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Endogenous Growth in a Small Open Economy: Optimal Fiscal Policy and the Informal Sector

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Abstract and Acknowledgments

Endogenous growth theory has paved the way of modern research on the long run outcomes of policy decisions. Within this context, we develop an endogenous framework to study the welfare outcomes of government fiscal policy with multiple fiscal policy instruments. First, we propose a Romer *type model for a developed economy, with increasing returns arising from free aggregate capital externalities, facing an exogenously given world interest rate, and where the labour supply is elastic and endogenously given. Then, we extend this benchmark economy by assuming that increasing returns are now a consequence of government productive spending. We show that a first best fiscal policy outcome is achievable and that it reproduces the dynamics for the benchmark decentralized economy. Finally, we introduce an informal sector in our welfare analysis, by assuming that there exist microeconomic arbitrage conditions for the development of underground activities, which arise from factor income taxation during transitions. We discuss specific fiscal policy outcomes for both horizons and describe the specific dynamics, both analytically and numerically. In all models, we also discuss the introduction of convexity, in our endogenous environment, by assuming investment adjustment costs. Based on the optimal control conditions for these problems, we define the set of higher order systems that govern these economies, upon specific initial hypotheses, and simulate numerically the transitions of the respective scaled systems, by assuming a common balanced growth path, in order to described the dynamic convergence behaviour of the variables under specific initial macroeconomic assumptions.*

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Glossary of Terms, Parameters and Variables

y -Individual firm output

Y -Aggregate Output

k -Individual firm capital

K -Aggregate firm capital

c -Household consumption

C -Aggregate household consumption

i -Household Investment

I -Aggregate household Investment

b -Household net foreign assets/debt

B -Aggregate net foreign assets/debt

W -Aggregate net wealth

l -Leisure share

ℓ_1 -Labour allocation in formal sector share

Z_1 -Average productivity of capital variable

Z_2 -Consumption to net wealth variable

q -Shadow price of capital

λ -Marginal value net foreign assets/debt

G -Total government spending

N -Total exogenous population

A -Exogenous technology formal sector

ρ -Time discount factor

α -Elasticity of individual formal firm capital

β -Elasticity of aggregate capital (chapter 4.) and Government spending (chapter 5. and 6.)

ϕ -Labour elasticity formal sector

γ -Intertemporal elasticity of substitution

θ - Impact of leisure on the utility

δ -Depreciation rate

h -Adjustment costs parameter

r -Exogenous world interest rate

r_k -Household capital income

w -Household labour income

g -Output share of government spending

τ_c -Consumption Tax

τ_w -Labour income tax

τ_k -Capital income tax

τ -Household lump-sum tax

T - Aggregate lump-sum taxes

D -Exogenous technology informal sector

μ -Fiscalization parameter

ξ - Labour elasticity informal sector

η - Elasticity of individual informal firm capital

e -Euler's number

U -Utility function

Φ -Investment adjustment costs function

H^* -Current value Hamiltonian

J -Jacobian matrix

λ_i -Eigenvalues for Jacobian matrix

ν_i -Eigenvectors for Jacobian matrix

Θ_1 -Support parameter expression

Ψ -Common growth rate (BGP)

$PmgK$ -Marginal productivity of aggregate capital

ω_{dom} -Share of domestic net wealth parameter

Υ -Foreign debt/wealth to domestic capital parameter

Ω -Support parameter expression

Θ_2 -Support parameter expression

ω_{for} -Share of foreign net wealth parameter

Ψ_K - Growth rate of formal capital parameter

Θ_3 - Support parameter expression

ω_T - Share of lump sum taxes to net wealth parameter

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1. Introduction

As the title suggests this thesis will deal with the subject of endogenous growth applied to small open developed economies. For this purpose we will present a set of models for a growing economy, where endogenous growth is a result of increasing returns arising from positive aggregate externalities. This class of models was first introduced in the well known article, Romer (1986), where Romer showed how positive externalities from aggregate capital can sustain the dynamics of endogenous growth, in the neo-classical class of closed economy growth models. Since the seminal paper from Stockey and Lucas (1983), a shift has occurred in research and brought on new extensions to long run economic growth models. The consequence of this shift has been a vast literature dealing with policy implications of economic growth models and specifically with fiscal policy and public spending issues, which is now commonly known as optimal fiscal policy theory. This framework was then extended to endogenous growth models by the work of Barro (1990), Jones and Manuelli (1990) and Rebelo (1991), which attributed an important role to policy in the long run growth outcomes. While this was the starting point for a wide research relating policy outcomes and rules in an endogenous set of models, those conclusions were subsequently contested by the empirical work of Easterly and Rebelo (1993), using a cross-section of countries and a time series analysis of public investment, and Jones (1995), using a time series analysis of various macroeconomic variables. Even though there is still a wide discussion about policy implications on long run outcomes, it is our opinion that there exists some evidence that both public infrastructure and aggregate capital externalities have a relevant role in the increasing returns mechanism of endogenous growth. Taking this in consideration, we propose to extend the formalization proposed in Turnovsky (1999) of a small open economy optimal fiscal policy model, in order to develop the welfare and policy implications of this class of models in two specific directions. First, we will deal with the welfare outcomes of choosing different sources of positive externalities to obtain the increasing returns mechanics of endogenous growth. Second, we will propose an extension of this class of models that accommodates the existence of an informal sector taking advantage of government taxing and lack of regulation during short to medium run transitions.

In order to deliver the intuition described above, three different models will be developed in this document. The first will be a similar version of the endogenous growth model developed in Turnovsky (1999), but without the government sector. This formalization can be portrayed as an extension of Romer (1986) endogenous growth model with capital externalities to a small open economy framework with an elastic labour supply. This model is just a benchmark to give a formal introduction to the framework that will be presented later on, when the government sector is included. The main difference in this model is in the form that the *AK* technology for the aggregate production function is attained, which is a necessary condition for the existence of endogenous growth. In this model there will not exist a flow of services provided by government spending, so an aggregate capital effect will be considered in the production function, in order to replicate the externalities needed to obtain an *AK* type production function. The second model is based on the model presented in Turnovsky (1999), where government investment is a flow, as proposed in Barro (1990) and Barro and Sala-i-Martin (1992), and increasing returns arise from public spending and are no longer a free public good. We deliver a formal presentation of the long run dynamics implied in this model and to the problems arising from deriving a set of first best rules for optimal fiscal policy. We show that the slight changes that we imposed in the Turnovsky (1999) framework do not change the policy results obtained in the original article. We will also suggest a different framework to evaluate, both analytically and numerically, the economy proposed by Turnovsky (1999), when we extend the equilibrium conditions for current account equilibrium, from a simple static macroeconomic rule, to an environment with convex investment adjustment costs. In the third and last model the informal sector will be introduced in the framework developed for the second model discussed above. We suggest that the informal economy can be portrayed by an independent neo-classical productive sector, as first suggested in Jones and Manuelli (1990). This hypothesis is based on the presumption that in the long run, formal activities will still be the dominant engine of growth in a developed economy, but if we consider that the long run outcome is composed by a sum of short to medium run periods, then, within this time frame, entrepreneurial opportunities for informal activities may arise and assume an important role during transitions. We show how information about the relationship between both sectors can be determinant in the neo-classical framework, as suggested in Peñalosa and Turnovsky (2004), in order to reduce our model in terms of the formal sector variables and maintain our long run growth hypothesis, and also to determine if fiscal policy is still given by a set of first best optimal rules or if only a second best outcome scenario may be achieved.

In the next three sections we will briefly discuss some specific aspects about the economic background, methodological implications and related literature of these proposals, within the endogenous theory of growth. We will take the option of discussing these specific issues in these subsequent sections and leave the rest of the introduction to describe the main outcomes we have achieved in our research. Overall, our results can be summed up in three main categories: modelling methodology; analytical solutions and numerical analysis; economic and policy results. Within the first main category it is worthwhile to mention that all the models that we developed, both centralized and decentralized optimal control problems, represent comparable hypotheses and/or extensions of the same class of model. Therefore, we may compare welfare, economic and mathematical outcomes between the three models as differences and extensions are introduced. This is an important result that satisfies our initial research proposal and is decisive to the description of the remaining categories of results. Using this methodology, we will show that within this class of endogenous growth models, with increasing returns, the best possible outcome is the centralized result for the economy with aggregate capital externalities, which is both unattainable in the decentralized framework or in a framework where externalities arise through public expenditure. First best optimal fiscal policy follows two main rules. The first rule is the maximizing growth rule, which states that all productive factors should be taxed equally and at a level that maximizes growth. The second rule reproduces the *Ramsey* optimal taxation principle, which states that policy should minimize the excess burden of taxes and the decrement of utility, without redistributive concerns. If this main fiscal policy rules are followed and government spending follows the neo-classical rule for the equality between the elasticity of output with respect to the government input and the government output spending share, then, in the simpler case, the decentralized outcome of the economy with aggregate capital externalities can be achieved. Following this set of rules by our non debt issuing government guarantees that the outcome with public externalities can be achieved without imposing any distortions to growth and labour markets. When we extend this analysis to the two sector model a first best set of rules for policy is not achieved when agents lack information about investment relations between sectors. Fiscal policy in this case is only a second best policy and the achieved outcome is a result of preferences and choices between growth maximization, utility enhancing and labour market distortions. When information is complete, fiscal policy is again a first best set of rules in the long run and only transitions are influenced by the presence of the informal sector.

The decision to tackle these issues using a comparable set of models, within the same class of endogenous growth models, has also influenced dramatically the choice for solution strategies and numerical simulations that we have adopted. In the last paragraph, we described the set of economic and policy results that briefly described our main results. Most of these results were obtained by assuming a static macroeconomic condition to deal with the absence of convexity in our open economy endogenous framework. This has proved to be a straightforward approach to provide a clear policy and economic insights and allowed us to use the same type of solution strategies for all models. However, this approach has revealed to be quite rigid when dealing with the economy with an informal sector. Not only, we had to assume strict assumptions about the relations between sectors, in order to deliver a comparable set of results, but also, when solving for the transformed model of transitions an additional set of restrictive assumptions had to be adopted, with the aim of achieving a feasible outcome. This is a clear downfall to our proposals as the results for the transitions models only depict a set of results based upon a restricted set of hypotheses. Nevertheless, we show that it is possible to apply the Jones and Manuelli (1990) methodology in a small open economy environment, originating from two different sectors and based upon microeconomic assumptions. Our results show that the phase plane dynamics for the benchmark centralized closed economy model are very similar to our two sector centralized economy transitions model. We can extend this result to our full information decentralized economy based upon a feasible set of parameters. However, this result may not always hold, as saddle path dynamics, in this case, are sensitive to the choice of parameters. With the objective of not limiting our results to a set of restrictive solution strategies we decided to extend our optimization framework by adopting a classical convex function of investment adjustment costs. This option has the advantage of introducing transitional dynamics in all the models considered but has the disadvantage of adding additional dimensions to our dynamical systems. In order to simplify our approach and define all the four dimensional systems that described this extended version of our economies, we relaxed the assumptions of motion for labour/leisure decisions and balanced budget in all models, and labour allocation for the two sector economy. Next, we decided to scale, evaluate and simulate the decentralized economies with externalities arising from public expenditure, to study convergence dynamics from an initial value to the long run regime. We show that it is possible to tackle large scaled systems following a straightforward methodology, using the variational system resulting from the usual linearization procedure, and that these results can be used to produce meaningful economic based numerical simulations. Unfortunately, due to the wide scope of our research, we had to limit these simulations to a set of specific examples in chapter 5. and 6., based upon a simple initial value routine for ordinary differential equations, thus limiting this approach to be just a methodological proposal and not yet a tool for evaluating policy decisions and welfare outcomes. Nevertheless, our simulations have showed that the introduction of an additional neo-classical sector seems to increase the range of initial values that guarantee convergence to a long run regime, where all variables share the same common growth rate. Although this result is yet to be fully confirmed, it is our conviction that this is a strong result and that further simulations will strengthen this evidence. We consider this outcome to be a consequence of the additional capital and labour incomes arising from informal activities during transitions, due to the positive effect that these additional domestic incomes impose on the intertemporal international budget constraint imbalances during transitions. If this is the case, then, the informal sector, although limiting potential growth during transitions, may act as a stabilizer for external financial imbalances, by providing additional income opportunities, which agents use to adjust their portfolios and consumption decisions.

1.1. Optimal Fiscal Policy and Endogenous Growth

After two decades of endogenous growth theory there has been a vast literature interested in studying the welfare and policy implications of this specific class of mathematical growth models. Within this specific framework a particular importance has been given to optimal fiscal policy models, with the purpose of extending public finance and taxation theory to a long run endogenous growth environment. We have already discussed briefly our main policy results in an economy where government has multiple fiscal instruments but cannot finance public spending through debt issuing. This framework is based in the Turnovsky (1996a, 1999) articles and extends the policy links between growth maximization fiscal policy and factor accumulation taxation proposed earlier by

Lucas (1990)[2], Rebelo (1991) and Jones, Manuelli and Rossi (1993), to accommodate an optimal taxation program that allocates taxes efficiently over time, as proposed by Ramsey (1927), guarantying a minimal decrement in utility and therefore limiting the excess burden of taxes. Policy in this class of models will be given by a set of endogenous rules that comprise a first best optimum choice of fiscal instruments, which are consistent with both the standard assumptions for growth maximization in endogenous theory and a non-distorting *Ramsey* taxation program, as we demonstrate in chapter *5*.. However, these instruments are all interdependent, meaning that additional hypotheses or extensions might drive the feasible set of fiscal policies to be just second best outcomes in the long run. In this case, fiscal policy becomes a problem of public choice theory and preferences between growth maximizing policies and *Ramsey* efficient allocation taxation programs must be taken into account.

As all models of optimal fiscal policy, our benchmark framework and extensions come short when dealing with other relevant implications of taxation and government spending in an endogenous framework. This limitation is due to the wide range of extensions that can be considered for the study of fiscal policy outcomes. One interesting extension of fiscal policy models deals with the public sector socially optimal intertemporal debt structure. Saint-Paul (1992) describes the implications of considering a debt issuing government, for a growing economy, in a continuous time deterministic model and challenges the results of intertemporal debt efficient allocations in overlapping generations endogenous growth models and neoclassical growth models, by defining debt issuing policies to be sub-optimal for growth maximization. Turnovsky (1996b), on the other hand, considers that a first best optimum is achievable by a government that acts as a net lender but there is a trade-off between the dimension of taxes and the structure of debt. Other relevant extensions include fiscal policy models, where government spending is used, not only to provide public services, but also to invest in human capital formation, such as the articles by Ortigueira (1998) and Agénor (2005), which follow a similar modelling approach to ours. Park and Philippopoulos (2003) discuss optimal fiscal policy and dynamic determinacy in a continuous deterministic endogenous growth model with increasing returns, generated by public infrastructure and additional non productive public spending, specifically public consumption services and redistributive transfers. The list of fiscal policy models in an endogenous framework is long and growing rapidly, it includes a wide variety of initial assumptions and extensions, applied to stochastic and non-stochastic economies, both in continuous and discrete time, with the objective of performing analytical and numerical analysis of welfare outcomes. For that reason, we will limit the references in this section to these specific articles that follow a very similar approach to ours and that are relevant to the methodology that we will develop further on.

[2] To our knowledge the Lucas (1990) article is the first article that extends this problem fully to an endogenous growth framework, based upon previous fiscal policy models and, among others, the earlier versions of Rebelo (1991) article on policy in an endogenous growth framework.

1.2. Small Open Economies and Endogenous Growth

The small open economy class of growth models has gained a renewed interest in recent years by researchers, due to the increasing economic integration promoted by globalization and regional political agreements. As the majority of countries can be designated as a developed or a developing small open economy, this framework as become increasingly important when tackling policy issues. The framework that we will use in our models will be the conventional hypotheses of an economy that faces an intertemporal international financial budget constraint, following the usual current account specification. Additionally, we will consider that our economies face an exogenously given international interest rate. This specification has many times been considered as an excessive simplification of the external financial conditions faced by small open economies, nevertheless, with the increasing openness of capital markets and economic integration the real world has taken a clear step towards this theoretical assumption. This is particularly true when we consider that small developed open economies have limited financial and economic resources, and therefore require an access to international capital markets, to diversify their investment portfolios and guarantee foreign financing of domestic economic activities. Taking in consideration that contracts in developed international financial markets are usually denominated in currencies from large monetary areas, it is reasonable to consider that these economies act as price takers and to some extent the financial price that they face is linked to monetary conditions of large currency areas. When we confine our range of possible economies to the small countries that belong to the European Monetary Union, which have exchanged their monetary policy for better external financial conditions, then this framework seems to match perfectly the external financial conditions faced by this specific range of economies.

The downfall of this approach in an endogenous framework is that we have to consider an additional state condition in our optimization problem. This will introduce two main problems in our approach. The first one has to do with the absence of the usual convexity properties in an endogenous framework that are a necessary condition to deal with optimal control problems. We will tackle this issue using two strategies. First, we will consider a static macroeconomic condition that guarantees the shadow prices of domestic and foreign capital will always equalize. This procedure will be used to solve the simplified models, with the aim of developing some analytical insight about the different economies that we will deal with. The second approach uses an investment adjustment costs convex function in order to introduce convexity in the open economy intertemporal budget constraint. This option will be sufficient to guarantee that the necessary conditions for optimal control exist and relax the all period equality for shadow prices by a convergent intertemporal equilibrium that allows for transitions to the long run regime. This option seems more interesting to study but will introduce additional dimensions to our dynamical systems, decreasing its economic insight and increasing its mathematical complexity. The other difficulty about introducing an additional state condition for open economy models, in an endogenous growth framework, refers to the complexities that arise when introducing additional economic productive sectors. If the introduction of an additional productive sector implies an additional state condition, then, the complexity problems that were discussed for the one sector growing economy become increasingly intricate, as a new additional dimension is added to the dynamic specification. We will discuss this subject thoroughly when introducing the informal sector in chapter 6. and show that in order to deliver a simplified dynamic analysis of our hypothesis, using the Jones and Manuelli (1990) methodology, we will require a set of restrictive assumptions. Turnovsky (2002) discusses these dynamic issues that arise when dealing with small open economies, in a framework very similar to the one we follow here, and compares the equilibrium conditions in open economy endogenous growth models with other varieties of open economy growth models.

1.3. The Informal Sector and Endogenous Growth

The informal sector has not been a central theme to long run growth theory, at least, when we consider the causes and policy implications of this phenomena in developed countries. On the other hand, there has been a wide interest of both causes and policy implications of the shadow economy in developing countries. The reason for this is clear, in developed countries production is mainly based in the formal sector of the economy and the opposite is usually observed in developing economies. However, recent decades have showed a clear positive trend of the size of these activities relative to GDP in industrialized countries. Schneider (2005) reports an average size for the informal sector relative to GDP of 13,2% in 21 OECD economies, for the years 1989/90, using the currency demand and DYMIMIC method[3]. These values increase to an average size of 16,4% for the years 2002/03, representing an increase of total output share of about 24% for informal activities, during this recent thirteen years time frame, which represents an average annual growth of output share of about 1,6%. If we extend our time frame some decades to the past, using Frey and Schneider (2001) results, we can emphasize the increasing role of informality in developed economies. They report that Nordic economies, except for Finland, had an average informal sector size of less than 5% relative to GNP in 1960, according to the currency demand method. This output share increases to values of about 15% to 20% when we consider the year 1995. This increased output share of informal activities in major industrialized economies varies according to the country we consider, but we can state that an increasing share of output has been located in the informal sector in every industrialized economy. Moreover, we can state also that this specific sector is no longer residual and in some cases, as in south European countries, it represents between one fifth to almost one third of total output shares.

[3] Dynamic multiple-indicators multiple-causes method. Refer to Schneider (2005) for further details in this recent modelling technique.

What are the causes for this clear trend in developed economies and what are the consequences of this increasing output share for policy outcomes? There are two key causes that may explain this evidence and at the same time are able to capture this trend, even if we consider the specific socio-cultural and political backgrounds of different developed economies. These causes are increasing government spending and bureaucracy, in industrialized countries, during the past four decades. Both these causes are widely accepted for explaining this trend because they match basic industrial economics theory. Increasing bureaucracy and taxes augment both fixed and marginal costs, distort market prices, consequently, creating legal and economic barriers to entry in the formal sector, decreasing present and future profits for formal sector firms and widening the range of opportunities for illegal activities. We try to tackle these two issues in an endogenous framework for a small developed economy, where informal activities arise in the short to medium run transitions, taking advantage of these arbitrage conditions at the micro level. Peñalosa and Turnovsky (2004) explore these same sources of opportunities for informal activities in a developing economy, considering a *CES* production function and increasing returns arising from the usual aggregate capital hypothesis. Although, our framework has many similarities with the hypotheses proposed in Peñalosa and Turnovsky (2004), the main scope of their article is the second optimal policies that arise in the presence of a government that lacks revenues and only has access to a limited formal sector for taxation. Other examples of articles that follow similar hypotheses for developing economies in an endogenous growth framework are Braun and Loayza (1994) and Sarte (1999), which tackle the issue of rent-seeking bureaucracies acting through excessive regulation and taxation. The consequences of a growing informal sector are thus evident for both macroeconomic and microeconomic government policy and embody a loss of effectiveness by public authorities to enforce their policies, when facing competition from a sector that acts as a substitute to government regulation. In chapter *6.* we will discuss a hypothesis, within our specific framework, that symbolizes this loss of power by public authorities. In our example, we will show that under some strict assumptions, fiscal policy might not achieve a first best optimum in the long run and may be limited to a set of second best policy options, thus exemplifying the loss of policy effectiveness.

