\mathrm{bg}})\right)$$

Rate function

Notation:

$$\pi_{L,N}\left(\rho_{\text{bg}}\right) := \pi_{L,N}\left(N - \max_{x\in\Lambda_L}\eta_x = \rho_{\text{bg}}\left(L-1\right)\right)$$
$$= \pi_{L,N}\left(\max_{x\in\Lambda_L}\eta_x = \eta_{\text{m}}\right)$$

where $\rho_{\text{bg}}(L-1) = N - \eta_{\text{m}}$.

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where $\rho_{\text{bg}}(L-1) = N - \eta_{\text{m}}$.

We have:

$$\pi_{L,N}\left(\rho_{\text{bg}}\right) = \frac{1}{Z(L,N)} L w(\eta_{\text{m}}) \sum_{\boldsymbol{\eta}\in\tilde{X}} \prod_{x=1}^{L-1} w(\eta_x)$$

where $\tilde{X} = \{\boldsymbol{\eta} : \sum_{x=1}^{L-1}\eta_x = \rho_{\text{bg}}(L-1),\ \eta_1,\ldots,\eta_{L-1} \le N-\rho_{\text{bg}}(L-1)\}$.

Rate function

$$\tilde{X} = \{\boldsymbol{\eta} : \sum_{x=1}^{L-1} \eta_x = \rho_{\text{bg}}(L-1),\ \eta_1, \ldots, \eta_{L-1} \leq N - \rho_{\text{bg}}(L-1)\}$$

Rate function

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$$= \frac{1}{Z(L,N)} L w(\eta_{\mathrm{m}}) \phi^{-\rho_{\mathrm{bg}}(L-1)} z_{\eta_{\mathrm{m}}}^{L-1}(\phi) \nu_{\phi,\eta_{\mathrm{m}}}^{L-1}(\Sigma_{L-1} = \rho_{\mathrm{bg}}(L-1))$$

Rate function

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$$= \frac{1}{Z(L,N)} L w(\eta_{\mathrm{m}}) \phi^{-\rho_{\mathrm{bg}}(L-1)} z_{\eta_{\mathrm{m}}}^{L-1}(\phi) \nu_{\phi,\eta_{\mathrm{m}}}^{L-1}(\Sigma_{L-1} = \rho_{\mathrm{bg}}(L-1))$$

where,

$$\nu_{\phi,\eta_{\mathrm{m}}}^{L-1}(\Sigma_{L-1} = \rho_{\mathrm{bg}}(L-1)) = \nu_{\phi}^{L-1}(\Sigma_{L-1} = \rho_{\mathrm{bg}}(L-1) | \eta_x \leq \eta_{\mathrm{m}})$$

Rate function

$$\log \pi_{N,L}\left(\rho_{\mathrm{bg}}\right) = (L-1)\left(\log z_{\eta_{\mathrm{m}}}(\phi) - \rho_{\mathrm{bg}} \log \phi\right) + \\ + \log w(\eta_{\mathrm{m}}) + \log \nu^{L-1}_{\phi,\eta_{\mathrm{m}}}(\Sigma_{L-1} = \rho_{\mathrm{bg}}(L-1)) - \\ - \log Z(L,N) + \log L.$$

Holds for all $\phi \in [0,\infty)$. Choose ϕ so that,

$$R_{\eta_{\mathrm{m}}}(\phi) := \frac{1}{z_{\eta_{\mathrm{m}}}(\phi)} \sum_{k=1}^{\eta_{\mathrm{m}}} k w(k) \phi^k = \rho_{\mathrm{bg}}$$

Rate function

$$\log \pi_{N,L}(\rho_{bg}) = (L-1)\left(\log z_{\eta_m}(\Phi_{\eta_m}(\rho_{bg})) - \rho_{bg} \log \Phi_{\eta_m}(\rho_{bg})\right) + \\ + \log w(\eta_m) + \log \nu^{L-1}_{\Phi_{\eta_m}(\rho_{bg}),\eta_m}(\Sigma_{L-1} = \rho_{bg}(L-1)) - \\ - \log Z(L,N) + \log L.$$

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$$R_{\eta_m}(\phi) := \frac{1}{z_{\eta_m}(\phi)} \sum_{k=1}^{\eta_m} k w(k) \phi^k = \rho_{bg}$$

Define: $\Phi_{\eta_m} = R^{-1}_{\eta_m}$.