2. Methodology and Background[4]

In chapter *1.* we described, in general terms, both the theoretical framework and also a part of the relevant literature that is significant for our research proposals. In this chapter and the next we will resume the specific methodology and assumptions that we will employ and the organization scheme that we will follow in general terms. The framework followed in here will be similar to the ideas presented in Turnovsky (1999), where an optimal deterministic control problem of continuous time

[4] For a full coverage of the mathematical methodology and analysis applied to continuous time economic models that will be discussed in the forthcoming chapters, the reader should refer to the following textbooks: Chiang (1992), Turnovsky (1997) and Barro and Sala-i-Martin (1999).

dynamics, with two state and three control variables, consumption, investment and labour/leisure, is considered. As it is usual in these specifications, we will assume that these economies face an infinite horizon and have a large number of identical households, who live infinitely as self-perpetuating dynasties. There is no population growth so the total number of agents or households is fixed and given exogenously. The productive part of these economies will be composed by a large number of identical firms that face perfect competition from each other and produce a homogenous good with a Cobb-Douglas production technology. Since these firms face a perfectly competitive market they will pay capital and labour returns equal to the marginal product of factors, following the usual neo-classical assumptions. The technology considered in the Cobb-Douglas production functions is given by an exogenous parameter. Households may choose between leisure, which is an utility enhancing activity and labour, which will provide labour incomes and increased production, but will act as an utility diminishing activity. This will be achieved be introducing labour as an input in our Cobb-Douglas production function and using an isoelastic utility function, where consumption and leisure are substitutes. In chapter 6. we will introduce labour allocation as an input in the production functions of both sectors and as an additional control variable to our optimization problem. We will consider that households define their labour and leisure preferences and then allocate their labour time according to those initial choices.

The methodology that we will use to tackle the described economic background will be based on the standard optimal control conditions that define intertemporal optimization problems. In order to develop manageable dynamical systems to evaluate, we will consider the following assumptions. First, we will consider that the control variable leisure is given endogenously by the model equilibrium dynamics. Second, the investment optimality condition will define the arbitrage condition, where both accumulation factors, domestic capital and foreign net debt, are chosen indifferently and converge to a feasible equilibrium, according to the different assumptions for adjustment costs. We will name this concept indifference in accumulation. This set of assumptions will be sufficient to build a set of dynamical systems that will define the dynamics of each specific economy. As we are dealing with models of endogenous growth we will use the linearization method of the scaled systems in the neighbourhood of a feasible equilibrium to define the balanced growth path, relevant equilibrium ratios between variables and additional growth and mathematical conditions, such as optimal fiscal policy rules. The method for scaling the original systems will be achieved by taking trends and assuming a common growth rate for all variables. In chapter 6. we will introduce transition dynamics in the simplified model framework by using a similar transformation of the original one proposed by Jones and Manuelli (1990). As the resulting models are not models of endogenous growth but stationary ones, we will extend the analytic analysis of this system thoroughly for the centralized economy, based as always in the linearized equilibrium defined by the variational system.

Additionally, we will use Matlab based routines and software to evaluate some dynamical systems and extend some of the analytical results previously obtained.

3. Paper Structure and Presentation

In the subsequent sections we will follow the following organization scheme. First the model will be presented and then some analytical results for the centrally planned economy and the decentralized one will be derived. Differences arising from both optima will then be discussed. As we go on, differences from alternative formalizations and hypothesis, between and within models, will be stressed, and relevant references for the problems discussed will be presented. For most of the document we will keep the presentation of these three economies based on the main text with some footnote references that in most cases are not determinant to our results or conclusions. The only two exceptions to this rule refer to the first two sections of the appendix that we will use to broaden the description about two specific subjects. The first of these exceptions is related to the discussion of the no adjustment costs case of chapter *4.* and the options to deal with this specific optimal control problem. We extend this framework in section *1.* of the appendix because it will be important for the specific solutions that we are going to develop in all the subsequent models. The second of these exceptions deals with the optimal control problem for the economy with an informal sector. In section *2.* of the appendix we will discuss the options for specific functional forms of investment adjustment costs and central planner information about the relations between both sectors. We use this section as a support section for some of the options that we will undertake in the main text, when presenting the centrally planned economy of chapter *6.*. This set of options will also be important when deciding on the two information hypotheses that we will present for the decentralized economy. Therefore, it is advisable that the reader takes a brief look at this section, since it is used to define the set of assumptions that will be dealt in the main text. Last, but not less important, it is worth to mention that throughout the document we will excuse from using the time subscripts for motion variables with the purpose of simplifying the mathematical notation presented.

4. Endogenous Growth in a Small Open Economy with Elastic Labour Supply and no Government Sector

4.1. Analytical Framework for the Centralized and Decentralized Economies

In this section a small open economy without a Government sector is introduced. This economy will consist of N identical individuals. Aggregate conditions will imply that $X = Nx$ individuals can allocate their time between labour, $(1-l)$, and leisure, l, which will be given endogenously by the model. Output for the individual firm, y, is given by the following Cobb-Douglas production function:

$$y = AK^{\beta}(1-l)^{\phi}k^{\alpha},\ 0<\beta<1,\ 0<\phi<1,\ \phi<\beta,\ 0<\alpha<1 \qquad \mathbf{(A1)}$$

As usual in economic growth formalizations, k denotes the individual firm capital stock and K the aggregate capital stock, which produces an externality compatible with the existence of endogenous growth. This specification will replicate the behaviour which will arise when productive government spending is considered in place of aggregate capital externalities. As a result, this Cobb-Douglas production function is an ideal candidate to derive a benchmark model for comparison with the optimal fiscal policy models that will be presented in the following chapters.

Aggregate production is derived substituting the aggregate conditions $Y = Ny$ and $k = K / N$, in the individual firm production function:

$$Y = AK^{\beta+\alpha}(1-l)^{\phi} N^{1-\alpha} \qquad \textbf{(A2)}$$

In order to obtain the necessary condition for endogenous growth, this implies a homogeneous linear production function in the parameters (this result can be derived when solving for both equilibria also). So we can impose the following restriction on the parameters, $\alpha+\beta=1$. This result can be derived because we are not assuming population growth, and because it is assumed that decisions between leisure and labour are given endogenously by the model parameters. Assuming this restriction, aggregate production now comes:

$$Y = AK(1-l)^{\phi} N^{\beta} \qquad \textbf{(A3)}$$

The representative agent's welfare is given by the following intertemporal isoelastic utility function, as in Turnovsky (1999). This utility function will be used in all models presented here. Equations (A4) and (A5) define the individual's utility and the aggregate utility for this economy:

$$U = \int_0^{\infty} \frac{1}{\gamma}\left(c\, l^{\theta}\right)^{\gamma} e^{-\rho t} dt \qquad \textbf{(A4)}$$

$$U = \int_0^{\infty} \frac{1}{\gamma}\left([C / N]\, l^{\theta}\right)^{\gamma} e^{-\rho t} dt \qquad \textbf{(A5)}$$

$$\theta > 0 \;;\; -\infty < \gamma < 1 \;;\; 1 > \gamma(1+\theta) \;;\; 1 > \gamma\theta$$

As usual C stands for aggregate private consumption, c is the individual consumption, and parameters γ and θ, are related with the intertemporal elasticity of substitution and the impact of leisure on the utility of the representative agent, respectively. The constraints imposed on the parameters are necessary to ensure that the utility function is concave in c and l.

Firms face investment decisions, $i = I / N$, which involve adjustment costs, h. The function used is the classical quadratic, convex function, where adjustment costs can also be interpreted as installation costs. Equations (A6) and (A7) define individual and aggregate investment costs:

$$\Phi\left(i,k\right)=i+\frac{h\,i^{2}}{2k}=i\left(1+\frac{h\,i}{2k}\right) \quad \textbf{(A 6)}$$

$$\Phi\left(I,K\right)=I+\frac{h\,I^{2}}{2K}=I\left(1+\frac{h\,I}{2K}\right) \quad \textbf{(A 7)}$$

This type of cost functions implies that for each unit of investment there is an adjustment cost that is proportional to the investment and capital ratio. This framework is commonly used in models of small open economies in order to avoid degenerate dynamics, as referred in Turnovsky (1999), in the case of endogenous growth models, and in Barro and Sala-i-Martin (1999), for an open economy version of the Ramsey model[5].

In the models presented here it will be assumed that capital depreciates at the constant rate, δ. The aggregate capital accumulation constraint is defined as:

$$\dot{K}=I-\delta K \quad \textbf{(A 8)}$$

And the individual firm capital accumulation constraint is just:

$$\dot{k}=i-\delta k \quad \textbf{(A 9)}$$

In a small open economy framework it is usual to assume that individuals and firms have full access to international capital markets and can accumulate debt (or foreign bonds) at an exogenously given world interest rate. So the intertemporal budget constraint facing the representative agent in this economy will be equal to individual consumption, c, individual investment, i, and debt interest payments, rb, minus capital and labour incomes, $r_k k$ and $w(1-l)$, assuming that the representative agent is a net borrower[6]. The aggregate intertemporal budget constraint for this economy is given by the macroeconomic aggregate conditions, described in the previous equations which deliver the usual national accounts equality of (A11).

$$\dot{b}=c+\Phi\left(i,k\right)+rb-w\left(1-l\right)-r_k k \quad \textbf{(A 10)}$$

$$\dot{B}=C+\Phi\left(I,K\right)+rB-Y \quad \textbf{(A 11)}$$

Assuming that both capital and labour are paid their marginal productivity, the expressions for the wage rate and return on capital satisfy the marginal product conditions:

$$r_k=\frac{dy}{dk}=\left(1-\beta\right)AK^{\beta}\left(1-l\right)^{\phi}k^{-\beta}=\left(1-\beta\right)\frac{y}{k} \quad \textbf{(A 12)}$$

$$w=\frac{dy}{d\left(1-l\right)}=\phi AK^{\beta}\left(1-l\right)^{\phi-1}k^{1-\beta}=\phi\frac{y}{\left(1-l\right)} \quad \textbf{(A 13)}$$

Because there are positive externalities on capital, the aggregate return on capital is greater then the one obtained in (A13). The same does not happen with the wage rate. Therefore there exists

[5] These problems of degenerate dynamics that arise in models of small open economies when not considering adjustment costs in investment are tackled in the first section of the appendix and in sections *4.2.3.*, *4.3.3.*, *5.2.3.* and *5.3.3.*.

[6] This type of intertemporal budget constraints leaves the possibility of analyzing agents and economies that act as net lenders also.

incomplete information about the real return on capital which influences prices in the national capital market. This will affect financial decisions taken by agents in the decentralized economy.

4.2. Analytical Results for the Centralized Economy

4.2.1. Optimal Control Conditions for the Centralized Economy

The social planner's problem is to maximize $\boldsymbol{C}$, $\boldsymbol{l}$ and $\boldsymbol{I}$, subject to the intertemporal budget constraint, (A11), and the capital accumulation equation, (A8).

$$\underset{C,l,I}{MAX} \quad U = \int_0^\infty \frac{1}{\gamma}\left([C / N]\, l^\theta\right)^\gamma e^{-\rho t} dt \tag{A14}$$

subject to :

$$\begin{cases} \dot{B} = C + I\left(1 + \dfrac{hI}{2K}\right) + rB - AK\,(1-l)^\phi\, N^\beta & \text{(A15)} \\ \dot{K} = I - \delta K & \text{(A16)} \end{cases}$$

The present value Hamiltonian for this optimization problem is:

$$H^* = \frac{1}{\gamma}\left([C / N]\, l^\theta\right)^\gamma + \lambda\left[C + I\left(1 + \frac{hI}{2K}\right) + rB - AK\,(1-l)^\phi N^\beta\right] + q\,[I - \delta K]$$

The Pontryagin maximum conditions for this optimal control problem are:

Optimality Conditions

$$C^{\gamma-1} N^{-\gamma} l^{\theta\gamma} + \lambda = 0 \tag{A17}$$

$$\theta C^{\gamma} N^{-\gamma} l^{\theta\gamma-1} + \lambda \phi AK\,(1-l)^{\phi-1} N^\beta = 0 \tag{A18}$$

$$\lambda\left(1 + h\,\frac{I}{K}\right) + q = 0 \tag{A19}$$

Admissibility Conditions

$B_0 = B_{(0)}$ and $K_0 = K_{(0)}$

Multipliers Conditions

$$\dot{\lambda} = \lambda(\rho - r) \tag{A20}$$

$$\dot{q} = q\,(\rho + \delta) + \lambda\left[\frac{h}{2}\left(\frac{I}{K}\right)^2 + A\,(1-l)^\phi\, N^\beta\right] \tag{A21}$$

State Conditions

$$\dot{B} = C + I\left(1 + \frac{hI}{2K}\right) + rB - AK\,(1-l)^\phi\, N^\beta \tag{A22}$$

$$\dot{K} = I - \delta K \tag{A23}$$

Transversality Conditions

$$\lim_{t\to\infty} \lambda \boldsymbol{B} \boldsymbol{e}^{-\rho t} = 0 \qquad \textbf{(A24)}$$

$$\lim_{t\to\infty} \boldsymbol{q} \boldsymbol{K} \boldsymbol{e}^{-\rho t} = 0 \qquad \textbf{(A25)}$$

4.2.2. Labour and Leisure Decisions and Indifference in Accumulation

Using the optimality condition in (A17) and substituting in (A18) we can derive the expressions that will determine the investment and leisure decisions in this economy[7]:

$$\frac{\boldsymbol{C}}{\boldsymbol{Y}} = \frac{\phi}{\theta} \frac{\boldsymbol{l}}{(1-\boldsymbol{l})} \qquad \textbf{(A26)}$$

$$\lambda = -\boldsymbol{q}\boldsymbol{K}\left(\boldsymbol{K} + \boldsymbol{h}\boldsymbol{I}\right)^{-1} \qquad \textbf{(A27)}$$

From the same optimality condition and using the co-state condition (A20) it is straightforward to obtain the differential equation for aggregate consumption:

$$\dot{\boldsymbol{C}} = \frac{(\rho - \boldsymbol{r})}{(\gamma - 1)} \boldsymbol{C} \qquad \textbf{(A28)}$$

Since equation (A28) was obtained using the co-state equation (A20), one possible strategy to solve the accumulation problem in a small open economy is to follow the strategy presented in section 1. of the appendix and obtain a condition that guarantees that there is indifference in accumulation. Substituting expression (A27) in the consumption optimum, (A17), solving for $\boldsymbol{q}$ and taking the time derivative we obtain the following two expressions that can be used to substitute in the co-state equation (A21):

$$\boldsymbol{q} = \left(1 + \boldsymbol{h}\boldsymbol{I}\boldsymbol{K}^{-1}\right)\boldsymbol{C}^{\gamma-1}\boldsymbol{N}^{-\gamma}\boldsymbol{l}^{\theta\gamma} \qquad \textbf{(A29)}$$

$$\dot{\boldsymbol{q}} = (\gamma - 1)\left(1 + \boldsymbol{h}\boldsymbol{I}\boldsymbol{K}^{-1}\right)\Theta_1 \boldsymbol{C}^{-1}\dot{\boldsymbol{C}} + \boldsymbol{h}\boldsymbol{K}^{-1}\Theta_1 \dot{\boldsymbol{I}} - \boldsymbol{h}\boldsymbol{I}\boldsymbol{K}^{-2}\Theta_1 \dot{\boldsymbol{K}} \qquad \textbf{(A30)}$$

Where $\Theta_1 = \boldsymbol{C}^{\gamma-1}\boldsymbol{N}^{-\gamma}\boldsymbol{l}^{\theta\gamma}$. Substituting equations (A30), (A29), (A27) and the capital accumulation equation, (A23), in the co-state equation (A21) and dividing everything by Θ_1 we obtain the following differential equation for consumption:

$$\dot{\boldsymbol{C}} = \frac{\boldsymbol{C}}{\gamma - 1}\Big[\left(1 + \boldsymbol{h}\boldsymbol{I}\boldsymbol{K}^{-1}\right)^{-1}\Big[(\rho + \delta)\left(1 + \boldsymbol{h}\boldsymbol{I}\boldsymbol{K}^{-1}\right) -$$
$$- \boldsymbol{h}\boldsymbol{K}^{-1}\dot{\boldsymbol{I}} + \boldsymbol{h}\boldsymbol{I}\boldsymbol{K}^{-2}\left(\boldsymbol{I} - \delta\boldsymbol{K}\right) - \boldsymbol{h}\boldsymbol{I}^2\boldsymbol{K}^{-2} - \boldsymbol{Y}\boldsymbol{K}^{-1}\Big]\Big] \qquad \textbf{(A31)}$$

We can now apply the equality of numerators condition, used in section 1. of the appendix, in order to obtain a differential equation for investment, which guarantees that there will be indifference in accumulation of factors used in consumption. In a small open economy, where there are two possible equations that satisfy the Keynes-Ramsey rule for consumption, this dynamic equivalence guarantees

[7] The polynomial intertemporal equation for the labour and leisure decisions is obtained in section 1. of the appendix.

that agents will be indifferent between both the domestic and foreign accumulation of assets, when choosing their optimal path for consumption.

$$\dot{I} = \frac{I^2}{2K} + rI + \left(\delta + r - \frac{Y}{K}\right)\frac{K}{h} \qquad \textbf{(A32)}$$

The time path of this economy is therefore well described by the four dimensional dynamical system composed by the consumption differential equation, (A28), the two state differential equations, (A22) and (A23), and the quadratic non-linear differential equation for investment, (A32).

Before ongoing with the characterization and simulation of the four dimensional dynamic system described above, we can use the results obtained in section *1.* of the appendix to solve a simple endogenous growth problem. The results obtained from solving the no adjustment costs model will provide some valuable notions for simulating the models with adjustment costs. In order to solve for the simple case we just need to use the usual strategies that are used to solve endogenous growth models. Assuming that consumption, capital and debt all growth at the same rate and that the steady state condition for indifference in accumulation, (D15) holds, we can describe this economy by using the method of taking trends to variables and solving for a common growth rate, Ψ.

4.2.3. Solution of the Centralized Economy with no Adjustment Costs

As we have no clear intuition about investment decisions in the case without adjustment costs, except the optimality condition for investment, which is obtained in condition (D11) from section *1.* of the appendix, stating that in equilibrium the marginal value of foreign assets must be equal to the shadow price of domestic capital, from where the indifference in accumulation steady state condition (D15) is obtained. One possible strategy to tackle this problem is to use the capital accumulation equation, (A23), as the only source of information about investment decisions and substitute it in the economy aggregate budget constraint. Indifference in accumulation is guaranteed by substituting the steady state condition, (D15), in the differential equations for consumption and the foreign debt/wealth. By taking these assumptions, we can reduce the dynamics of this economy in a two dimension dynamical system, where all information about investment decisions and equilibrium in accumulation are included.

$$\begin{cases} \dot{C} = \dfrac{\rho + \delta - A\left(1 - l\right)^{\phi} N^{\beta}}{\gamma - 1} C & \textbf{(A33)} \\ \dot{B} - \dot{K} = C + \left[A\left(1 - l\right)^{\phi} N^{\beta} - \delta\right]\left(B - K\right) & \textbf{(A34)} \end{cases}$$

We can simplify equation (A34) even further, by assuming that total aggregate net wealth in this economy is given by $W = K - B$. Where W represents total assets in the case individuals hold foreign assets (bonds), or net balances if individuals hold foreign based liabilities. Equation (A34) simplifies to an expression that depends only in C and W:

$$\dot{W} = \left[A\left(1-l\right)^{\phi} N^{\beta} - \delta\right]W - C \tag{A35}$$

Taking a brief look at the dynamical system given by (A33) and (A35) it is obvious that equilibrium[8] in this system will only exist when assuming corner solutions. This system can only be solved by taking trends to variables and assuming a constant growth rate for all time varying variables. This will allow us to solve the system as the usual degenerate case that arises in endogenous growth models.

Taking trends to variables using the usual formulation:

$$\begin{cases} X_t = \tilde{x}_t e^{\Psi_x t} & \text{(A36)} \\ \dot{X}_t = \dot{\tilde{x}}_t e^{\Psi_x t} + \Psi_x \tilde{x}_t e^{\Psi_x t} & \text{(A37)} \end{cases}$$

The dynamical system with detrend variables, assuming that all growth rates equalize will become:

$$\begin{cases} \dot{\tilde{c}} = \left[\dfrac{\rho + \delta - A\left(1-l\right)^{\phi} N^{\beta}}{\gamma - 1} - \Psi\right]\tilde{c} & \text{(A38)} \\ \dot{\tilde{w}} = \left[A\left(1-l\right)^{\phi} N^{\beta} - \delta - \Psi\right]\tilde{w} - \tilde{c} & \text{(A39)} \end{cases}$$

Equilibrium in this system is obtained by solving for the common growth rate:

$$\begin{cases} \overline{\Psi} = \dfrac{\rho + \delta - A\left(1-l\right)^{\phi} N^{\beta}}{\gamma - 1} & \text{(A40)} \\ \overline{\tilde{c}} = \dfrac{\left(A\left(1-l\right)^{\phi} N^{\beta} - \delta\right)\gamma - \rho}{\gamma - 1}\overline{\tilde{w}} & \text{(A41)} \end{cases}$$

We can now linearize this dynamical system around the estimated equilibrium in the usual form:

$$\begin{bmatrix} \dot{\tilde{c}} \\ \dot{\tilde{w}} \end{bmatrix} = \begin{bmatrix} \left.\dfrac{d\,\dot{\tilde{c}}}{d\,\tilde{c}}\right|_{\overline{\Psi}} & \left.\dfrac{d\,\dot{\tilde{c}}}{d\,\tilde{w}}\right|_{\overline{\Psi}} \\ \left.\dfrac{d\,\dot{\tilde{w}}}{d\,\tilde{c}}\right|_{\overline{\Psi}} & \left.\dfrac{d\,\dot{\tilde{w}}}{d\,\tilde{w}}\right|_{\overline{\Psi}} \end{bmatrix} \begin{bmatrix} \tilde{c} - \overline{\tilde{c}} \\ \tilde{w} - \overline{\tilde{w}} \end{bmatrix}, \text{ where } J = \begin{bmatrix} \left.\dfrac{d\,\dot{\tilde{c}}}{d\,\tilde{c}}\right|_{\overline{\Psi}} & \left.\dfrac{d\,\dot{\tilde{c}}}{d\,\tilde{w}}\right|_{\overline{\Psi}} \\ \left.\dfrac{d\,\dot{\tilde{w}}}{d\,\tilde{c}}\right|_{\overline{\Psi}} & \left.\dfrac{d\,\dot{\tilde{w}}}{d\,\tilde{w}}\right|_{\overline{\Psi}} \end{bmatrix}$$

The dynamics of this system can now be characterized in the neighbourhood of the equilibrium as determined in expressions (A40) and (A41), by observing the properties of the Jacobian matrix, J. Solving for the partial derivatives and substituting the common growth rate we can derive the trace, determinant and roots of the characteristic equation that will describe the specific dynamics for this economy.

[8] When referring to dynamic system equilibrium we will be always referring to the dynamic systems theory of equilibrium which implies that equilibrium in dynamical systems is obtained by considering, $\dot{X}_i = 0$

$$\boldsymbol{J} = \begin{bmatrix} 0 & 0 \\ -1 & \dfrac{\left(\boldsymbol{A}\left(1-\boldsymbol{l}\right)^{\phi} \boldsymbol{N}^{\beta} - \delta\right)\gamma - \rho}{\gamma - 1} \end{bmatrix}, \text{ where } \boldsymbol{tr}(\boldsymbol{J}) = \frac{\left(\boldsymbol{A}\left(1-\boldsymbol{l}\right)^{\phi} \boldsymbol{N}^{\beta} - \delta\right)\gamma - \rho}{\gamma - 1},$$

$\det(\boldsymbol{J}) = 0$ and the roots of the characteristic equation are $\lambda_1 = \boldsymbol{tr}(\boldsymbol{J})$ and $\lambda_2 = 0$

Imposing the restrictions for the common growth rate, $\bar{\Psi}$, consumption, $\bar{\tilde{\boldsymbol{c}}}$, and wealth, $\bar{\tilde{\boldsymbol{w}}}$, to be always positive we can derive the final conditions that will describe the dynamics of this economy:

$$\begin{cases} \bar{\Psi} \succ 0 \Rightarrow \rho + \delta - \boldsymbol{A}\left(1-\boldsymbol{l}\right)^{\phi} \boldsymbol{N}^{\beta} \prec 0 \\ \bar{\tilde{\boldsymbol{c}}}, \bar{\tilde{\boldsymbol{w}}} \succ 0 \Rightarrow \left(\boldsymbol{A}\left(1-\boldsymbol{l}\right)^{\phi} \boldsymbol{N}^{\beta} - \delta\right)\gamma - \rho \prec 0 \end{cases} \Leftrightarrow \rho + \delta \prec \boldsymbol{PmgK} \prec \frac{\rho}{\gamma} + \delta$$

As we have imposed a condition for wealth and consumption to be positive this will mean that *tr(J)* will always be positive also. This happens because both conditions are exactly the same. With *tr(J)* being positive this means that there are no transitional dynamics in this economy and the phase diagram is similar to the simpler case of endogenous growth in a closed economy.

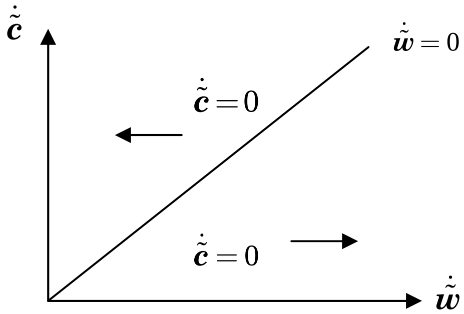

Fig.1 – Phase diagram for the centralized economy with no adjustment costs

4.3. Analytical Results for the Decentralized Economy

4.3.1. Optimal Control Conditions for the Decentralized Economy

The representative agent's problem is to maximize $\boldsymbol{c}$, $\boldsymbol{l}$ and $\boldsymbol{i}$ subject to the individual intertemporal budget constraint (A10) and the firm capital accumulation equation (A9).

$$\underset{\boldsymbol{c},\boldsymbol{l},\boldsymbol{i}}{\boldsymbol{MAX}} \ \boldsymbol{U} = \int_0^{\infty} \frac{1}{\gamma}\left(\boldsymbol{c}\boldsymbol{l}^{\theta}\right)^{\gamma} \boldsymbol{e}^{-\rho t} \boldsymbol{dt} \tag{A42}$$

subject to :

$$\begin{cases} \dot{\boldsymbol{b}} = \boldsymbol{c} + \boldsymbol{i}\left(1 + \dfrac{\boldsymbol{hi}}{2\boldsymbol{k}}\right) + \boldsymbol{rb} - \boldsymbol{w}\left(1-\boldsymbol{l}\right) - \boldsymbol{r}_k \boldsymbol{k} & \textbf{(A43)} \\ \dot{\boldsymbol{k}} = \boldsymbol{i} - \delta \boldsymbol{k} & \textbf{(A44)} \end{cases}$$

The present value Hamiltonian for this optimization problem is:

$$\boldsymbol{H}^* = \frac{1}{\gamma}\left(\boldsymbol{c}\boldsymbol{l}^{\theta}\right)^{\gamma} + \lambda\left[\boldsymbol{c} + \boldsymbol{i}\left(1 + \frac{\boldsymbol{hi}}{2\boldsymbol{k}}\right) + \boldsymbol{rb} - \boldsymbol{w}\left(1-\boldsymbol{l}\right) - \boldsymbol{r}_k \boldsymbol{k}\right] + \boldsymbol{q}\left[\boldsymbol{i} - \delta\boldsymbol{k}\right]$$

The Pontryagin maximum conditions for this optimal control problem are:

Optimality Conditions

$$\boldsymbol{c}^{\gamma-1}\boldsymbol{l}^{\theta\gamma}+\lambda=0 \tag{A45}$$

$$\theta\boldsymbol{c}^{\gamma}\boldsymbol{l}^{\theta\gamma-1}+\lambda\boldsymbol{w}=0 \tag{A46}$$

$$\lambda\left(1+\boldsymbol{h}\,\frac{\boldsymbol{i}}{\boldsymbol{k}}\right)+\boldsymbol{q}=0 \tag{A47}$$

Admissibility Conditions

$\boldsymbol{b}_0=\boldsymbol{b}_{(0)}$ and $\boldsymbol{k}_0=\boldsymbol{k}_{(0)}$

Multipliers Conditions

$$\dot{\lambda}=\lambda(\rho-\boldsymbol{r}) \tag{A48}$$

$$\dot{\boldsymbol{q}}=\boldsymbol{q}\left(\rho+\delta\right)+\lambda\left[\frac{\boldsymbol{h}}{2}\left(\frac{\boldsymbol{i}}{\boldsymbol{k}}\right)^2+\boldsymbol{r}_k\right] \tag{A49}$$

State Conditions

$$\dot{\boldsymbol{b}}=\boldsymbol{c}+\boldsymbol{i}\left(1+\frac{\boldsymbol{h}\,\boldsymbol{i}}{2\boldsymbol{k}}\right)+\boldsymbol{r}\boldsymbol{b}-\boldsymbol{w}\left(1-\boldsymbol{l}\right)-\boldsymbol{r}_k\boldsymbol{k} \tag{A50}$$

$$\dot{\boldsymbol{k}}=\boldsymbol{i}-\delta\boldsymbol{k} \tag{A51}$$

Transversality Conditions

$$\lim_{\boldsymbol{t}\to\infty}\lambda\boldsymbol{b}\boldsymbol{e}^{-\rho\boldsymbol{t}}=0 \tag{A52}$$

$$\lim_{\boldsymbol{t}\to\infty}\boldsymbol{q}\boldsymbol{k}\boldsymbol{e}^{-\rho\boldsymbol{t}}=0 \tag{A53}$$

Applying market clearing conditions from equations (A12) and (A13) and then aggregate conditions to the maximum conditions above, the dynamic general equilibrium equations are obtained for this economy:

Optimality Conditions

$$\boldsymbol{C}^{\gamma-1}\boldsymbol{l}^{\theta\gamma}\boldsymbol{N}^{1-\gamma}+\lambda=0 \tag{A54}$$

$$\theta\boldsymbol{C}^{\gamma}\boldsymbol{l}^{\theta\gamma-1}\boldsymbol{N}^{-\gamma}+\lambda\phi\,\frac{\boldsymbol{Y}}{\left(1-\boldsymbol{l}\right)\boldsymbol{N}}=0 \tag{A55}$$

$$\lambda\left(1-\boldsymbol{h}\,\frac{\boldsymbol{I}}{\boldsymbol{K}}\right)+\boldsymbol{q}=0 \tag{A56}$$

Multipliers Conditions

$$\dot{\lambda}=\lambda(\rho-\boldsymbol{r}) \tag{A57}$$

$$\dot{\boldsymbol{q}}=\boldsymbol{q}\left(\rho+\delta\right)+\lambda\left[\frac{\boldsymbol{h}}{2}\left(\frac{\boldsymbol{I}}{\boldsymbol{K}}\right)^2+\left(1-\beta\right)\frac{\boldsymbol{Y}}{\boldsymbol{K}}\right] \tag{A58}$$

State Conditions

$$\dot{\boldsymbol{B}} = \boldsymbol{C} + \boldsymbol{I}\left(1 + \frac{\boldsymbol{h}\boldsymbol{I}}{2\boldsymbol{K}}\right) + \boldsymbol{r}\boldsymbol{B} - \phi\boldsymbol{Y} - (1-\beta)\boldsymbol{Y} \tag{A59}$$

$$\dot{\boldsymbol{K}} = \boldsymbol{I} - \delta\boldsymbol{K} \tag{A60}$$

4.3.2. Labour and Leisure Decisions and Indifference in Accumulation

Following the same strategy employed in the centralized economy of section *4.2.2.*, the same motion equation for C, (A28), and the same labour/leisure condition, (A26), are obtained. For the case where there are no adjustment costs, the steady state condition for accumulation indifference now comes:

$$\boldsymbol{r} = (1-\beta)\boldsymbol{A}\,(1-\boldsymbol{l})^{\phi}\,\boldsymbol{N}^{\beta} - \delta \tag{A61}$$

The investment differential equation for the case with adjustment costs comes:

$$\dot{\boldsymbol{I}} = \frac{\boldsymbol{I}^2}{2\boldsymbol{K}} + \boldsymbol{r}\boldsymbol{I} + \left(\delta + \boldsymbol{r} - (1-\beta)\frac{\boldsymbol{Y}}{\boldsymbol{K}}\right)\frac{\boldsymbol{K}}{\boldsymbol{h}} \tag{A62}$$

4.3.3. Solution of the Decentralized Economy with no Adjustment Costs

The decentralized dynamical system for this economy is obtained by following the same strategy as in section *4.2.3.* and section *1.* of the appendix. First we have to set adjustment costs, $\boldsymbol{h}$, equal to zero in the intertemporal budget constraint, (A59), and substitute investment by the usual capital accumulation equation. Finally, the two dimensional dynamical system is obtained by substituting the exogenous international interest rate, in both differential equations, by the indifference accumulation equation given by (A61). The dynamical system comes as follows:

$$\begin{cases} \dot{\boldsymbol{C}} = \dfrac{\rho + \delta - (1-\beta)\boldsymbol{A}\,(1-\boldsymbol{l})^{\phi}\,\boldsymbol{N}^{\beta}}{\gamma - 1}\boldsymbol{C} & \text{(A63)} \\ \dot{\boldsymbol{B}} - \dot{\boldsymbol{K}} = \boldsymbol{C} + \left[(1-\beta)\boldsymbol{A}\,(1-\boldsymbol{l})^{\phi}\,\boldsymbol{N}^{\beta} - \delta\right](\boldsymbol{B} - \boldsymbol{K}) - \phi\boldsymbol{A}\,(1-\boldsymbol{l})^{\phi}\,\boldsymbol{N}^{\beta}\boldsymbol{K} & \text{(A64)} \end{cases}$$

Contrary to the results obtained in section *4.2.3.*, the system that is obtained for the decentralized economy does not simplify automatically, which does not allow us to apply a direct solution strategy for wealth and consumption, as in section *4.2.3.*. This is a consequence of not considering aggregate labour income when the steady state condition for indifference in accumulation is obtained, although it is always considered in the intertemporal aggregate budget constraint. This feature reflects an economy where agents determine their foreign financial position and accumulation decisions by taking in consideration only the marginal productivity of aggregate capital and not the marginal productivity of labour. Although in the aggregate, both productive factor incomes will affect their intertemporal budget constraint and therefore affect their total wealth in all periods. We can tackle this issue by applying two strategies. The first strategy consists in considering that the steady state condition for accumulation, (A61), is incomplete and that individuals would eventually take into account this asymmetry and assume an indifference accumulation condition, which would consider the aggregate marginal productivity of labour. This result would be consistent if we consider it as a result of rational expectations but not consistent with the maximum conditions derived in section *4.3.1.*. The other hypothesis is to consider that the first strategy does not apply and that the mechanics for solving the system in W and C must arise from another mechanism, which is already reflected in the dynamical system described in equations (A63) and (A64). In this section we will only assume the second hypothesis as valid, because it is consistent with the aggregate maximum conditions for this dynamic general equilibrium problem.

In order to follow this intuition we shall rearrange equation (A64) in a way that is consistent to solving this model in W and C, as it was done in section 4.2.3.. The intertemporal aggregate budget constraint can now be expressed as follows:

$$\dot{K} - \dot{B} = \left[(1-\beta)A(1-l)^{\phi} N^{\beta} - \delta + \phi A (1-l)^{\phi} N^{\beta} \frac{K}{K-B}\right](K-B) - C \qquad \mathbf{(A65)}$$

Substituting by the usual wealth equality, $W = K - B$, in the equation we obtain:

$$\dot{W} = \left[(1-\beta)A(1-l)^{\phi} N^{\beta} - \delta + \phi A (1-l)^{\phi} N^{\beta} \frac{K}{W}\right]W - C \qquad \mathbf{(A66)}$$

From equation (A66), it becomes clear that the additional term of the intertemporal aggregate budget constraint, which relates aggregate labour income to time decisions on accumulation of wealth, depends also on the ratio between total domestic assets and total net wealth, $\frac{K}{W}$. This term implies that this economy will accumulate wealth taking in consideration, not only aggregate factor productivities of capital and labour, but also the relative position of domestic assets to total net assets.

When considering the case for endogenous growth, as we did in section 4.2.3., the adjustment term described above will be constant for all time periods, because we are considering that all growth rates of time varying variables equalize. Therefore, the ratio $\frac{K}{W}$ can be reduced to a constant parameter that describes the relative internal asset position relative to total net wealth. We will call this parameter ω_{dom}. The detrend dynamical system can now be obtained by following the formulation expressed in equations (A36) and (A37) and substituting the $\frac{K}{W}$ ratio by parameter ω_{dom}.

$$\begin{cases} \dot{\tilde{c}} = \left[\dfrac{\rho + \delta - (1-\beta) \boldsymbol{A} (1-\boldsymbol{l})^{\phi} \boldsymbol{N}^{\beta}}{\gamma - 1} - \Psi\right] \tilde{c} & \textbf{(A67)} \\ \dot{\tilde{w}} = \left[\left((1-\beta) + \phi \omega_{dom}\right) \boldsymbol{A} (1-\boldsymbol{l})^{\phi} \boldsymbol{N}^{\beta} - \delta - \Psi\right] \tilde{w} - \tilde{c} & \textbf{(A68)} \end{cases}$$

Solving for the common growth rate, $\bar{\Psi}$, equilibrium in this dynamical system comes as usual:

$$\begin{cases} \bar{\Psi} = \dfrac{\rho + \delta - (1-\beta) \boldsymbol{A} (1-\boldsymbol{l})^{\phi} \boldsymbol{N}^{\beta}}{\gamma - 1} & \textbf{(A69)} \\ \bar{\tilde{c}} = \dfrac{\left[(1-\beta)\gamma + \phi \omega_{dom} (\gamma - 1)\right] \boldsymbol{A} (1-\boldsymbol{l})^{\phi} \boldsymbol{N}^{\beta} - \delta\gamma - \rho}{\gamma - 1} \bar{\tilde{w}} & \textbf{(A70)} \end{cases}$$

We can now linearize this system around equilibrium, as expressed by equations (A69) and (A70), and describe the dynamic features of this economy using the Jacobian matrix, as we did in section *4.2.3.*.

$$\boldsymbol{J} = \begin{bmatrix} 0 & 0 \\ -1 & \dfrac{\left[(1-\beta)\gamma + \phi \omega_{dom} (\gamma - 1)\right] \boldsymbol{A} (1-\boldsymbol{l})^{\phi} \boldsymbol{N}^{\beta} - \delta\gamma - \rho}{\gamma - 1} \end{bmatrix}$$, where $\det(\boldsymbol{J}) = 0$,

$\boldsymbol{tr}(\boldsymbol{J}) = \dfrac{\left[(1-\beta)\gamma + \phi \omega_{dom} (\gamma - 1)\right] \boldsymbol{A} (1-\boldsymbol{l})^{\phi} \boldsymbol{N}^{\beta} - \delta\gamma - \rho}{\gamma - 1}$ and the roots of the characteristic equation are $\lambda_1 = \boldsymbol{tr}(\boldsymbol{J})$ and $\lambda_2 = 0$

Once more the determinant is equal to zero, implying that the dynamics of this system are of the type of degenerate dynamics that arise in models of endogenous growth. Imposing the final restrictions on $\bar{\Psi}$, $\bar{\tilde{c}}$ and $\bar{\tilde{w}}$ to be always positive, the final conditions for the decentralized economy come:

$$\begin{cases} \bar{\Psi} \succ 0 \Rightarrow \rho + \delta - (1-\beta) \boldsymbol{A} (1-\boldsymbol{l})^{\phi} \boldsymbol{N}^{\beta} \prec 0 \\ \bar{\tilde{c}}, \bar{\tilde{w}} \succ 0 \Rightarrow \left[(1-\beta)\gamma + \phi \omega_{dom} (\gamma - 1)\right] \boldsymbol{A} (1-\boldsymbol{l})^{\phi} \boldsymbol{N}^{\beta} - \delta\gamma - \rho \prec 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \frac{\rho + \delta}{(1-\beta)} \prec \boldsymbol{PmgK} \prec \frac{\rho + \delta\gamma}{(1-\beta)\gamma + \phi \omega_{dom} (\gamma - 1)}$$

Conditions imposed imply that the $\boldsymbol{tr}(\boldsymbol{J})$ will be always positive, meaning that again there are no transitional dynamics. The phase diagram for this economy is similar to figure 1 in section *4.2.3.*, although equilibrium conditions differ in the parameters between the centralized and decentralized economies. It is clear that the dynamic general equilibrium arising from the representative agent's maximization problem is not a first best optimum, when comparing to endogenous growth conditions for the centralized equilibrium of section *4.2.3.*. Parameter restrictions and the absence of discretionary policy instruments assure that it is impossible to reproduce first best optimum conditions. Growth in the decentralized economy will be slower due to incomplete information in domestic capital markets. Additionally, endogenous growth conditions are affected by this asymmetry and by the adjustment of the relative domestic assets parameter expression, which arises when considering maximum conditions, where aggregate labour income is not taken into account in the condition for indifference in accumulation.