Rate function

$$
\begin{aligned}
\log \pi_{N,L}\left(\rho_{\mathrm{bg}}\right) = &(L-1)\left(\log z_{\eta_{\mathrm{m}}}(\Phi_{\eta_{\mathrm{m}}}(\rho_{\mathrm{bg}})) - \rho_{\mathrm{bg}} \log \Phi_{\eta_{\mathrm{m}}}(\rho_{\mathrm{bg}})\right) + \\
&+ \log w(\eta_{\mathrm{m}}) + \log \nu^{L-1}_{\Phi_{\eta_{\mathrm{m}}}(\rho_{\mathrm{bg}}),\eta_{\mathrm{m}}}(\Sigma_{L-1} = \rho_{\mathrm{bg}}(L-1)) - \\
&- \log Z(L,N) + \log L.
\end{aligned}
$$

- Looks like the thermodynamic entropy, except with truncation at η_{m} so exists for all ρ_{bg}.

Rate function

$$\log \pi_{N,L}\left(\rho_{\mathrm{bg}}\right) = (L-1)\left(\log z_{\eta_{\mathrm{m}}}\left(\Phi_{\eta_{\mathrm{m}}}\left(\rho_{\mathrm{bg}}\right)\right) - \rho_{\mathrm{bg}} \log \Phi_{\eta_{\mathrm{m}}}\left(\rho_{\mathrm{bg}}\right)\right) + \\ + \log w(\eta_{\mathrm{m}}) + \log \nu^{L-1}_{\Phi_{\eta_{\mathrm{m}}}(\rho_{\mathrm{bg}}),\eta_{\mathrm{m}}}\left(\Sigma_{L-1} = \rho_{\mathrm{bg}}(L-1)\right) - \\ - \log Z(L,N) + \log L.$$

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- Contribution due to the condensate.

Rate function

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- Can be approximated by Gaussian density at 0.

Rate function

$$\log \pi_{N,L}\left(\rho_{\mathrm{bg}}\right) = (L-1)\left(\log z_{\eta_{\mathrm{m}}}(\Phi_{\eta_{\mathrm{m}}}(\rho_{\mathrm{bg}})) - \rho_{\mathrm{bg}} \log \Phi_{\eta_{\mathrm{m}}}(\rho_{\mathrm{bg}})\right) + \\ + \log w(\eta_{\mathrm{m}}) + \log \nu^{L-1}_{\Phi_{\eta_{\mathrm{m}}}(\rho_{\mathrm{bg}}),\eta_{\mathrm{m}}}(\Sigma_{L-1} = \rho_{\mathrm{bg}}(L-1)) - \\ - \log Z(L,N) + \log L.$$

- Looks like the thermodynamic entropy, except with truncation at η_{m} so exists for all ρ_{bg}.
- Contribution due to the condensate.
- Can be approximated by Gaussian density at 0.
- The final terms are constant for fixed N and L.

Rate function in the thermodynamic limit

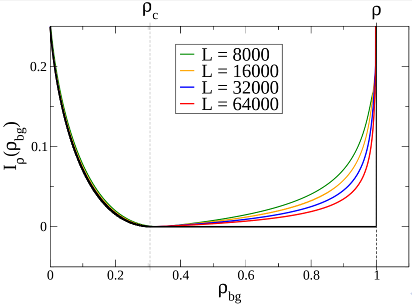
$$I_\rho^L(\rho_{\rm bg}) = -\frac{1}{L}\log \pi_{[\rho L],L}\left(\Sigma_L^{\rm bg} = [\rho_{\rm bg}(L-1)]\right)$$