4.4. Concluding Remarks

This chapter served the purpose of presenting the basic economic, mathematical structures and concepts that we will use to describe the more complex economies of chapter 5. and 6.. For this purpose we choose to extend the well known model of aggregate capital externalities model by Romer (1986) to a small open economy context with an elastic labour supply. We used it not only to present concepts, like indifference in accumulation and adjustment costs, but also to present the basic methodology that we will use further ahead, such as linearization, when solving for all other simplified cases. We will leave the analysis of the more complex dynamics arising, when considering adjustment costs, to the numerical analysis presented in those chapters.

In the following chapters endogenous growth will arise, not by considering aggregate capital externalities, which are cost free for agents, but as a public good provided by government spending. This will have strong implications, as we will show that the results obtained for the centralized case, which we presented in this chapter, represent an unachievable optimum when positive externalities have a real cost to agents. On the other hand, the decentralized economy results from this chapter represent the first best optimum that is feasible in a framework with public goods. This makes this framework an interesting one to consider when making welfare comparisons with models that imply policy decisions.

5. Endogenous Growth and Optimal Fiscal Policy in a Small Open Economy with Elastic Labour Supply

5.1. Analytical Framework for the Centralized and Decentralized Economies

In this economy positive externalities are no longer the result of aggregate capital externalities but from public productive capital. Output for the individual firm is determined by a Samuelson type function with non-excludable and non-rival public goods, described in Barro and Sala-I-Martin (1992). The formulation followed here is a variation to assume an elastic labour supply as presented in Turnovsky (1999).

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{G}^{\beta}\left(1-\boldsymbol{l}\right)^{\phi}\boldsymbol{k}^{1-\beta} \qquad \mathbf{(B1)}$$

$$\boldsymbol{y} = \boldsymbol{A}\left(\boldsymbol{G}/\boldsymbol{k}\right)^{\beta}\left(1-\boldsymbol{l}\right)^{\phi}\boldsymbol{k} \qquad \mathbf{(B2)}$$

In order to obtain an *AK* technology in the aggregate framework it is convenient to tie government expenditure, $\boldsymbol{G}$, to aggregate output, $\boldsymbol{Y}$, where $\boldsymbol{g}$ acts as an exogenously given fraction of government expenditure relative to aggregate output. Applying aggregate conditions to the individual's firm capital and output, we can derive the aggregate output for this economy. Parameter restrictions will follow those in equation (A1), plus a restriction to guarantee that labour productivity is diminishing in the aggregate $\phi < 1 - \beta$.

$$G = gY \qquad\qquad 0 < g < 1 \tag{B3}$$

$$Y = \left(A\left[gN\right]^{\beta}\right)^{\frac{1}{1-\beta}} (1-l)^{\frac{\phi}{1-\beta}} K \tag{B4}$$

The method of derivation of rules of policy in optimal fiscal models of economic growth is basically the same that was used to determine the differences between a centrally planned economy and a dynamic general equilibrium outcome, now first best optimal fiscal policies will replicate equilibrium for the centrally planned economy. In this framework, the main difference is on the form that the aggregate intertemporal budget constraint will take for the central planner, in this case, it will have total government expenditure given in (B3) as part of the expenditure side. Where, in the other hand, individual budget constraints will face taxes on wage income, τ_w, capital income, τ_k, consumption, τ_c, and finally a lump sum tax given by $\tau = \dfrac{T}{N}$. Taxes on foreign bonds, τ_b, are dismissed for now, because we continue to consider that this economy and its representative agents are net borrowers.

$$\dot{b} = (1+\tau_c)c + \Phi(i,k) + rb + \tau - (1-\tau_w)w(1-l) - (1-\tau_k)r_k k \tag{B5}$$

$$\dot{B} = C + \Phi(I,K) + rB - (1-g)Y \tag{B6}$$

The aggregate and individual investment decisions are again given by the adjustment cost function, Φ, as in equations (A6) and (A7). The parameters, r_k and w, represent again the incomes from capital and labour.

From equation (B5) it's easy to derive the government balanced budget constraint for this economy:

$$\tau_c C + \tau_w w(1-l)N + \tau_k r_k K + T = gY \tag{B7}$$

Assuming again that capital and labour incomes satisfy the marginal productivity conditions, we can take the partial derivatives for the production factors, deriving from the individual firm production function, to obtain the expressions for r_k and w. These results are expressed in equations (B8) and (B9):

$$r_k = \frac{dy}{dk} = (1-\beta) A G^{\beta} (1-l)^{\phi} k^{-\beta} = (1-\beta)\frac{y}{k} \tag{B8}$$

$$w = \frac{dy}{d(1-l)} = \phi A G^{\beta} (1-l)^{\phi-1} k^{1-\beta} = \phi \frac{y}{(1-l)} \tag{B9}$$

It is arguable that aggregate marginal productivity will reflect the result obtained for individual marginal productivity represented by equations (B8) and (B9). Turnovsky (1999) derives aggregate marginal productivity for both factors by aggregating (B1) and then taking partial derivatives on both capital and labour, reaching an equivalent aggregate condition for both returns on productive factors. He does not consider the relation for government expenditures as determined by (B3). If this relation is considered then aggregate marginal productivity should be given by taking partial derivatives for both productive factors from equation (B4). This result will imply the same conclusion that was taken in our previous model. Positive capital externalities will involve incomplete information affecting the price for the real return on capital, which will affect the domestic capital market and therefore influence financial decisions taken by agents.

In order to finish the analytical framework for the centralized and decentralized economies all that is needed is a utility function and a capital accumulation equation. Equations (A4), (A5), (A8) and (A9) will serve perfectly for this purpose and will preserve much of the analytical formulation from the previous model. This will allow us to develop some relevant comparisons between these different economies.

5.2. Analytical Results for the Centralized Economy

5.2.1. Optimal Control Conditions for the Centralized Economy

The social planner's problem is to maximize $\boldsymbol{C}$, $\boldsymbol{l}$ and $\boldsymbol{I}$, subject to the intertemporal budget constraint, (B6), and the capital accumulation equation, (A8).

$$\underset{C,l,I}{MAX}\quad U = \int_0^{\infty} \frac{1}{\gamma}\left([C/N]l^{\theta}\right)^{\gamma} e^{-\rho t} dt \qquad \textbf{(B10)}$$

$\boldsymbol{subject\ to}$:

$$\begin{cases} \dot{B} = C + I\left(1+\dfrac{hI}{2K}\right) + rB - (1-g)\left(A\left[gN\right]^{\beta}\right)^{\frac{1}{1-\beta}}(1-l)^{\frac{\phi}{1-\beta}}K & \textbf{(B11)} \\ \dot{K} = I - \delta K & \textbf{(B12)} \end{cases}$$

The present value Hamiltonian for this optimization problem is:

$$H^{*} = \frac{1}{\gamma}\left([C/N]l^{\theta}\right)^{\gamma} + \lambda\left[C + I\left(1+\frac{hI}{2K}\right) + rB - (1-g)\left(A[gN]^{\beta}\right)^{\frac{1}{1-\beta}}(1-l)^{\frac{\phi}{1-\beta}}K\right] + q[I - \delta K]$$

The Pontryagin maximum conditions for this optimal control problem are:

Optimality Conditions

$$C^{\gamma-1}N^{-\gamma}l^{\theta\gamma} + \lambda = 0 \qquad \textbf{(B13)}$$

$$\theta C^{\gamma}N^{-\gamma}l^{\theta\gamma-1} + \lambda\frac{\phi}{1-\beta}(1-g)\left(A\left[gN\right]^{\beta}\right)(1-l)^{\frac{\phi+\beta-1}{1-\beta}}K = 0 \qquad \textbf{(B14)}$$

$$\lambda\left(1+h\frac{I}{K}\right) + q = 0 \qquad \textbf{(B15)}$$

Admissibility Conditions

$$\boldsymbol{B}_0 = \boldsymbol{B}_{(0)} \text{ and } \boldsymbol{K}_0 = \boldsymbol{K}_{(0)}$$

Multipliers Conditions

$$\dot{\lambda} = \lambda(\rho - \boldsymbol{r}) \qquad \textbf{(B16)}$$

$$\dot{\boldsymbol{q}} = \boldsymbol{q}(\rho + \delta) + \lambda\left[\frac{\boldsymbol{h}}{2}\left(\frac{\boldsymbol{I}}{\boldsymbol{K}}\right)^2 + (1-\boldsymbol{g})\left(\boldsymbol{A}\left[\boldsymbol{g}\boldsymbol{N}\right]^{\beta}\right)^{\frac{1}{1-\beta}}(1-\boldsymbol{l})^{\frac{\phi}{1-\beta}}\right] \qquad \textbf{(B17)}$$

State Conditions

$$\dot{\boldsymbol{B}} = \boldsymbol{C} + \boldsymbol{I}\left(1 + \frac{\boldsymbol{h}\boldsymbol{I}}{2\boldsymbol{K}}\right) + \boldsymbol{r}\boldsymbol{B} - (1-\boldsymbol{g})\left(\boldsymbol{A}\left[\boldsymbol{g}\boldsymbol{N}\right]^{\beta}\right)^{\frac{1}{1-\beta}}(1-\boldsymbol{l})^{\frac{\phi}{1-\beta}}\boldsymbol{K} \qquad \textbf{(B18)}$$

$$\dot{\boldsymbol{K}} = \boldsymbol{I} - \delta\boldsymbol{K} \qquad \textbf{(B19)}$$

Transversality Conditions

$$\lim_{\boldsymbol{t}\to\infty} \lambda\boldsymbol{B}\boldsymbol{e}^{-\rho\boldsymbol{t}} = 0 \qquad \textbf{(B20)}$$

$$\lim_{\boldsymbol{t}\to\infty} \boldsymbol{q}\boldsymbol{K}\boldsymbol{e}^{-\rho\boldsymbol{t}} = 0 \qquad \textbf{(B21)}$$

5.2.2. Labour and Leisure Decisions and Indifference in Accumulation

Again following the same strategy employed in the centralized economy of section *4.2.2.*, the same motion equation for $\boldsymbol{C}$, (A28), is obtained. The labour/leisure condition now comes:

$$\frac{\boldsymbol{C}}{\boldsymbol{Y}} = \frac{\phi(1-\boldsymbol{g})}{\theta(1-\beta)}\frac{\boldsymbol{l}}{(1-\boldsymbol{l})} \qquad \textbf{(B22)}$$

The steady state condition for the case with no adjustment costs and the investment differential equation for the case with adjustment costs, that guarantee indifference in accumulation for the centralized economy now become:

$$\boldsymbol{r} = (1-\boldsymbol{g})\frac{\boldsymbol{Y}}{\boldsymbol{K}} - \delta \qquad \textbf{(B23)}$$

$$\dot{\boldsymbol{I}} = \frac{\boldsymbol{I}^2}{2\boldsymbol{K}} + \boldsymbol{r}\boldsymbol{I} + \left(\delta + \boldsymbol{r} - (1-\boldsymbol{g})\frac{\boldsymbol{Y}}{\boldsymbol{K}}\right)\frac{\boldsymbol{K}}{\boldsymbol{h}} \qquad \textbf{(B24)}$$

5.2.3. Solution of the Centralized Economy with no Adjustment Costs

As in section *4.3.3.*, we will follow closely the intuition already presented in section *4.2.3.* and in section *1.* of the appendix, when solving for the no adjustment costs and no government sector centralized economy. The strategy for solution of the no adjustment costs economy with a government sector is very similar to the results presented in section *4.2.3.*, therefore we will not present any excessive details in this section and refer the reader to these sections for further information.

Again we will need to impose $\boldsymbol{h} = 0$ to the maximum conditions presented for this model in order to obtain the maximum conditions for the case with no adjustment costs, all other conditions remain equal. We have already determined the steady state condition of indifference in accumulation for this economy, (B23), so we can use our standard assumptions to build the dynamical system in $\boldsymbol{W}$ and $\boldsymbol{C}$.

$$\begin{cases} \dot{\boldsymbol{C}} = \dfrac{\rho + \delta - (1 - \boldsymbol{g})\left(\boldsymbol{A}\left[\boldsymbol{g}\boldsymbol{N}\right]^{\beta}\right)^{\frac{1}{1-\beta}}(1 - \boldsymbol{l})^{\frac{\phi}{1-\beta}}}{\gamma - 1}\boldsymbol{C} & \textbf{(B25)} \\ \dot{\boldsymbol{W}} = \left[(1 - \boldsymbol{g})\left(\boldsymbol{A}\left[\boldsymbol{g}\boldsymbol{N}\right]^{\beta}\right)^{\frac{1}{1-\beta}}(1 - \boldsymbol{l})^{\frac{\phi}{1-\beta}} - \delta\right]\boldsymbol{W} - \boldsymbol{C} & \textbf{(B26)} \end{cases}$$

Once more, equilibrium in this dynamical system exists only for corner solutions. We have to take trends in order to solve this system for a common growth rate. Following the usual formulation expressed in section *4.2.3.*, from expressions (A36) and (A37), the scaled dynamical system comes:

$$\begin{cases} \dot{\tilde{\boldsymbol{c}}} = \left[\dfrac{\rho + \delta - (1 - \boldsymbol{g})\left(\boldsymbol{A}\left[\boldsymbol{g}\boldsymbol{N}\right]^{\beta}\right)^{\frac{1}{1-\beta}}(1 - \boldsymbol{l})^{\frac{\phi}{1-\beta}}}{\gamma - 1} - \varPsi\right]\tilde{\boldsymbol{c}} & \textbf{(B27)} \\ \dot{\tilde{\boldsymbol{w}}} = \left[(1 - \boldsymbol{g})\left(\boldsymbol{A}\left[\boldsymbol{g}\boldsymbol{N}\right]^{\beta}\right)^{\frac{1}{1-\beta}}(1 - \boldsymbol{l})^{\frac{\phi}{1-\beta}} - \delta - \varPsi\right]\tilde{\boldsymbol{w}} - \tilde{\boldsymbol{c}} & \textbf{(B28)} \end{cases}$$

Solving for the common growth rate, $\overline{\varPsi}$, equilibrium in this dynamical system comes as usual:

$$\begin{cases} \overline{\varPsi} = \dfrac{\rho + \delta - (1 - \boldsymbol{g})\left(\boldsymbol{A}\left[\boldsymbol{g}\boldsymbol{N}\right]^{\beta}\right)^{\frac{1}{1-\beta}}(1 - \boldsymbol{l})^{\frac{\phi}{1-\beta}}}{\gamma - 1} & \textbf{(B29)} \\ \overline{\tilde{\boldsymbol{c}}} = \dfrac{\left[(1 - \boldsymbol{g})\left(\boldsymbol{A}\left[\boldsymbol{g}\boldsymbol{N}\right]^{\beta}\right)^{\frac{1}{1-\beta}}(1 - \boldsymbol{l})^{\frac{\phi}{1-\beta}} - \delta\right]\gamma - \rho}{\gamma - 1}\overline{\tilde{\boldsymbol{w}}} & \textbf{(B30)} \end{cases}$$

Linearising this system around equilibrium, as expressed by equations (B29) and (B30), we can describe the dynamic features of this economy using the Jacobian matrix.

$$\boldsymbol{J} = \begin{bmatrix} 0 & 0 \\ -1 & \dfrac{\left[(1 - \boldsymbol{g})\left(\boldsymbol{A}\left[\boldsymbol{g}\boldsymbol{N}\right]^{\beta}\right)^{\frac{1}{1-\beta}}(1 - \boldsymbol{l})^{\frac{\phi}{1-\beta}} - \delta\right]\gamma - \rho}{\gamma - 1} \end{bmatrix}$$, where $\det(\boldsymbol{J}) = 0$,

$$\boldsymbol{tr}(\boldsymbol{J}) = \dfrac{\left[(1 - \boldsymbol{g})\left(\boldsymbol{A}\left[\boldsymbol{g}\boldsymbol{N}\right]^{\beta}\right)^{\frac{1}{1-\beta}}(1 - \boldsymbol{l})^{\frac{\phi}{1-\beta}} - \delta\right]\gamma - \rho}{\gamma - 1}$$ and the roots of the characteristic equation are $\lambda_1 = \boldsymbol{tr}(\boldsymbol{J})$ and $\lambda_2 = 0$

Again, as the determinant is equal to zero, the dynamics of this system are of the type of degenerate dynamics that arises in models of endogenous growth. We can now conclude the solution of this model by imposing the final restrictions on $\bar{\Psi}$, $\bar{\tilde{c}}$ and $\bar{\tilde{w}}$ to be always positive:

$$\begin{cases} \overline{\Psi} \succ 0 \Rightarrow \rho + \delta - (1 - \boldsymbol{g})\left(\boldsymbol{A}\,[\boldsymbol{g}\,\boldsymbol{N}\,]^{\beta}\right)^{\frac{1}{1-\beta}}(1 - \boldsymbol{l})^{\frac{\phi}{1-\beta}} \prec 0 \\ \overline{\tilde{c}}, \overline{\tilde{w}} \succ 0 \Rightarrow \left[(1 - \boldsymbol{g})\left(\boldsymbol{A}\,[\boldsymbol{g}\,\boldsymbol{N}\,]^{\beta}\right)^{\frac{1}{1-\beta}}(1 - \boldsymbol{l})^{\frac{\phi}{1-\beta}} - \delta\right]\gamma - \rho \prec 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \rho + \delta \prec (1 - \boldsymbol{g})\boldsymbol{P}\,\boldsymbol{m}\,\boldsymbol{g}\boldsymbol{K} \prec \frac{\rho}{\gamma} + \delta$$

As in the centralized economy without a government sector of section *4.2.3.*, the restrictions imposed imply that $tr(\boldsymbol{J})$ will be always positive, which will mean that there are no transitional dynamics in this economy. The phase diagram for this economy is similar to the one in figure1 of section *4.2.3.*, although the restrictions imposed and the equilibrium conditions change with the specific parameters. One interesting feature that arises in this economy, when comparing to restrictions imposed for the centrally planned economy without a government sector solved in section *4.2.3.*, is the relative size of government versus the private sector dimension, in our final admissibility condition for the marginal productivity of aggregate capital. This difference reflects the way that positive externalities on productive capital enter in each of these economies. In the first model positive externalities on capital arise as result of aggregate capital externalities, this can be compared to a free public good. In this model, positive externalities are a costly public good that has to be considered in the intertemporal budget constraint and, as a result, have an effect on endogenous growth conditions.

5.3. Analytical Results for the Decentralized Economy

5.3.1. Optimal Control Conditions for the Decentralized Economy

The representative agent's problem is to maximize $\boldsymbol{c}$, $\boldsymbol{l}$ and $\boldsymbol{i}$ subject to the individual intertemporal budget constraint (B10) and the firm capital accumulation equation (A9).

$$\underset{c,l,i}{MAX}\ \boldsymbol{U} = \int_0^{\infty} \frac{1}{\gamma}\left(\boldsymbol{c}\boldsymbol{l}^{\theta}\right)^{\gamma} \boldsymbol{e}^{-\rho t}\boldsymbol{dt} \qquad \textbf{(B31)}$$

subject to :

$$\begin{cases} \dot{\boldsymbol{b}} = (1+\tau_c)\boldsymbol{c} + \boldsymbol{i}\left(1+\dfrac{\boldsymbol{h}\boldsymbol{i}}{2\boldsymbol{k}}\right) + \boldsymbol{r}\boldsymbol{b} + \tau - (1-\tau_w)\boldsymbol{w}(1-\boldsymbol{l}) - (1-\tau_k)\boldsymbol{r}_k\boldsymbol{k} & \textbf{(B32)} \\ \dot{\boldsymbol{k}} = \boldsymbol{i} - \delta\boldsymbol{k} & \textbf{(B33)} \end{cases}$$

The present value Hamiltonian for this optimization problem is:

$$H^{*} = \frac{1}{\gamma}\left(c\,l^{\theta}\right)^{\gamma} + q\left[i - \delta k\right] +$$

$$\lambda\left[(1+\tau_c)c + i\left(1+\frac{h\,i}{2\,k}\right) + r\,b + \tau - (1-\tau_w)w\,(1-l) - (1-\tau_k)r_k k\right]$$

The Pontryagin maximum conditions for this optimal control problem are:

Optimality Conditions

$$c^{\gamma-1}l^{\theta\gamma} + \lambda\left(1+\tau_c\right) = 0 \qquad \textbf{(B34)}$$

$$\theta c^{\gamma}l^{\theta\gamma-1} + \lambda\left(1-\tau_w\right)w = 0 \qquad \textbf{(B35)}$$

$$\lambda\left(1+h\frac{i}{k}\right) + q = 0 \qquad \textbf{(B36)}$$

Admissibility Conditions

$b_0 = b_{(0)}$ and $k_0 = k_{(0)}$

Multipliers Conditions

$$\dot{\lambda} = \lambda(\rho - r) \qquad \textbf{(B37)}$$

$$\dot{q} = q\left(\rho+\delta\right) + \lambda\left[\frac{h}{2}\left(\frac{i}{k}\right)^{2} + \left(1-\tau_k\right)r_k\right] \qquad \textbf{(B38)}$$

State Conditions

$$\dot{b} = (1+\tau_c)c + i\left(1+\frac{h\,i}{2k}\right) + r\,b + \tau - (1-\tau_w)w\,(1-l) - (1-\tau_k)r_k k \qquad \textbf{(B39)}$$

$$\dot{k} = i - \delta k \qquad \textbf{(B40)}$$

Transversality Conditions

$$\lim_{t\to\infty} \lambda b e^{-\rho t} = 0 \qquad \textbf{(B41)}$$

$$\lim_{t\to\infty} q k e^{-\rho t} = 0 \qquad \textbf{(B42)}$$

Applying market clearing conditions from equations (B8) and (B9) and then aggregate conditions to the maximum conditions above, the dynamic general equilibrium equations are obtained for this economy:

Optimality Conditions

$$C^{\gamma-1}N^{1-\gamma}l^{\theta\gamma} + \lambda\left(1+\tau_c\right) = 0 \qquad \textbf{(B43)}$$

$$\theta C^{\gamma}N^{-\gamma}l^{\theta\gamma-1} + \lambda\phi\left(1-\tau_w\right)\frac{Y}{(1-l)N} = 0 \qquad \textbf{(B44)}$$

$$\lambda\left(1+h\frac{I}{K}\right) + q = 0 \qquad \textbf{(B45)}$$

Multipliers Conditions

$$\dot{\lambda} = \lambda(\rho - \boldsymbol{r}) \tag{B46}$$

$$\dot{\boldsymbol{q}} = \boldsymbol{q}(\rho + \delta) + \lambda\left[\frac{\boldsymbol{h}}{2}\left(\frac{\boldsymbol{I}}{\boldsymbol{K}}\right)^2 + (1 - \tau_k)(1 - \beta)\frac{\boldsymbol{Y}}{\boldsymbol{K}}\right] \tag{B47}$$

State Conditions

$$\dot{\boldsymbol{B}} = (1 + \tau_c)\boldsymbol{C} + \boldsymbol{I}\left(1 + \frac{\boldsymbol{hI}}{2\boldsymbol{K}}\right) + \boldsymbol{rB} + \boldsymbol{T} - \phi(1 - \tau_w)\boldsymbol{Y} - (1 - \beta)(1 - \tau_k)\boldsymbol{Y} \tag{B48}$$

$$\dot{\boldsymbol{K}} = \boldsymbol{I} - \delta\boldsymbol{K} \tag{B49}$$

5.3.2. Labour and Leisure Decisions and Indifference in Accumulation

Using the same strategy employed in the centralized economy of section *4.2.2.*, the same motion equation for $\boldsymbol{C}$, (A28), is obtained. The labour/leisure condition now comes:

$$\frac{\boldsymbol{C}}{\boldsymbol{Y}} = \frac{\phi(1 - \tau_w)}{\theta(1 + \tau_c)}\frac{\boldsymbol{l}}{(1 - \boldsymbol{l})} \tag{B50}$$

The steady state condition for the case with no adjustment costs and the investment differential equation for the case with adjustment costs now come:

$$\boldsymbol{r} = (1 - \tau_k)(1 - \beta)\frac{\boldsymbol{Y}}{\boldsymbol{K}} - \delta \tag{B51}$$

$$\dot{\boldsymbol{I}} = \frac{\boldsymbol{I}^2}{2\boldsymbol{K}} + \boldsymbol{rI} + \left(\delta + \boldsymbol{r} - (1 - \tau_k)(1 - \beta)\frac{\boldsymbol{Y}}{\boldsymbol{K}}\right)\frac{\boldsymbol{K}}{\boldsymbol{h}} \tag{B52}$$

5.3.3. Solution of the Decentralized Economy with no Adjustment Costs

In this section, a solution for the no adjustment costs dynamic general equilibrium model is presented. These results will enable us to address the implications of fiscal policy in our endogenous growth framework, when comparing these results, to the results obtained for the centralized equilibrium of section *5.2.3.*. Following the strategies described in section *4.2.3.* and section *1.* of the appendix, we can set adjustment costs, $\boldsymbol{h}$, equal to zero and use the aggregate maximum conditions already obtained, in order to solve this simplified version in the usual form.

In the formalization of the two dimensional dynamical system for this economy, the same problems discussed in the decentralized economy without a government sector, of section *4.3.3.*, again arise. Not considering aggregate net labour income in the steady state condition (B51), will not enable us to simplify our system in the usual fashion, as we did for the centralized economies of sections *4.2.3.* and *5.2.3.*. In this section, we will only present the usual results for the formalization of our endogenous growth problem, the complete formalization and detailed description of the strategy employed to tackle this issue can be found in section *4.3.3.*. It should be stressed that in both models the basic assumptions for simplifying the wealth differential equation are exactly the same.

Following the usual methodology the two dimensional dynamical system for this economy comes as usual[9]:

$$\begin{cases} \dot{C} = \left[\dfrac{\rho + \delta - (1-\tau_k)(1-\beta)\dfrac{Y}{K}}{\gamma - 1} \right] C & \textbf{(B53)} \\ \dot{K} - \dot{B} = \left[(1-\beta)(1-\tau_k)\dfrac{Y}{K} - \delta + \phi(1-\tau_w)\dfrac{Y}{K}\dfrac{K}{K-B} \right](K-B) - (1+\tau_c)C - T & \textbf{(B54)} \end{cases}$$

Again assuming the case for endogenous growth, where all variables growth at the same rate, we can take the trends from this dynamical system and assume that the ratio $\dfrac{K}{W}$ simplifies to the constant parameter ω_{dom}, which determines the relative size of domestic assets to total net wealth.

$$\begin{cases} \dot{\tilde{c}} = \left[\dfrac{\rho + \delta - (1-\tau_k)(1-\beta)\dfrac{Y}{K}}{\gamma - 1} - \Psi \right] \tilde{c} & \textbf{(B55)} \\ \dot{\tilde{w}} = \left[\left[(1-\beta)(1-\tau_k) + \phi(1-\tau_w)\omega_{dom} \right] \dfrac{Y}{K} - \delta - \Psi \right] \tilde{w} - (1+\tau_c)\tilde{c} - Te^{-\Psi t} & \textbf{(B56)} \end{cases}$$

From equation (B56) it becomes clear that further assumptions about the independent term on aggregate lump-sum taxation must be taken, in order to simplify this system. It is straightforward to observe that if the constant growth rate is positive, then this independent term will converge to zero in the long run. This will mean that in a growing economy aggregate lump-sum taxation will have an ever diminishing effect on wealth accumulation. We can take this mathematical result and set our first rule of fiscal policy, by setting lump-sum taxation to be equal to zero for all periods in this simplified model. Taking this final assumption we can derive equilibrium expressions for this dynamical system as usual:

$$\begin{cases} \bar{\Psi} = \dfrac{\rho + \delta - (1-\tau_k)(1-\beta)\dfrac{Y}{K}}{\gamma - 1} & \textbf{(B57)} \\ \bar{\tilde{c}} = \dfrac{\left[(1-\beta)(1-\tau_k)\gamma + \phi(1-\tau_w)\omega_{dom}(\gamma - 1) \right] \dfrac{Y}{K} - \delta\gamma - \rho}{(\gamma - 1)(1+\tau_c)} \bar{\tilde{w}} & \textbf{(B58)} \end{cases}$$

Linearising this system around equilibrium, as expressed by equations (B57) and (B58), we can describe the dynamic features of this economy using the Jacobian matrix.

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[9] In order to simplify this presentation, we will use in this section the expression $\dfrac{Y}{K} = pmgK = \left(A \left[gN \right]^{\beta} \right)^{\frac{1}{1-\beta}} (1-l)^{\frac{\phi}{1-\beta}}$ and extend this expression only for relevant results.

$$\boldsymbol{J} = \begin{bmatrix} 0 & 0 \\ -(1+\tau_c) & \dfrac{\left[(1-\beta)(1-\tau_k)\gamma + \phi(1-\tau_w)\omega_{dom}(\gamma-1)\right]\dfrac{\boldsymbol{Y}}{\boldsymbol{K}} - \delta\gamma - \rho}{(\gamma-1)} \end{bmatrix},$$

where $\det(\boldsymbol{J}) = 0$, $\boldsymbol{tr}(\boldsymbol{J}) = \dfrac{\left[(1-\beta)(1-\tau_k)\gamma + \phi(1-\tau_w)\omega_{dom}(\gamma-1)\right]\dfrac{\boldsymbol{Y}}{\boldsymbol{K}} - \delta\gamma - \rho}{(\gamma-1)}$

and the roots of the characteristic equation are $\lambda_1 = \boldsymbol{tr}(\boldsymbol{J})$ and $\lambda_2 = 0$

Again the dynamics of this dynamical system are of the type of the usual special degenerate case, where *det(J)*=0. Following the methodology presented in sections *4.2.3.*, *4.3.3.* and *5.2.3.* we just need to apply the final restrictions for $\bar{\Psi}$, $\bar{\tilde{\boldsymbol{c}}}$ and $\bar{\tilde{\boldsymbol{w}}}$ to be always positive, in order to conclude the description of this model:

$$\begin{cases} \bar{\Psi} \succ 0 \Rightarrow \rho + \delta - (1-\tau_k)(1-\beta)\dfrac{\boldsymbol{Y}}{\boldsymbol{K}} \prec 0 \\ \bar{\tilde{\boldsymbol{c}}}, \bar{\tilde{\boldsymbol{w}}} \succ 0 \Rightarrow \left[(1-\beta)(1-\tau_k)\gamma + \phi(1-\tau_w)\omega_{dom}(\gamma-1)\right]\dfrac{\boldsymbol{Y}}{\boldsymbol{K}} - \delta\gamma - \rho \prec 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \frac{\rho+\delta}{(1-\tau_k)(1-\beta)} \prec \boldsymbol{PmgK} \prec \frac{\rho+\delta\gamma}{(1-\beta)(1-\tau_k)\gamma + \phi(1-\tau_w)\omega_{dom}(\gamma-1)}$$

Restrictions imposed imply that *tr(J)* will be always positive, therefore there are no transitional dynamics in this economy and the phase diagram for this model is similar to the one in figure *1*,of section *4.2.3.*, only varying in the parameters.

5.3.3.1. First Best Optimal Fiscal Policy for the Decentralized Economy with no Adjustment Costs

Having described the necessary conditions for the existence of endogenous growth in both the centralized and decentralized economies, for the case without adjustment costs, we can now obtain the conditions for a first best optimum fiscal policy. In order to deliver this intuition we will take on two approaches to the concept of a first best optimum fiscal policy in a growing economy. The first approach will restrict conditions for a first best optimum to the growth mechanism, which will enable us to simplify our analysis and produce a straightforward rule for government spending and taxing decisions. The second approach will extend the first best optimum to all optimality conditions, in order to obtain a set of endogenous rules for fiscal policy.

Using solely the endogenous growth rates, $\bar{\Psi}$, to determine optimal fiscal policy , it is straightforward to observe from equation (B57), that only taxes on capital affect growth in the decentralized equilibrium. If we exclude the possibility of subsidies on capital, growth is maximized by setting taxes on capital to zero. Choosing a growth maximizing policy will reduce the first best optimum to a spending rule, obtained when we equalize equation (B29) to (B57). This spending rule will mean that government size relative to aggregate output, $\boldsymbol{g}$, should be equal to the public good factor intensity, β[10]. This rule will imply that labour/leisure optimality condition for the centralized economy with a government sector is equal to the labour/leisure optimality condition, (A26), from the centralized and decentralized economies without a government sector.

We can extend our analysis on optimal fiscal policy by solving a system of equations that guarantees optimal equilibrium conditions for both models equalize and that is possible to replicate the centralized equilibrium trough fiscal policy rules. In order to build this system, we will use the endogenous growth equilibrium conditions, $\overline{\Psi}$, from equations (B29) and (B57), the labour/leisure optimal conditions, (B22) and (B50), and the balanced government budget constraint, (B7). Rearranging these conditions, the three dimensional system for the first best optimal fiscal policy can be reduced to this system of equations:

$$\begin{cases} (1-\tau_k)(1-\beta) = (1-\boldsymbol{g}) & \textbf{(B59)} \\ \dfrac{(1-\tau_w)}{(1+\tau_c)} = \dfrac{(1-\boldsymbol{g})}{(1-\beta)} & \textbf{(B60)} \\ \tau_c \boldsymbol{C} + \tau_w \boldsymbol{w}(1-\boldsymbol{l})\boldsymbol{N} + \tau_k \boldsymbol{r}_k \boldsymbol{K} + \boldsymbol{T} = \boldsymbol{g}\boldsymbol{Y} & \textbf{(B61)} \end{cases}$$

Before formalizing the set of rules for fiscal policy, we should recall the assumptions taken about lump sum taxes in section *5.3.3.*. In that section we considered that in the case of endogenous growth, the independent term for lump sum aggregate taxation of equation (B56), could be set to zero, in order to simplify our model. The reason for this was the diminishing impact of this term in the long run for wealth accumulation. This does not mean that lump sum taxation is a bad choice for a fiscal instrument, as a matter of fact, this same argument insures that lump sum taxation is less distorting in the long run than other available fiscal instruments. Nevertheless, our hypothesis of considering this independent term as irrelevant, is consistent with lump sum taxation that is constant in the long run, or at least that does not growth at a rate bigger than $\overline{\Psi}$. A simple example of a rule for lump sum taxation, that would be consistent for solving our decentralized economy, is given by setting $\boldsymbol{T} = \phi(1-\tau_w)\dfrac{\boldsymbol{Y}}{\boldsymbol{K}}\boldsymbol{B}$, and substituting it in equation (B56). This complex rule for lump sum taxation will simplify our original dynamical system changing the results presented in section *5.3.3.*, by setting $\omega_{dom} = 1$. Other rules for lump sum taxation could be considered also, provided that they share the same common growth rate as our system variables[11]. One special case that could be applied to our model is a rule that would correct the labour income asymmetry on accumulation indifference,

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[10] Turnovsky (1999) describes this optimal spending rule as the equality between the elasticity of output with respect to the government input.

[11] Turnovsky (1999) considers the following admissible rule for lump sum taxation, $\boldsymbol{T}_{(t)} = (\boldsymbol{aK}_0 + \boldsymbol{bB}_0)\boldsymbol{e}^{\overline{\Psi}t}$, where *a* and *b* are constants derived from the balanced growth equilibrium. Because of our initial assumptions on the intertemporal budget constraint the rule for this economy should be $\boldsymbol{T}_{(t)} = (\boldsymbol{aK}_0 - \boldsymbol{bB}_0)\boldsymbol{e}^{\overline{\Psi}t} \Leftrightarrow \boldsymbol{T}_{(t)} = (\boldsymbol{cW}_0)\boldsymbol{e}^{\overline{\Psi}t}$.

described in section *4.3.3.*. Setting $\boldsymbol{T} = \phi(1-\tau_{\boldsymbol{w}})\dfrac{\boldsymbol{Y}}{\boldsymbol{K}}\boldsymbol{K}$, in section *5.3.3.* results, will simplify the term $\phi(1-\tau_{\boldsymbol{w}})\omega_{\boldsymbol{dom}}(\gamma-1)$ to be equal to zero.

Substituting the rules for maximum growth and optimal spending in the system of equations that represents our first best optimum, the rules for optimal fiscal policy can be formalized as:

$$\begin{cases} \tau_{\boldsymbol{k}} = 0 \Rightarrow \boldsymbol{g} = \beta & \textbf{(B62)} \\ -\tau_{\boldsymbol{w}} = \tau_{\boldsymbol{c}} & \textbf{(B63)} \\ \tau_{\boldsymbol{c}}\boldsymbol{C} + \tau_{\boldsymbol{w}}\boldsymbol{w}(1-\boldsymbol{l})\boldsymbol{N} + \boldsymbol{T} = \beta\boldsymbol{Y} & \textbf{(B64)} \end{cases}$$

Our second rule of policy implies that taxes on consumption and labour should be set in accordance to equation (B63). Following the conclusions in Turnovsky (1999), when we consider taxes in labour as a negative tax on leisure, this result will require that consumption and leisure, the two utility enhancing goods, should be taxed uniformly, because there are no externalities on the optimal tax structure. We can view this rule of policy as a dynamic extension of the Ramsey optimal taxation principle. Ramsey (1927) tackles the issue of setting a fiscal policy that minimizes the decrement of utility, in order to minimize the economic distortion of taxation (excess burden), not taking into account the equity and redistributive aspects that may arise from fiscal policy. Although Ramsey's problem in its simpler version is not dynamic, the solution of his hypothesis is similar to our results, where taxes on labour income are symmetric to taxes on consumption. This result holds optimal taxation for consumption and leisure, when the utility function is multiplicatively separable in $\boldsymbol{C}$ and $\boldsymbol{l}$, as it happens in our formulation. Both these rules of policy imply that labour/leisure optimal conditions, (B22) and (B50), for the centralized and decentralized economies with a government sector, will be equal to the optimal condition for labour/leisure decisions, (A26), of the centralized and decentralized economies without a government sector. This rule guarantees that there are no distortions imposed in labour markets due to taxation and government spending.

We can now use equations (B62), (B63) and (B64) to obtain a set of endogenous policies for taxing consumption and labour income, in accordance to the three hypotheses we presented for lump sum taxation and the optimal condition for leisure and labour, (A26). These set of policies are shown in table 1:

	Lump sum optimal taxing policy hypotheses		
T	0	$T=\phi(1-\tau_w)\dfrac{Y}{K}B$	$T=\phi(1-\tau_w)\dfrac{Y}{K}K$
τ_c	$\dfrac{\beta}{\left(\dfrac{l}{(1-l)\theta}-1\right)\phi} \succ 0$	$\dfrac{\beta-\phi\Upsilon}{\left(\dfrac{l}{\theta(1-l)}-1+\Upsilon\right)\phi}$	$\dfrac{(\beta-\phi)\theta(1-l)}{\phi l}$
τ_w	$\dfrac{\beta}{\left(1-\dfrac{l}{(1-l)\theta}\right)\phi} \prec 0$	$\dfrac{\phi\Upsilon-\beta}{\left(\dfrac{l}{\theta(1-l)}-1+\Upsilon\right)\phi}$	$\dfrac{(\phi-\beta)\theta(1-l)}{\phi l}$

Table 1 – Optimal taxes in consumption and wages according to lump sum taxing hypotheses. Where, Υ is equal to the constant equilibrium ratio of debt/foreign assets to domestic capital, $\dfrac{B}{K}$.

The results for our three hypotheses in table 1 and equations (B62) and (B63) summarize the first best optimal fiscal and spending policies for this economy, considering three specific lump sum rules, which helped us to simplify the dynamical system in section 5.3.3.. We can generalize our rules for policy by solving equation (B64) and substituting C by the optimal labour/leisure condition, (A26), as we did for results in table 1.

$$\tau_c = \frac{\beta Y - T}{\left(\dfrac{l}{(1-l)\theta}-1\right)\phi Y} = -\tau_w \qquad \textbf{(B65)}$$

Equation (B66) guarantees that optimal taxes on labour income and consumption are always endogenous to the model parameters and/or constants when T is a linear and homogenous function of output, Y. Recalling the admissible rule for lump sum taxation, described in the previous foot note, we can extend this rule taking this in consideration:

$$T_{(t)} = (c W_0) e^{\bar{\Psi} t} \Rightarrow T_{(t)}(Y) = (f(Y) W_0) e^{\bar{\Psi} t} \Rightarrow T = f(Y) \qquad \textbf{(B66)}$$

Where, $f(Y)$, is a homogenous linear function of Y.