Theorem

As $L \to \infty$ with $N/L = \rho$ and $\rho_{bg} \in (0,\rho)$,

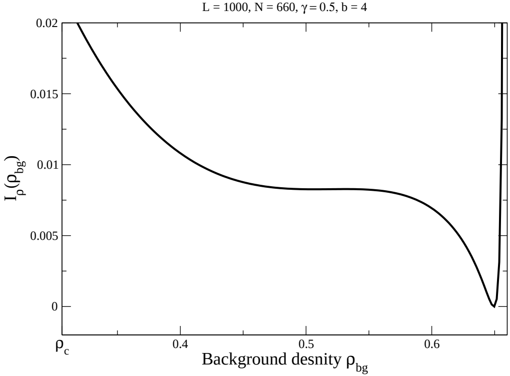
$$I_\rho(\rho_{bg}) := \lim_{L\to\infty} I_\rho^L(\rho_{bg}) = s(\rho) - s(\rho_{bg}) \quad (1)$$

and so $I_\rho(\rho_{bg}) = 0$ for each $\rho_{bg} \geq \rho_c$.



Rate function (finite lattices)

Ignoring the final two normalising terms we can approximate $-\log \pi_{N,L}\left(\rho_{\text{bg}}\right)$ very quickly. Still informative.



- ρ below sharp transition point.

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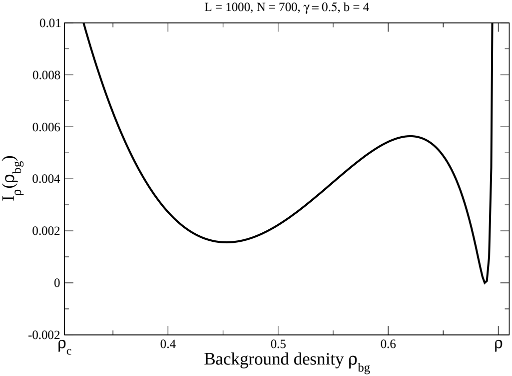

- ρ near sharp transition point.

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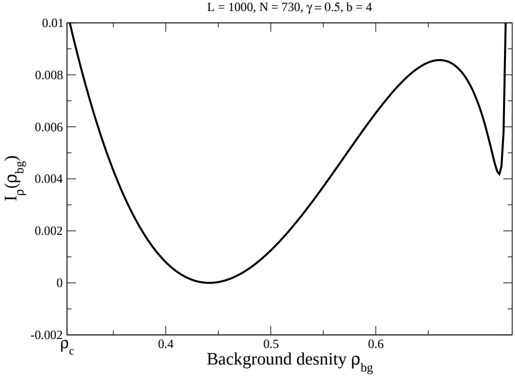


- ρ above sharp transition point.

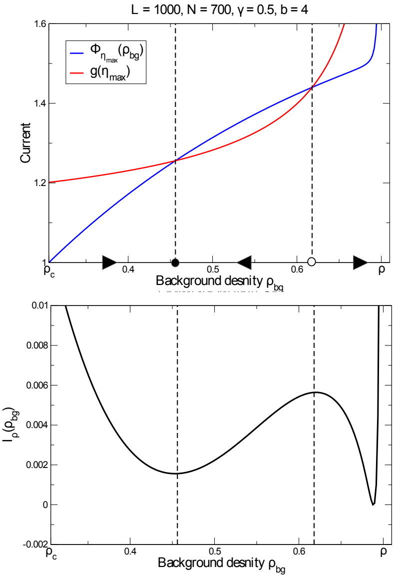
Max and min of distribution

Under controllable approximations (to leading order in L):