5.3.4. Simulation and Results for the Decentralized Economy with Adjustment Costs

In order to develop a specification to analyse this economy numerically, when adjustment costs are considered, we decided to consider only the four dimensional system as it shown bellow. This implies assuming that leisure is given by an exogenous parameter and not by a motion equation derived from the optimal control equation (B50). This choice of specification simplifies this analysis and offers the opportunity to explore a framework that is not tackled in Turnovsky (1999). In this article, Turnovsky shows that equilibrium in the presence of adjustment costs is feasible only when the dynamic system for leisure and investment converges to a single equilibrium[12], which is the case for his specific model. In our analysis we discard the dynamics for leisure and chose to analyse the four dimensional model of consumption, foreign debt, capital and investment. This option is feasible because we have already shown that when consumption and output growth at the same rate, leisure decisions are reduced to an endogenous parameter in equilibrium. The best option however, is to consider the additional equation of motion for leisure and analyse the five dimensions system that describes this economy. We won't perform this analysis here, because this would imply a new indifference in accumulation condition for investment decisions, which reflected the hypothesis of leisure decisions having transitional dynamics.

$$\begin{cases} \dot{\boldsymbol{C}} = \dfrac{(\rho - \boldsymbol{r})}{(\gamma - 1)}\boldsymbol{C} & \textbf{(B67)} \\ \dot{\boldsymbol{B}} = (1+\tau_{\boldsymbol{c}})\boldsymbol{C} + \boldsymbol{I}\left(1+\dfrac{\boldsymbol{hI}}{2\boldsymbol{K}}\right) + \boldsymbol{rB} + \boldsymbol{T} - & \\ -\left[\phi(1-\tau_{\boldsymbol{w}}) + (1-\beta)(1-\tau_{\boldsymbol{k}})\right]\left(\boldsymbol{A}\left[\boldsymbol{gN}\right]^{\beta}\right)^{\frac{1}{1-\beta}}(1-\boldsymbol{l})^{\frac{\phi}{1-\beta}}\boldsymbol{K} & \textbf{(B68)} \\ \dot{\boldsymbol{K}} = \boldsymbol{I} - \delta\boldsymbol{K} & \textbf{(B69)} \\ \dot{\boldsymbol{I}} = \dfrac{\boldsymbol{I}^2}{2\boldsymbol{K}} + \boldsymbol{rI} + \left(\delta + \boldsymbol{r} - (1-\tau_{\boldsymbol{k}})(1-\beta)\left(\boldsymbol{A}\left[\boldsymbol{gN}\right]^{\beta}\right)^{\frac{1}{1-\beta}}(1-\boldsymbol{l})^{\frac{\phi}{1-\beta}}\right)\dfrac{\boldsymbol{K}}{\boldsymbol{h}} & \textbf{(B70)} \end{cases}$$

Since our main assumption for endogenous growth remains that all variables must share the same common growth rate, we have to detrend the system described by (B67) to (B70) in the usual form:

$$\begin{cases} \dot{\tilde{\boldsymbol{c}}} = \left(\dfrac{(\rho - \boldsymbol{r})}{(\gamma - 1)} - \Psi\right)\tilde{\boldsymbol{c}} & \textbf{(B71)} \\ \dot{\tilde{\boldsymbol{b}}} = (1+\tau_{\boldsymbol{c}})\tilde{\boldsymbol{c}} + \tilde{\boldsymbol{i}}\left(1+\dfrac{\boldsymbol{h}\tilde{\boldsymbol{i}}}{2\tilde{\boldsymbol{k}}}\right) + (\boldsymbol{r} - \Psi)\tilde{\boldsymbol{b}} + \boldsymbol{T}\boldsymbol{e}^{-\Psi \boldsymbol{t}} - & \\ -\left[\phi(1-\tau_{\boldsymbol{w}}) + (1-\beta)(1-\tau_{\boldsymbol{k}})\right]\left(\boldsymbol{A}\left[\boldsymbol{gN}\right]^{\beta}\right)^{\frac{1}{1-\beta}}(1-\boldsymbol{l})^{\frac{\phi}{1-\beta}}\tilde{\boldsymbol{k}} & \textbf{(B72)} \\ \dot{\tilde{\boldsymbol{k}}} = \tilde{\boldsymbol{i}} - (\delta + \Psi)\tilde{\boldsymbol{k}} & \textbf{(B73)} \\ \dot{\tilde{\boldsymbol{i}}} = \dfrac{\tilde{\boldsymbol{i}}^2}{2\tilde{\boldsymbol{k}}} + (\boldsymbol{r} - \Psi)\tilde{\boldsymbol{i}} + \left(\delta + \boldsymbol{r} - (1-\tau_{\boldsymbol{k}})(1-\beta)\left(\boldsymbol{A}\left[\boldsymbol{gN}\right]^{\beta}\right)^{\frac{1}{1-\beta}}(1-\boldsymbol{l})^{\frac{\phi}{1-\beta}}\right)\dfrac{\tilde{\boldsymbol{k}}}{\boldsymbol{h}} & \textbf{(B74)} \end{cases}$$

Linearising the system, we obtain that only the main diagonal terms of the *Jacobian* matrix impacts the determinant. These terms are just the four partial derivatives of each variable and one of them

[12] Turnovsky (1999) solves this problem by considering a control in investment as we do and substituting it in the co-state differential equation for the shadow price of capital. He then considers the saddle path dynamic equilibrium conditions for the two dimensional system of leisure and the shadow price of capital.

must be equal to zero for endogenous dynamics to arise. We describe below the four expressions in terms of the endogenous growth rate, $\bar{\Psi}$:

$$\bar{\Psi} = \frac{(\rho - \boldsymbol{r})}{(\gamma - 1)} \quad \vee \quad \bar{\Psi} = \boldsymbol{r} \quad \vee \quad \bar{\Psi} = -\delta \quad \vee \quad \bar{\Psi} = \frac{\tilde{\boldsymbol{i}}}{\tilde{\boldsymbol{k}}} + \boldsymbol{r}$$

The first three hypotheses are endogenous rules, while the fourth hypothesis is only endogenous in the dynamic system equilibrium. This last conclusion is easily obtained by estimating the equilibrium for the scaled investment differential equation in terms of the ratio $\frac{\tilde{\boldsymbol{i}}}{\tilde{\boldsymbol{k}}}$. In equilibrium this ratio is given by an endogenous expression of parameters when assuming this rate of growth:

$$\frac{\tilde{\boldsymbol{i}}}{\tilde{\boldsymbol{k}}} = \left[\left(\delta + \boldsymbol{r} - (1 - \tau_{\boldsymbol{k}})(1 - \beta)\left(\boldsymbol{A}\,[\boldsymbol{g}\boldsymbol{N}]^{\beta}\right)^{\frac{1}{1-\beta}}(1 - \boldsymbol{l})^{\frac{\phi}{1-\beta}}\right)\frac{2}{\boldsymbol{h}}\right]^{\frac{1}{2}} \qquad \textbf{(B75)}$$

Both this expressions are also consistent with endogenous dynamical equilibrium for a two dimensional system between scaled investment and capital, which are independent from other variables of the extended model.

Using this rule allows for transitional dynamics to arise in the short run path for convergence, while in the other cases convergence is not achieved or has an oscillatory behaviour, with an ever increasing variance. The third hypothesis for the rate of growth was not considered for obvious reasons.

With the purpose of portraying convergence in this dynamical system, under these assumptions, we decided to use the initial value problem solver for Matlab, ODE45, which we present in generic terms in section *4.1.* of the appendix. In a recent article, Trimborn, Koch and Steger (2006) propose a new method for simulating endogenous growth models, the relaxation method, and discuss the implications of using different approaches to tackle endogenous growth models numerically. They show the advantages of their method when compared to other methods, such as multiple shooting and time-eliminating procedures, using a methodology that allows for comparative dynamics analysis and other extensions. In our example one of the downfalls that we have to deal with is the equilibrium that is generated by the scaled system, which is equal to zero in the computed steady state. This limits the options for performing a comparative dynamic analysis of this system and restricts the analysis to convergence from initial values to long run equilibrium. In spite of that, we can parameterize the solver in order to obtain the time path for the transitions of the common growth rate that is obtained from the convergence results of the scaled system.

Using a feasible set of numerical parameters, presented in table *2* bellow, we can use the assumptions discussed for the balanced growth path dynamics and the restrictions imposed by the scaled investment to capital ratio, to described two different states for this economy and their convergence to steady state, according to different initial value assumptions. For this purpose we chose a set of conservative values for fiscal and spending policy and relax the balanced budget assumption.

Parameter	ρ	γ	τ_c	T	ϕ	τ_w	β
Value	0,02	0,3	0,2	0,1	0,5	0,2	0,3
Parameter	A	g	N	l	δ	τ_k	θ
Value	0,1	0,4	400	0,25	0,06	0,2	0,2

Table 2- Numerical values for base parameters

We assume that this economy is performing bellow long run conditions and initial values for scaled capital and consumption are negative representing this scenario. Foreign debt is growing at an initial state above its long run regime. The world interest rate faced by this small open economy is 5% making it now more profitable to hold domestic assets than foreign assets. Investment however has recovered making the scaled investment to capital ratio to be negative at an initial value. Simulated convergence results when adjustment costs are equal to -2 are presented bellow:

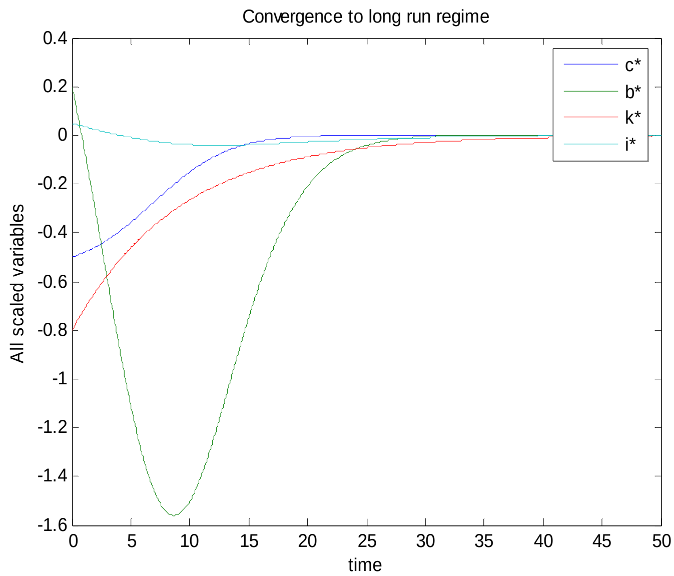

Fig. 2- Convergence to long run regime

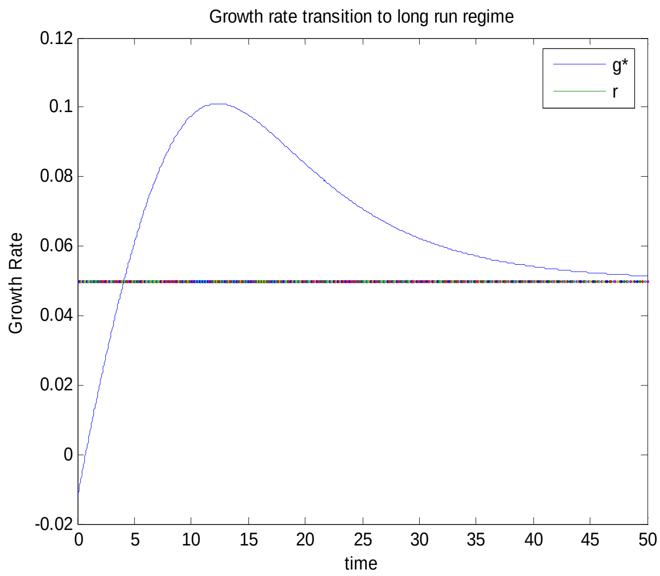

Fig. 3- Growth rate transition

Another straightforward hypothesis is to consider an economy that is growing above its long run regime but now faces a 10% world interest rate, which makes more attractive for agents to exchange domestic assets for foreign assets. Assuming now adjustment costs equal to 2, due to the restrictions imposed by (B75), we obtain the following convergence time paths for the scaled variables and the growth rate:

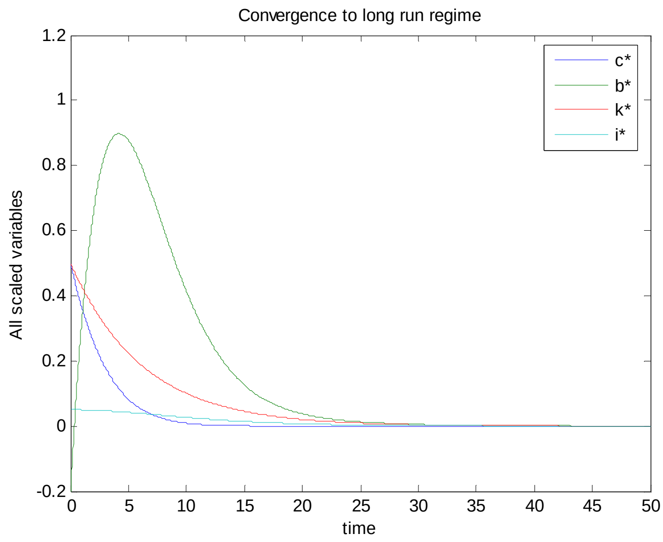

Fig. 4- Convergence to long run regime

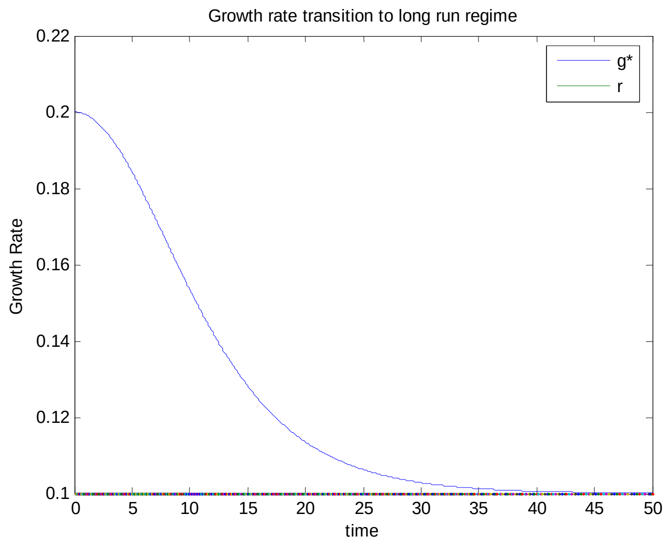

Fig. 5- Growth rate transition

In both simulations there exists an over-shoot effect in holdings of foreign assets/debt, reflecting the specific financial effects that determine each initial state. All other variables converge smoothly to long run equilibrium. In the first hypothesis there is an overshoot in growth in the short to medium run. The same does not occur in the booming economy case, where the grow rate decays slowly in the short run and accelerates until decaying exponentially in the transition to the long run regime. This is a result of the proposed initial imbalances for the adjustment mechanism, defined by the scaled investment to capital ratio. In these specific cases, final simulated results are somewhat sensible to initial conditions, in particular when considering a negative initial ratio of scaled investment to capital.

5.4. Concluding Remarks

Despite following a different approach than the one used in Turnovsky (1999) for solving this model and obtaining optimal fiscal policy rules, using a slightly different range of fiscal instruments and choosing an intertemporal budget constraint expressed in terms of foreign debt and not in foreign assets, the results obtained for both policy and welfare remain exactly the same to the ones described in Turnovsky (1999). The only exception is the additional rule for taxation of holdings of foreign assets that must equal the rule for capital taxation, which was not considered here.

One of the most interesting features about these results relates the policy rules needed for achieving the first best optimum and its outcome, to the results obtained for the decentralized model of chapter *4.*. First best fiscal and spending policies guarantee that the growth rate for this economy is equal to the one obtained in the case with aggregate capital externalities. Only the consumption to net wealth ratio outcome might differ between these economies. This means that with only a minimal welfare effect government spending can achieve the same endogenous growth rate obtained in the opposite theoretical hypotheses of a completely free public good. As the evidence suggests that some public goods and services produce important macroeconomic externalities, this result reinforces the hypothesis that government spending and taxing have important implications for long run growth outcomes.

In the last section of this chapter we tackled the issue of an extended dynamical system when considering adjustment investment costs, again, our approach differed from the one in Turnovsky (1999), as we briefly discussed in section *5.3.4.*. The purpose of this approach was to provide a new possible specification that still entailed some intuition about labour/leisure decisions but focused on the intertemporal financial adjustment process. The results from Turnovsky (1999) gave us the opportunity to relax the leisure assumption without loosing much intuition about labour/leisure decisions, which allowed us to focus on the development of an analytical specification and numerical routine that described convergence from an initial state to the long run growth regime. The methodology described and results obtained in section *5.3.4.* will prove to be important for convergence analysis of the economy with an informal sector that will be presented in chapter *6.*. The framework provided in this section can also be considered as a straightforward methodology for tackling large scale endogenous growth models, when a common growth rate is considered, in order to achieve some preliminary insight about both convergence and long run equilibrium.

6. Endogenous Growth and Optimal Fiscal Policy in a Small Open Economy with Elastic Labour Supply and an Informal Sector

6.1. Analytical Framework for the Centralized and Decentralized Economies

In this section, as the title suggests, an informal sector will be introduced in this optimal fiscal policy framework that was developed in the last chapter. Before ongoing the discussion about the analytical framework, for both the centralized and decentralized economies, it is convenient to introduce some notions and facts about the informal economy and the objectives that will be pursued in this section.

The informal economy, also known as the hidden or underground economy, is a part of the real economy that arises in order to take advantage of numerous factors that affect the formal sector of an economy. In our model the informal economy arises because of taxation from the government sector, as presented in chapter *5.*. There are other reasons for this type of activities to surface. One of the most common are the criminal activities that take advantage of the existence of a market that is heavily regulated or even illegal, to take on the opportunity of profiteering. Other activities lie in the formal sector but take advantage of asymmetric information in order to avoid or evade taxes. A brief presentation of the subject of the informal sector, estimation procedures, methodology and results can be found in Schneider (2005), Schneider and Enste (2000) and Frey and Schneider

(2001)[13]. In table 3 we reproduce a table presented in Schneider and Enste (2000), which better illustrates the type of activities that usually are developed in the informal sector and their specific categories.

Type of Activity	Monetary Transactions		Nonmonetary Transactions	
ILLEGAL ACTIVITIES	Trade in stolen goods; drug dealing and manufacturing; prostitution; gambling; smuggling, and fraud		Barter: drugs, stolen goods, smuggling etc. Produce or growing drugs for own use. Theft for own use.	
	Tax Evasion	**Tax Avoidance**	**Tax Evasion**	**Tax Avoidance**
LEGAL ACTIVITIES	Unreported income from self-employment; Wages, salaries and assets from unreported work related to legal services and goods	Employee discounts, fringe benefits	Barter of legal services and goods	All do-it-yourself work and neighbour help

Table 3 - Taxonomy of Types of Underground Economic Activities

Usually, the economics literature about the informal sector is concerned with measuring its quantitative dimension, more commonly, its size relative to some macro aggregate like the GDP or GNP, or with the modelling of the aspects that contribute to this phenomenon, with the intention to produce analysis or to contribute to better measuring techniques. The most common methods to produce estimates of the informal sector are those which are based in currency demand models or consumption models of general industrial goods, such as electricity. The reported values are then compared with formal sector figures by a regression type analysis. This type of methodology has some problems, such as measurement errors and omitted variables, though they are still the most reliable ones to account for the shadow economy. Recently a model type approach has gained some relevance, in order to tackle not only the econometric problems[14] referred and to produce better indicators, but also to explore analytically the dynamics that are relevant for this subject. Some examples of this type of analytical approach are Marcouillier and Young (1995), Antunes and Cavalcanti (2004), Antunes (2006) and Amaral and Quintin (2006).

Although this chapter follows an analytical approach to an economy with an informal sector, we are not interested in discussing the causes or determinants that give rise to informality. Our work will follow closely the ideas presented in Peñalosa and Turnovsky (2004), focussing on the impacts that the introduction of an informal sector will produce on an optimal fiscal policy model of endogenous growth. The main difference from the ideas developed in Peñalosa and Turnovsky (2004) is the type of economy that will be considered. We consider an open economy model for a developed economy, whereas Peñalosa and Turnovsky (2004) consider a closed economy model for a developing country.

[13] The literature on this subject is vast and is not limited to the economic science. The references presented here are useful to provide the reader with an introduction to this particular subject.
[14] One example is models that include latent variables to tackle the issue of omitted variables. For a more complete description of this subject refer to Schneider (2005).

In order to formalize an endogenous growth model of a small open economy with a formal and an informal sector, one should take some time in describing the specific technology in each sector. Before tackling that issue, both the notation employed and aggregate conditions should be revised. From now on variables with the subscript *1*, will be specific of the formal sector variables, and variables with the subscript 2, will be specific of the informal sector variables. Aggregate conditions will now be given by the following conditions:

$$\boldsymbol{X}_i = \boldsymbol{x}_i \boldsymbol{N} \quad , \quad \boldsymbol{X} = \sum_{\boldsymbol{i}=1}^{\boldsymbol{n}} \boldsymbol{X}_i \quad , \quad \boldsymbol{i} = 1,2 \qquad \textbf{(C 1)}$$

Conditions in (C1) just state that we must first aggregate a specific sector and that total aggregate is equal to the sum of both sector aggregates. This rule is reasonable and will prove to be useful further on.

Formal sector technology will follow the *AK* type aggregate technology that was developed in the previous section, which was represented by equations (B1) to (B4). To use those functions in this framework, we need to take in account aggregate conditions as presented in (C1) and assume that the government sector can only observe and tax activities that occur in the formal sector.

$$\boldsymbol{y}_1 = \boldsymbol{A}\boldsymbol{G}^{\beta}\left[\ell_1\left(1-\boldsymbol{l}\right)\right]^{\phi}\boldsymbol{k}_1^{1-\beta} \qquad \textbf{(C2)}$$

$$\boldsymbol{y}_1 = \boldsymbol{A}\left(\boldsymbol{G}/\boldsymbol{k}_1\right)^{\beta}\left[\ell_1\left(1-\boldsymbol{l}\right)\right]^{\phi}\boldsymbol{k}_1 \qquad \textbf{(C3)}$$

$$\boldsymbol{G} = \boldsymbol{g}\boldsymbol{Y}_1 \qquad 0 < \boldsymbol{g} < 1 \qquad \textbf{(C4)}$$

$$\boldsymbol{Y}_1 = \left(\boldsymbol{A}\left[\boldsymbol{g}\boldsymbol{N}\right]^{\beta}\right)^{\frac{1}{1-\beta}}\left[\ell_1\left(1-\boldsymbol{l}\right)\right]^{\frac{\phi}{1-\beta}}\boldsymbol{K}_1 \qquad \textbf{(C5)}$$

This formulation is exactly the same used for the one sector model presented in the previous sector, except for the inclusion of the endogenous variable ℓ_1, which refers to the percentage of time devoted to working in the formal sector by our representative agent. This endogenous variable is included to insure the normalization of the labour input to one is maintained and because it gives a straightforward option for taking an optimal control rule for labour allocation between sectors. As it is our intention to maintain the labour input normalized to one, ℓ_1 will vary between zero and one and labour allocation in the informal sector will be just $1-\ell_1$. Production technology for the formal sector firm is given by (C2) and government expenditures are tied only to the observed aggregate output, $\boldsymbol{Y}_1$. Using aggregate conditions stated in (C1), total output for the formal sector is obtained in equation (C5). This formalization for the formal sector reproduces exactly the AK type technology presented in the one sector model. For now parameter restrictions will remain the ones already considered.

Before continuing to the formalization of the informal sector technology, some issues concerning production in this sector must be discussed first. The representative firm in the informal sector will also have its output given by a Cobb-Douglas type production function, following the same structure of (C2), where all productive factor intensities, including the elasticity of the government input, are lower than the ones considered for the formal sector production function. These hypotheses are necessary in order to obtain static equilibrium conditions for capital and labour decisions between sectors. They are also reasonable if we consider that informality arises in order to take advantage of the absence of government regulation and taxation. Therefore, in spite of using a worst technology, they are still able to participate with success in a competitive market framework.

Following this short intuition the production firm for the representative firm in the informal sector comes:

$$\boldsymbol{y}_2 = \boldsymbol{D}\boldsymbol{G}^{\beta-\mu}\left[(1-\ell_1)(1-\boldsymbol{l})\right]^{\xi}\boldsymbol{k}_2^{\eta} \qquad \textbf{(C6)}$$

$$0 \prec \mu \prec 1 \ , \ \beta \succ \mu \ , \ 0 \prec \xi \prec 1 \ , \ \phi \succ \xi \ , \ 0 \prec \eta \prec 1 \ , \ 1-\beta \succ \eta$$

$$\boldsymbol{A} \succ \boldsymbol{D}$$

Where $\boldsymbol{D}$, represents the usual exogenous technological infrastructure, which is smaller than the one in the formal sector, $\boldsymbol{A}$. The elasticities for the capital and labour inputs are different and restricted to be smaller than the ones in the formal sector. Labour allocation by the representative agent is given by expression $(1-\ell_1)$. Informal sector firms are still able to benefit from public goods and services, but face fiscalization of its use by the authorities, which diminishes the factor intensity for public services to be just, $\beta-\mu$. Parameter μ can therefore be used to determine the impact of fiscalization by authorities. This specification could be extended to other productive factors, where one could distinguish between factor fiscalization. In this framework, for reasons of simplification, only the dimension described above will be considered.

Substituting equations (C1) and (C4) in (C6), aggregate production for the informal sector is obtained as follows:

$$\boldsymbol{Y}_2 = \boldsymbol{D}(\boldsymbol{g}\boldsymbol{Y}_1)^{\beta-\mu}\left[(1-\ell_1)(1-\boldsymbol{l})\right]^{\xi}\boldsymbol{K}_2^{\eta}\boldsymbol{N}^{1-\eta} \qquad \textbf{(C7)}$$

$$\boldsymbol{Y}_2 = \boldsymbol{D}(\boldsymbol{A}\boldsymbol{g})^{\frac{\beta-\mu}{1-\beta}}\boldsymbol{N}^{\frac{\beta(\beta-\mu)+(1-\eta)(1-\beta)}{1-\beta}}\ell_1^{\frac{\phi(\beta-\mu)}{1-\beta}}(1-\ell_1)^{\xi}(1-\boldsymbol{l})^{\frac{\phi(\beta-\mu)+\xi(1-\beta)}{1-\beta}}\boldsymbol{K}_1^{\beta-\mu}\boldsymbol{K}_2^{\eta} \qquad \textbf{(C8)}$$

It is straightforward to observe that output in the informal sector will depend on output from the formal sector, due to the capacity of using public capital, though restricted by government fiscalization. As all productive factor intensities in the informal individual firm technology are lower than the ones from firms in the formal sector, investment and labour decisions in this economy will only arise if the net factor payments are equal for both economies. These conditions come as follows:

$$(1-\tau_w)\boldsymbol{w}_1 = \boldsymbol{w}_2 \ , \quad (1-\tau_k)\boldsymbol{r}_{k_1} = \boldsymbol{r}_{k_2} \qquad \textbf{(C9)}$$

Informal sector firms can only arise in this framework if capital and labour taxation are both positive.

Government spending will be given by the usual intertemporal balanced budget constraint, as in (B7), but now it has to face the constraints suggested in condition (C4) and by the framework discussed previously, which restricts factor taxation to the formal sector:

$$\tau_c \boldsymbol{C} + \tau_w \boldsymbol{w}_1 (1-\boldsymbol{l})\ell_1 \boldsymbol{N} + \tau_k \boldsymbol{r}_{k_1} \boldsymbol{K}_1 + \boldsymbol{T} = \boldsymbol{g} \boldsymbol{Y}_1 \qquad \textbf{(C10)}$$

Applying the usual marginal productivity conditions to the static sector equilibrium conditions described in (C9), the additional static equilibrium conditions for this economy are obtained:

$$(1-\tau_w)\phi \boldsymbol{y}_1 = \xi \boldsymbol{y}_2 \qquad \textbf{(C11)}$$

$$(1-\tau_k)(1-\beta)\frac{\boldsymbol{y}_1}{\boldsymbol{k}_1} = \eta \frac{\boldsymbol{y}_2}{\boldsymbol{k}_2} \qquad \textbf{(C12)}$$

Solving (C11) for $\boldsymbol{y}_1$ and then substituting it in (C12) gives the static relation for individual firm capital between both sectors:

$$\boldsymbol{k}_2 = \frac{\eta(1-\tau_w)\phi}{(1-\tau_k)(1-\beta)\xi}\boldsymbol{k}_1 \qquad \textbf{(C13)}$$

We can now use this static equilibrium equation of (C13) and substitute it in the informal firm technology, in order to obtain a production function, which depends exclusively on the parameters, on government spending and on individual firm capital used in the formal sector.

$$\boldsymbol{y}_2 = \boldsymbol{D}\boldsymbol{G}^{\beta-\mu}\left[(1-\ell_1)(1-\boldsymbol{l})\right]^{\xi}\left(\frac{\eta(1-\tau_w)\phi}{(1-\tau_k)(1-\beta)\xi}\right)^{\eta}\boldsymbol{k}_1^{\eta} \qquad \textbf{(C14)}$$

Using again aggregate conditions (C1) and (C4), aggregate output for the informal sector comes:

$$\boldsymbol{Y}_2 = \boldsymbol{D}\boldsymbol{g}^{\beta-\mu}\left[(1-\ell_1)(1-\boldsymbol{l})\right]^{\xi}\left(\frac{\eta(1-\tau_w)\phi}{(1-\tau_k)(1-\beta)\xi}\right)^{\eta}\boldsymbol{Y}_1^{\beta-\mu}\boldsymbol{K}_1^{\eta}\boldsymbol{N}^{1-\eta} \qquad \textbf{(C15)}$$

Substituting $\boldsymbol{Y}_1$ in expression (C15) and rearranging the terms, aggregate output for the informal sector is given by the model parameters, formal sector aggregate capital and exogenous population employed in each sector:

$$\boldsymbol{Y}_2 = \boldsymbol{D}\Omega\left(\frac{\eta(1-\tau_w)\phi}{(1-\tau_k)(1-\beta)\xi}\right)^{\eta}\boldsymbol{K}_1^{\beta+\eta-\mu} \qquad \textbf{(C16)}$$

Where parameter Ω is:

$$\Omega = (\boldsymbol{A}\boldsymbol{g})^{\frac{\beta-\mu}{1-\beta}}\boldsymbol{N}^{\frac{\beta(\beta-\mu)+(1-\eta)(1-\beta)}{1-\beta}}\ell_1^{\frac{\phi(\beta-\mu)}{1-\beta}}(1-\ell_1)^{\xi}(1-\boldsymbol{l})^{\frac{\phi(\beta-\mu)+\xi(1-\beta)}{1-\beta}} \qquad \textbf{(C17)}$$

The following parameter restrictions must now be imposed in order to assure that capital and labour are diminishing in the aggregate:

$$\beta + \eta - \mu \prec 1 \qquad \textbf{(C18)}$$

$$\phi\left(\beta - \mu\right) + \xi\left(1 - \beta\right) \prec 1 - \beta \qquad \textbf{(C19)}$$

Applying the static market equilibrium conditions necessary for the existence of these two sectors in a competitive economy has produced two technologies that depend only on the parameters, exogenous population employed in both sectors and capital employed in the formal sector. This style of formalization provides a clear strategy for modelling an economy where decisions, such as capital accumulation and investment, are based solely on the formal sector variables. Informal sector capital inputs enter this economy through the static equilibrium condition given by (C13). We will consider that formal aggregate capital employed in production is always bigger than aggregate informal capital. This restriction is consistent with data for the informal sector in developed countries. Recalling the market clearing condition for individual firm capital between sectors, (C13), we will just impose that $\eta\left(1-\tau_w\right)\phi < \left(1-\tau_k\right)\left(1-\beta\right)\xi$, in order to guarantee that this empirical evidence is always satisfied.

Following this strategy of modelling will allow us to develop a two sector continuous time dynamic model, without having to tackle with the difficulties that arise when dealing with a two sector economy maximum problem[15] and maintain the framework presented in previous sections. Considering a neoclassical production function for the informal sector that depends exclusively on formal capital will also be consistent with the existence of transitional dynamics, since total output for this economy is given separately[16]. We will deal with this subject later on when deriving an analytical solution for the no adjustment costs economy, however this subject can be described intuitively by a simple analysis of the aggregate marginal productivity of capital. Taking a partial derivative of $\boldsymbol{Y}_1$ on $\boldsymbol{K}_1$ we obtain:

$\boldsymbol{Y}'_{\boldsymbol{K}_1} = \dfrac{\boldsymbol{Y}_1}{\boldsymbol{K}_1} + \boldsymbol{Y}'_{2,\boldsymbol{K}_1}\left(\boldsymbol{K}_1\right)$, it is clear that this expression still depends on formal sector capital which in turn depends of time.

Considering the usual endogenous growth hypotheses that capital grows at a constant growth rate, it is straightforward to obtain that in the long run output will depend only on formal sector activities:

$\lim\limits_{\boldsymbol{k}_1 \to \infty} \boldsymbol{Y}'_{\boldsymbol{K}_1} = \dfrac{\boldsymbol{Y}_1}{\boldsymbol{K}_1}$, this result is consistent to restrictions imposed on parameters by equation (C18), which will imply that marginal aggregate output for the informal sector will decline asymptotically until it becomes negligible on the long run.

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[15] A formal presentation of the two sector economy maximization problem is presented in section 2. of the appendix.

[16] Barro and Sala-i-Martin(1999) discuss both this strategy and the CES production function formulation for the introduction of transitional dynamics in endogenous growth models in pages 161 to 167 of their book

Transitional dynamics in the short run will emerge due to informal sector production, which in turn depends on formal and government sector activities. In the long run, however, formal sector production will be the dominant force of growth. This formalization is consistent with problems arising from heavy taxation of some specific commodities in developed countries. An example is tobacco taxation, where rising even further taxes increases smuggling activities and in some cases overall tax income is lower with the new tax. On the long run, however, the effect of fiscalization and growing wealth will absorb this effect and lower the supply and demand for informal goods. We could extend this example to other effects of taxation in labour markets and even capital allocation.

Long run dynamics dominated by formal sector activities is also consistent with the view that it is in the formal sector, where generally proposed explanations for the existence of endogenous growth arise, such as learning-by-doing, ideas and innovation, human capital and education, product diversity and our hypotheses, capital and public capital externalities. Informal sector activities on the other hand are usually associated with the exploration of gaps in the system. Excessive bureaucracy, taxation, corruption, legal barriers, formal sector entry costs and illegal emigration are some examples of holes in the system that provide the opportunity for informal activities to arise and in some cases prosper. Although the dynamic effects arising from an informal sector in our model are limited to the existence of transitional dynamics in the short run, other effects as negative capital externalities arising from informal activities, which affect formal sector production, could be considered also. This issue although appealing will not be considered in our analytical analysis.

6.2. Analytical Results for the Centralized Economy

In the following sections we will present the analytical framework that will define our centrally planned economy. As maximum conditions are already defined in section *2.1.* of the appendix we will restrict our approach to the development of different central planner hypotheses as discussed in sections *2.3.1.* and *2.3.2.* of the appendix. In sections *6.2.1.* trough *6.2.3.* we consider a central planner optimization problem that is consistent with perfect information and assumes informal aggregate production to be given by equation (C16) or simply $Y_2\left(K_1\right)$. Section *6.2.4.* presents a simplified solution for the no adjustment costs case, where we will tackle the issue of long run growth versus short run transitional dynamics, and describe the first best optimal outcome for this economy.

6.2.1. Labour and Leisure Choices and Labour Allocation decisions in the Centrally Planned Economy

Using expression (D65) from section *2.3.2.* of the appendix and substituting the marginal value of foreign bonds by our usual optimal consumption condition we can obtain the equality that will determine labour/leisure decisions:

$$\frac{\theta}{\boldsymbol{l}}=\frac{1}{\boldsymbol{C}\,(1-\beta)(1-\boldsymbol{l})}\left[(1-\boldsymbol{g})\phi\boldsymbol{Y}_1(\boldsymbol{K}_1)+\left(\phi(\beta-\mu)+\xi(1-\beta)\right)\boldsymbol{Y}_2(\boldsymbol{K}_1)\right] \qquad \mathbf{(C20)}$$

Rearranging terms in (C20) in the usual form, labour/leisure decisions are given by:

$$\frac{\boldsymbol{l}}{(1-\boldsymbol{l})}=\frac{\theta(1-\beta)\boldsymbol{C}}{(1-\boldsymbol{g})\phi\boldsymbol{Y}_1(\boldsymbol{K}_1)+\left(\phi(\beta-\mu)+\xi(1-\beta)\right)\boldsymbol{Y}_2(\boldsymbol{K}_1)} \qquad \mathbf{(C21)}$$

This expression is equal to the labour/leisure ratio obtained for the centrally planned economy with a government sector, (B22), if we exclude the informal sector term from the right hand side denominator. On the long run this approach is plausible because in a growing economy this term will asymptotically have an ever diminishing relative impact in labour/leisure decisions as $\boldsymbol{Y}_1(\boldsymbol{K}_1)$ will growth exponentially and $\boldsymbol{Y}_2(\boldsymbol{K}_1)$ logarithmically. This result is slightly different from the one obtained for the aggregate marginal productivity of capital across sectors, where we could assume that the informal sector capital productivity will decay to zero asymptotically. We cannot extend this result here, but we can consider that the impact of this term in the long run will be relatively small and assume that in the long run, the standard condition for leisure and labour decisions, (B22), for the centrally planned economy will hold.

Labour allocation decisions in this economy are given by equation (D66). Substituting functional forms we obtain:

$$\frac{(1-\boldsymbol{g})\phi\boldsymbol{Y}_1(\boldsymbol{K}_1)_1}{(1-\beta)\ell_1}+\frac{\phi(\beta-\mu)\boldsymbol{Y}_2(\boldsymbol{K}_1)}{(1-\beta)\ell_1}=\frac{\xi\boldsymbol{Y}_2(\boldsymbol{K}_1)}{(1-\ell_1)} \qquad \mathbf{(C22)}$$

Rearranging terms, the ratio that will determine labour allocation across sectors comes:

$$\frac{\phi(1-\ell_1)}{(1-\beta)\ell_1}=\frac{\xi\boldsymbol{Y}_2(\boldsymbol{K}_1)}{(1-\boldsymbol{g})\boldsymbol{Y}_1(\boldsymbol{K}_1)+(\beta-\mu)\boldsymbol{Y}_2(\boldsymbol{K}_1)} \qquad \mathbf{(C23)}$$

The right hand side of this equation clearly is not a constant expression in time. This will impact labour allocation between sectors at all periods as time approaches infinity. As $\boldsymbol{Y}_1(\boldsymbol{K}_1)$ is part of the denominator and $\boldsymbol{Y}_2(\boldsymbol{K}_1)$ grows logarithmically, impacting relatively less in this expression over time, we can approach the right hand side of (C23) to zero in the long run. For this equality to maintain we need to approach the left hand side of (C23) to zero in the long run also. As formal output grows steadily over time the only possibility is to derive the following result for labour allocation between sectors in the long run:

$$\lim_{\boldsymbol{t}\to\infty}(1-\ell_1)=0 \Rightarrow \lim_{\boldsymbol{t}\to\infty}\ell_1=1$$

This result just states that in the long run labour allocation in the informal sector will tend to be equal to zero. This implies that labour will tend to allocate exclusively in the formal sector as time tends to infinity, which is in accordance to our long run hypothesis about the impact of informal sector production in aggregate output discussed in section *6.1*..

6.2.2. Indifference in Accumulation in the Centralized Economy with no Adjustment Costs

Although we have considered that our problem is correctly given by a three state dynamic maximization problem as presented in the appendix, we can use the information contained in the linear capital relation, (C13), to obtain a linear relation for investment. This implies that we can internalize all this information for the central planner maximization and reduce our dynamical optimization problem to a two state maximum problem, where we consider only formal capital accumulation, tackling the issue of considering a state condition that adds no useful information to the co-state by eliminating it. Doing this will not affect the results obtained in the previous section because none of those depend on informal investment decisions. Only our arbitrage condition for investment, the intertemporal open-economy budget constraint and the co-state condition for the shadow price of formal capital will be altered.