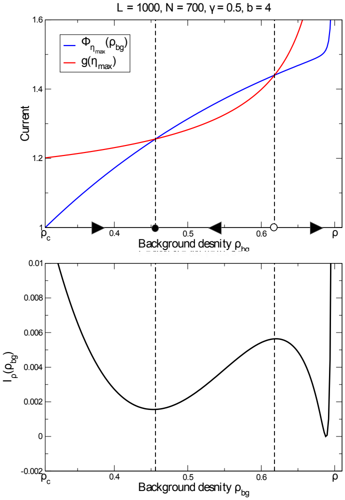
$$\log \pi_{N,L}\left(\rho_{\mathrm{bg}}\right) - \log \pi_{N,L}\left(\rho_{\mathrm{bg}} - 1/L\right) \simeq \log g(\eta_{\mathrm{m}}) - \log \Phi_{\eta_{\mathrm{m}}}\left(\rho_{\mathrm{bg}}\right)$$

- First minimum and maximum are given by current matching.
- The final minimum is effectively a boundary minimum.

Current matching



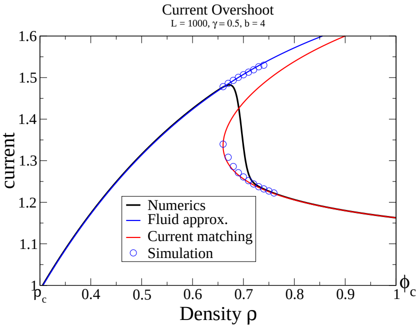
Lifetime of fluid and condensed ‘phases’



Fixed Density N/L

- Ratio of the areas will always give proportion of time spent in each ‘phase’
- For totally asymmetric process random walk argument very well to predict the life time.

Approximations



- Fluid current: From the average current in a cut-off grand canonical ensemble $\Phi_N(\rho)$.
- Condensed current: From current matching, solve

$$\Phi_{\eta_{\mathrm{m}}}(\rho_{\mathrm{bg}}) = g(\eta_{\mathrm{m}})$$

Scaling limit (Currents)

Examine behaviour on the scale of the current overshoot $\rho = \rho_c + \delta\rho L^{-\alpha}$.

Theorem

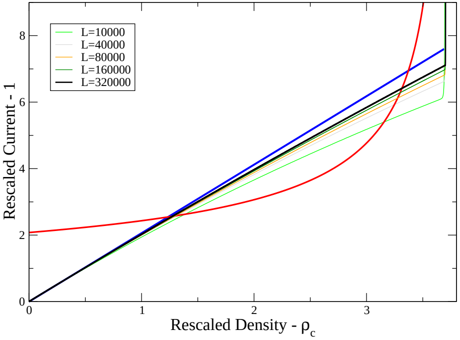
Condensate and background current are asymptotically:

$$g(\eta_m^L) = g\left((\delta\rho - \delta\rho_{bg})L^{1-\alpha}\right) = 1 + \frac{b}{(\delta\rho - \delta\rho_{bg})^{\gamma}} L^{\gamma(\alpha-1)}$$

$$\log \Phi_{\eta_m^L}(\rho_{bg}^L) = \log \Phi_{\eta_m^L}(\rho_c + \delta\rho_{bg} L^{-\alpha}) = \frac{1}{\sigma_c^2}\delta\rho_{bg} L^{-\alpha} + o(L^{-\alpha}).$$

- From this we can derive the scaling so that the two currents collapse $\alpha = \frac{\gamma}{1+\gamma}$.
- At this scale they cross at unique $\delta\rho_{\mathrm{bg}}$ for fixed $\delta\rho$

Scaling limit (Currents)



- Blue: Linear scaling limit for fluid current in background.
- Red: Current out of condensate (Collapsed in scaling limit).

Scaling limit (Rate function)

Examine behaviour of the system at the critical scaling.

$$I^{(2)}_{\delta\rho}(\delta\rho_{\mathrm{bg}}) := \lim_{L\to\infty} L^{1-\beta} I^{L}_{\rho_c+\delta\rho L^{-\alpha}}(\rho_c + \delta\rho_{\mathrm{bg}} L^{-\alpha})$$

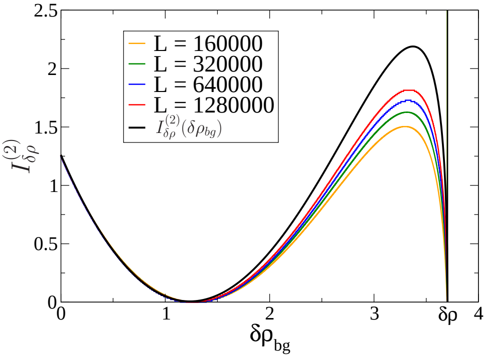
Theorem

If $\alpha = \frac{\gamma}{1+\gamma}$, $\beta = \frac{1-\gamma}{1+\gamma}$ *the limit exists and is given by*

$$I^{(2)}_{\delta\rho}(\delta\rho_{bg}) = \frac{\delta\rho_{bg}^2}{2\sigma_c^2} + \frac{b}{1-\gamma}(\delta\rho - \delta\rho_{bg})^{1-\gamma} + \min\{\frac{\delta\rho^2}{2\sigma_c^2}, f(\rho_{bg})\}$$

Poof: Bounds on remainder term in Taylor series of $\Phi_{\eta^L_{\mathrm{m}}}(\rho^L_{\mathrm{bg}})$ and local limit theorem for triangular arrays.

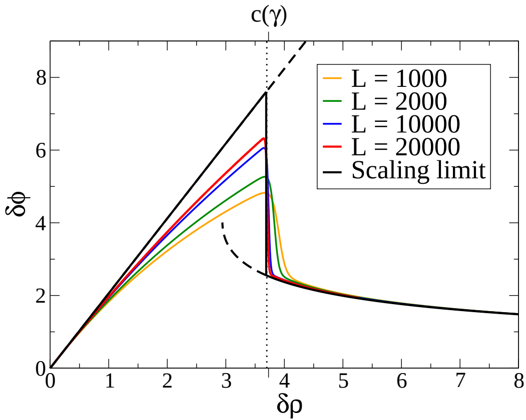
Scaling limit (Rate function)



$$I^{(2)}_{\delta\rho}(\delta\rho_{\text{bg}}) := \lim_{L\to\infty} \frac{1}{L^{\beta}} \log \pi_{L,[(\rho_c+\delta\rho L^{-\alpha})L]} \left(\rho_{\text{bg}} \asymp \rho_c + \delta\rho_{\text{bg}} L^{-\alpha}\right)$$

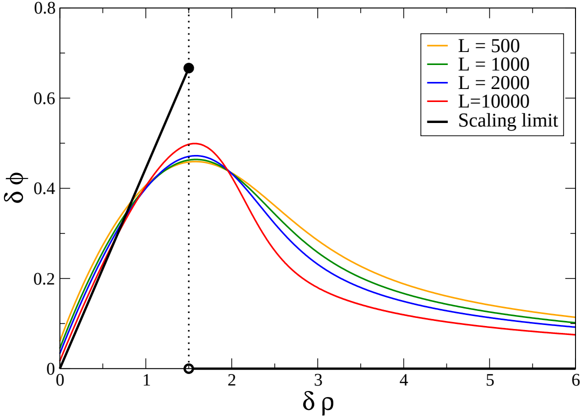
- Critical scaling and the position of global minima corresponds with recent results [Armendariz, Grosskinsky, Loulakis]

Data collapses



- Also for each vertical slice (fixed rescaled density) we have the scaling rate function.

Data collapses for $\gamma = 1$



Conclusions and Summary

- Large finite size effects observed for some parameter values.
- Observed phenomena on a finite system such as metastable switching are also observed in real world clustering.
- May have implications for understanding traffic flow patterns.

Results:

- Estimate the fluid current overshoot and condensed current on large finite systems.
- Estimate the density at which an observed change in ‘phase’ will occur on a large finite system.
- Predict asymptotic scaling.
- Describe metastable behaviour and predict switching times heuristically for simple systems.

Acknowledgements

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- Second supervisor Ellak Somfai.
- Discussions with Robin Ball.

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