To obtain a relation between formal and informal aggregate investment we need to take time derivatives from (C13) and then substitute the equality by the specific sector state conditions. The result obtained is the same to the one obtained in section 2.3.2. of the appendix from equation (D70). We rewrite this condition for convenience:

$$\boldsymbol{I}_2 = \frac{\eta\left(1-\tau_{\boldsymbol{w}}\right)\phi}{\left(1-\tau_{\boldsymbol{k}}\right)\left(1-\beta\right)\xi}\boldsymbol{I}_1 \tag{C24}$$

This result defines exactly the same relation for investment that was obtained for capital in equation (C13). We can now use our usual formulation to obtain the three conditions that are altered by this assumption. Substituting (C24) in the intertemporal aggregate budget constraint, we can obtain the new optimal arbitrage investment condition and the formal capital co-state expression for the no adjustment costs case:

$$\boldsymbol{q}_1 = -\lambda\,\frac{\left(1-\tau_{\boldsymbol{k}}\right)\left(1-\beta\right)\xi+\eta\left(1-\tau_{\boldsymbol{w}}\right)\phi}{\left(1-\tau_{\boldsymbol{k}}\right)\left(1-\beta\right)\xi} \tag{C25}$$

$$\dot{\boldsymbol{q}}_1 = \left(\rho+\delta\right)\boldsymbol{q}_1+\lambda\left(\left(1-\boldsymbol{g}\right)\frac{\boldsymbol{Y}_1\left(\boldsymbol{K}_1\right)}{\boldsymbol{K}_1}+\left(\beta+\eta-\mu\right)\frac{\boldsymbol{Y}_2\left(\boldsymbol{K}_1\right)}{\boldsymbol{K}_1}\right) \tag{C26}$$

Substituting (C25) in the optimal control condition for consumption, (D54), and taking time derivatives, we can obtain the usual differential equations for consumption:

$$\dot{\boldsymbol{C}} = \left(\frac{\rho-\boldsymbol{r}}{\gamma-1}\right)\boldsymbol{C} \tag{C27}$$

$$\dot{\boldsymbol{C}} = \left(\frac{\rho+\delta-\dfrac{\left(1-\tau_{\boldsymbol{k}}\right)\left(1-\beta\right)\xi}{\left(1-\tau_{\boldsymbol{k}}\right)\left(1-\beta\right)\xi+\eta\left(1-\tau_{\boldsymbol{w}}\right)\phi}\left[\left(1-\boldsymbol{g}\right)\dfrac{\boldsymbol{Y}_1\left(\boldsymbol{K}_1\right)}{\boldsymbol{K}_1}+\left(\beta+\eta-\mu\right)\dfrac{\boldsymbol{Y}_2\left(\boldsymbol{K}_1\right)}{\boldsymbol{K}_1}\right]}{\gamma-1}\right)\boldsymbol{C} \tag{C28}$$

From equations (C27) and (C28) it is straightforward to obtain our steady state condition for indifference in accumulation:

$$\boldsymbol{r} = \frac{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi}{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi + \eta(1-\tau_{\boldsymbol{w}})\phi}\left((1-\boldsymbol{g})\frac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1} + (\beta+\eta-\mu)\frac{\boldsymbol{Y}_2(\boldsymbol{K}_1)}{\boldsymbol{K}_1}\right) - \delta \quad \textbf{(C29)}$$

We can compare the effects of fully internalizing informal sector investment decisions by observing the steady state equation, (D75), where we did not considered a linear relation as described by (C24). Now the aggregate parameter term that defines total capital and total investment in terms of formal variables affects our steady state condition inversely on aggregate marginal productivities, while in the appendix this term impacted directly the rate of depreciation. The intuition for this is straightforward, our usual steady state condition for indifference in accumulation just states that the aggregate net marginal productivity of capital must be equal to the exogenous world interest rate. By internalizing informal investment decisions as a function of formal investment our central planner defines aggregate total marginal productivity in function of the formal sector activities and considers only formal capital accumulation. Whereas, in the three states dynamic problem presented, in the appendix, the central planner only considers the investment relation after choosing maximum conditions, this will distort the arbitrage condition for investment decisions and consequently the shadow price relations for foreign bonds and domestic capital. Total net marginal productivity is corrected through the information contained in the state condition for informal capital accumulation, given as a function of formal capital. In the simplified case with no adjustment costs, both hypotheses follow a similar pattern, though the resulting conditions differ on the scale in which they are defined.

On the long run we can extend the result given by (C29) by applying our usual assumptions about the asymptotic convergence for aggregate marginal productivity. On the long run the informal sector aggregate marginal productivity will decay asymptotically and we can express our long run equilibrium for indifference in accumulation as usual:

$$\lim_{t\to\infty} \boldsymbol{r} = \frac{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi}{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi + \eta(1-\tau_{\boldsymbol{w}})\phi}(1-\boldsymbol{g})\frac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1} - \delta \qquad \textbf{(C30)}$$

Long run indifference in accumulation, when compared to the steady state expression for indifference in accumulation for the central planned economy of section 5.2.2., (B22), is now related to the parameters governing the informal sector dimension. As we mentioned earlier, the impact of informality directly influences aggregate marginal productivity, in the long run indifference in accumulation, as opposed to the impact described in equation (D76), for the simplified hypothesis developed in the appendix, where informality impacts the rate of depreciation in both horizons.

6.2.3. Indifference in Accumulation in the Centralized Economy with Adjustment Costs

Following the framework described in the previous section for the no adjustment costs case, we can extend our investment relation hypothesis, (C24), to the case with adjustment costs. Applying this linear relation to both our alternatives for adjustment costs functions, (D52) and (D53), aggregate and specific to sector respectively, simplifies greatly our assumptions. Both alternative hypotheses simplify to one investment adjustment cost function given by:

$$\frac{(1-\tau_k)(1-\beta)\xi+\eta(1-\tau_w)\phi}{(1-\tau_k)(1-\beta)\xi}\boldsymbol{I}_1\left(1+\frac{\boldsymbol{h}}{2}\frac{\boldsymbol{I}_1}{\boldsymbol{K}_1}\right) \quad \textbf{(C31)}$$

We can now use expression (C31) to obtain the remaining maximum conditions for a two state maximum problem as in section *6.2.2.*. Substituting (C31) in the intertemporal aggregate budget constraint state condition, we can obtain the new optimal arbitrage investment condition and the formal capital co-state expression for the adjustment cost case:

$$\boldsymbol{q}_1=-\lambda\frac{(1-\tau_k)(1-\beta)\xi+\eta(1-\tau_w)\phi}{(1-\tau_k)(1-\beta)\xi}\left(1+\boldsymbol{h}\frac{\boldsymbol{I}_1}{\boldsymbol{K}_1}\right) \quad \textbf{(C32)}$$

$$\dot{\boldsymbol{q}}_1=(\rho+\delta)\boldsymbol{q}_1+$$
$$+\lambda\left[\frac{(1-\tau_k)(1-\beta)\xi+\eta(1-\tau_w)\phi}{(1-\tau_k)(1-\beta)\xi}\frac{\boldsymbol{h}}{2}\left(\frac{\boldsymbol{I}_1}{\boldsymbol{K}_1}\right)^2+(1-\boldsymbol{g})\frac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1}+(\beta+\eta-\mu)\frac{\boldsymbol{Y}_2(\boldsymbol{K}_1)}{\boldsymbol{K}_1}\right] \quad \textbf{(C33)}$$

Using the usual methodology for obtaining a formal investment differential equation as the one described in section *4.2.2.*, it is straightforward to obtain from (C32) and (C33) the quadratic equation for investment that will complete this section centrally planned four dimensional dynamical system:

$$\dot{\boldsymbol{I}}_1=\frac{\boldsymbol{I}_1^{\,2}}{2\boldsymbol{K}_1}+\boldsymbol{r}\boldsymbol{I}_1+$$
$$+\left[\delta+\boldsymbol{r}-\frac{(1-\tau_k)(1-\beta)\xi}{(1-\tau_k)(1-\beta)\xi+\eta(1-\tau_w)\phi}\left((1-\boldsymbol{g})\frac{\boldsymbol{Y}_1}{\boldsymbol{K}_1}+(\beta+\eta-\mu)\frac{\boldsymbol{Y}_2(\boldsymbol{K}_1)}{\boldsymbol{K}_1}\right)\right]\frac{\boldsymbol{K}_1}{\boldsymbol{h}} \quad \textbf{(C34)}$$

As we did in section *6.2.2.*, we can compare the results presented here with the results obtained in section *2.3.2.1.* of the appendix, for the case with specific to sector adjustment costs functions and without internalizing investment relations between sectors, before choosing maximum conditions. Both the structure of (C34) and (D77) are similar to the one obtained in all the previous models. In the case of (C34), we can extend section *6.2.2.* conclusions for the simplified case, by comparing the results obtained, in that section, to results obtained from the centralized and decentralized models presented so far. As to the specific parameter structure, which results from (D77), its analysis is not as straightforward as it is for equation (C34), though we can use conclusions discussed in section *6.2.2.* and extend them, in order to understand the differences arising from both alternative formulations. There is no reason to believe that the main conclusions, discussed for the simplified case, cannot be extended to the adjustment costs case also.

6.2.4. Solution for the Centralized Economy with no Adjustment Costs

Using the results obtained in all sub-sections of section 6.2. we can build the usual two dynamical system of consumption and net wealth accumulation. Before ongoing to the formalization of this problem we should recall some previous results, which must be taken into account when considering the proposed conclusions. Firstly, we must consider that in the short to medium run there are still transitional dynamics in labour allocation and leisure decisions as it is defined in sub-section 6.2.1.. Static equilibrium for labour/leisure decisions, which was the basis of our previous models analysis, is only obtained asymptotically. Secondly, net wealth is now given by formal plus informal domestic aggregate capital minus net foreign aggregate liabilities/assets. In order to simplify our approach and take into account this last assumption we will further assume that:

$$\Theta_2 = \frac{(1-\tau_k)(1-\beta)\xi + \eta(1-\tau_w)\phi}{(1-\tau_k)(1-\beta)\xi} \qquad \textbf{(C35)}$$

Using equation (C35) we can simplify our net wealth assumption to be just:

$$\boldsymbol{W} = \Theta_2 \boldsymbol{K}_1 - \boldsymbol{B} \Rightarrow \dot{\boldsymbol{W}} = \Theta_2 \dot{\boldsymbol{K}}_1 - \dot{\boldsymbol{B}} \qquad \textbf{(C36)}$$

Parameter expression (C35) can be also used to simplify our differential equation for consumption, (C28), when assuming that the steady state condition, (C29), for indifference in accumulation holds:

$$\dot{\boldsymbol{C}} = \left(\frac{\rho + \delta - \Theta_2^{-1}\left[(1-\boldsymbol{g})\frac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1} + (\beta+\eta-\mu)\frac{\boldsymbol{Y}_2(\boldsymbol{K}_1)}{\boldsymbol{K}_1}\right]}{\gamma - 1} \right)\boldsymbol{C} \qquad \textbf{(C37)}$$

We can now complete the two dimensional system by building the net wealth differential equation through some algebraic manipulations. Substituting, (C28), and the formal capital accumulation equation in the open economy intertemporal budget constraint, (D28) we obtain the following differential equation:

$$\begin{aligned} \dot{\boldsymbol{B}} &= \boldsymbol{C} + \Theta_2\left(\dot{\boldsymbol{K}}_1 - \delta \boldsymbol{K}_1\right) - (1-\boldsymbol{g})\boldsymbol{Y}_1(\boldsymbol{K}_1) - \boldsymbol{Y}_2(\boldsymbol{K}_1) + \\ &+ \left(\Theta_2^{-1}\left[(1-\boldsymbol{g})\frac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1} + (\beta+\eta-\mu)\frac{\boldsymbol{Y}_2(\boldsymbol{K}_1)}{\boldsymbol{K}_1}\right] - \delta\right)\boldsymbol{B} \qquad \textbf{(C38)} \end{aligned}$$

Rearranging expression (C38) in the usual form we obtain:

$$\begin{aligned} \Theta_2\dot{\boldsymbol{K}}_1 - \dot{\boldsymbol{B}} &= \left[(1-\boldsymbol{g})\frac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1} + \frac{\boldsymbol{Y}_2(\boldsymbol{K}_1)}{\boldsymbol{K}_1} - \Theta_2\delta\right]\left(\boldsymbol{K}_1 - \Theta_2^{-1}\boldsymbol{B}\right) - \\ &- \boldsymbol{C} - (\beta+\eta-\mu-1)\frac{\boldsymbol{Y}_2(\boldsymbol{K}_1)}{\boldsymbol{K}_1}\Theta_2^{-1}\boldsymbol{B} \qquad \textbf{(C39)} \end{aligned}$$

Substituting the expression for net wealth, (C36), we can obtain the differential equation for net wealth in the centrally planned economy:

$$\dot{W} = \left[(1-g)\frac{Y_1(K_1)}{K_1} + \frac{Y_2(K_1)}{K_1} - \Theta_2\delta - (\beta+\eta-\mu-1)\frac{Y_2(K_1)}{K_1}\frac{B}{W} \right] \Theta_2^{-1} W - C \quad \mathbf{(C40)}$$

With the introduction of an informal sector the same issue of a correcting term impacting net wealth accumulation arises. This issue was discussed in the decentralized economies of sections *4.3.3.* and *5.3.3.*, but in these economies this correction term was related to incomplete information, faced by agents, due to aggregate externalities. Now this issue arises not in terms of domestic capital relative to net wealth but in terms of net foreign debt/assets relative to net wealth. We can again consider the hypothesis that in a growing economy this term will be stable and therefore can be reduced to be just a parameter. Nevertheless, we should consider also that in the presence of transitional dynamics in the short to medium run this might not be true. In the long run, this issue is not important as aggregate informal capital marginal productivity will tend to be equal to zero. In order to simplify this analysis we will assume that this hypothesis holds so that the ratio $\frac{B}{W}$ is substituted by parameter ω_{for}. Net wealth accumulation now comes:

$$\dot{W} = \left[(1-g)\frac{Y_1(K_1)}{K_1} - \Theta_2\delta + \frac{Y_2(K_1)}{K_1}\left(1 - (\beta+\eta-\mu-1)\omega_{for}\right) \right] \Theta_2^{-1} W - C \quad \mathbf{(C41)}$$

From equation (C41) it is straightforward to reduce the analysis of the effect of our assumption to three simple cases. As this term is only related to transitional dynamics it will only impact net wealth accumulation in the short to medium run. Table 4 summarizes these results:

	Relative net foreign debt to net wealth assumptions		
	$\omega_{for} \prec \frac{1}{(\beta+\eta-\mu-1)}$	$\omega_{for} = \frac{1}{(\beta+\eta-\mu-1)}$	$\omega_{for} \succ \frac{1}{(\beta+\eta-\mu-1)}$
$\Delta\frac{Y_2(K_1)}{K_1} \rightarrow \Delta W$	$-$	0	$+$

Table 4- Relative net foreign debt to net wealth direct impact on short to medium run net wealth accumulation

As $(\beta+\eta-\mu-1) \prec 0$ due to the neo-classical assumptions about informal technology, we can generalize the impact of the informal sector marginal productivity as positively related to wealth when the small open economy runs a foreign debt in the short to medium run. When net foreign wealth is positive then all three cases considered may arise. If ω_{for} is not constant and varies in the short to medium run, then the presence of the informal sector may give rise to transitional dynamics, which may show some type of chaotic behaviour when converging to the long run endogenous equilibrium. This last hypothesis seems more realistic to arise when holdings of net foreign assets are positive. In the special case where $\omega_{for} = \frac{1}{(\beta+\eta-\mu-1)}$ holds in the short to medium run no transitional dynamics arise in wealth accumulation directly by the evolution of net wealth. Transitional dynamics affecting wealth accumulation only arise through transitional dynamics affecting the consumption path in the short to medium run. We can expand this result by considering the usual constant growth rate hypothesis and taking trends from the dynamical system defined by (C37) and (C41). It is straightforward to assume that this specific case, where ω_{for} is constant, guarantees the consumption to net wealth ratio transitional dynamics to be limited to the transition path arising from the short to medium run growth dynamics of consumption. In this case, all informal revenues are used on consumption goods. As empirical studies in developed economies suggest, this scenario is a likely one to consider. However, some investing capacity from informal savings must exist, so that informal entrepreneurs are capable to start new firms that take advantage of temporary gaps in the system.

We will not extend this analysis strategy further because the solution path for long run is not given recursively by our system and therefore this is not the correct approach to tackle transitional dynamics in this economy. Nevertheless, this intuition serves the purpose of assuming that different types of transitional dynamics might arise in the centrally planned economy when considering different hypothesis for ω_{for}. If ω_{for} was a discretionary policy instrument, then we could choose a policy or a set of policies that would maximize the short to medium run outcome effect of exogenous shocks. As uncertainty also bears costs and a set of policies could only adjust the relations implied between parameters and variables, previously to forthcoming shocks, then optimal policy might be to minimize the effect of these shocks in the short to medium run dynamics, by assuming a policy, which guaranteed the consumption to net wealth ratio to be always constant or at least that reduced the variance arising from exogenous shocks.

6.2.4.1. Long Run Dynamics

Using the asymptotic results for the no adjustment costs case we can solve this economy as usual for its long run equilibrium dynamics. The only assumption needed for this formalization is the asymptotic result obtained for labour allocation, $\lim_{t\to\infty} \ell_1 = 1$, as it eliminates completely the transitional dynamics arising from informal sector production. Labour/leisure decisions, (C21), are reduced to the stable ratio obtained in the one sector model economy, (B22), and we can use equations (C37) and (C41) to solve for the long run equilibrium in this economy.

$$\begin{cases} \dot{\boldsymbol{C}}_{l/r} = \left(\dfrac{\rho + \delta - \Theta_2^{-1}\left(1 - \boldsymbol{g}\right)\dfrac{\boldsymbol{Y}_1\left(\boldsymbol{K}_1\right)}{\boldsymbol{K}_1}}{\gamma - 1} \right) \boldsymbol{C}_{l/r} & \textbf{(C42)} \\ \dot{\boldsymbol{W}}_{l/r} = \left[\Theta_2^{-1}\left(1 - \boldsymbol{g}\right)\dfrac{\boldsymbol{Y}_1\left(\boldsymbol{K}_1\right)}{\boldsymbol{K}_1} - \delta \right] \boldsymbol{W}_{l/r} - \boldsymbol{C}_{l/r} & \textbf{(C43)} \end{cases}$$

Taking trends and solving for equilibrium we obtain:

$$\begin{cases} \overline{\Psi}_{l/r} = \dfrac{\rho+\delta-\Theta_2^{-1}(1-\boldsymbol{g})\dfrac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1}}{\gamma-1} & \textbf{(C44)} \\ \dfrac{\overline{\tilde{\boldsymbol{c}}}_{l/r}}{\overline{\tilde{\boldsymbol{w}}}_{l/r}} = \dfrac{\left(\Theta_2^{-1}(1-\boldsymbol{g})\dfrac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1}-\delta\right)\gamma-\rho}{\gamma-1} & \textbf{(C45)} \end{cases}$$

Linearising the scaled system around steady state, we can obtain as usual the *Jacobian* matrix that describes this system in equilibrium:

$$\boldsymbol{J} = \begin{bmatrix} 0 & 0 \\ -1 & \dfrac{\left[\Theta_2^{-1}(1-\boldsymbol{g})\dfrac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1}-\delta\right]\gamma-\rho}{\gamma-1} \end{bmatrix}, \text{ where } \det(\boldsymbol{J}) = 0,$$

$$\boldsymbol{tr}(\boldsymbol{J}) = \dfrac{\left[\Theta_2^{-1}(1-\boldsymbol{g})\dfrac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1}-\delta\right]\gamma-\rho}{\gamma-1}$$ and the roots of the characteristic equation are $\lambda_1 = \boldsymbol{tr}(\boldsymbol{J})$ and $\lambda_2 = 0$

To fully characterize this system we just have to impose the usual parameter restrictions on the long run growth rate, consumption and net wealth.

$$\begin{cases} \overline{\Psi}_{l/r} \succ 0 \Rightarrow \rho+\delta-\Theta_2^{-1}(1-\boldsymbol{g})\dfrac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1} \prec 0 \\ \overline{\tilde{\boldsymbol{c}}}_{l/r}, \overline{\tilde{\boldsymbol{w}}}_{l/r} \succ 0 \Rightarrow \left[\Theta_2^{-1}(1-\boldsymbol{g})\dfrac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1}-\delta\right]\gamma-\rho \prec 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \Theta_2(\rho+\delta) \prec (1-\boldsymbol{g})\dfrac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1} \prec \Theta_2\left(\dfrac{\rho}{\gamma}+\delta\right)$$

The results obtained for the long-run asymptotic equilibrium are very similar to the results obtained in section *5.2.3.* for the one sector centralized economy with a government sector. The only difference arising in equilibrium conditions comes from internalizing the complete information contained in the linear relation between sectors that was discussed previously. As we expected, this model serves both the purpose of offering a possible extension to the one sector endogenous modelling framework presented in chapters *4.* and *5.*, and also the possibility of deriving discretionary policy rules from the long run centralized optimum that are optimal in the long run equilibrium decentralized economy.

6.2.4.2. Transitional Dynamics

We can extend the analysis of this economy to the short to medium run dynamics by following the methodology[17] initially proposed by Jones and Manuelli (1990). This methodology was developed to describe the short to medium run dynamics of growth models, where output is given by an *AK* production function plus a neo-classical production function. Originally proposed for closed economy growth models, we will show how this method can be adapted to our open economy framework, under some strict assumptions, while maintaining the conclusions obtained for long run growth dynamics.

Basically the method proposed by Jones and Manuelli (1990) suggests that the transitional dynamics for this class of models can be obtained by studying a dynamical system of transformed variables that are constant in the steady state. Usually the choice of variables for closed economy models are the average product of capital, $\frac{f(K)}{K}$, which works as a state-like variable and the ratio of consumption to capital, $\frac{C}{K}$, which represents the control-like variable. In the open economy framework a good candidate for a control variable is the consumption to net wealth ratio, $\frac{C}{W}$, while the choice for a state-like variable remains the same. Despite these are obvious choices of variables for our model, some issues remain from using the average product of domestic formal capital for the state variable. These issues are best described by time differentiating the control variable, Z_1, and the state variable, Z_2:

$$Z_1 = \frac{C}{W} \Rightarrow \dot{Z}_1 = \frac{\dot{C}}{W} - Z_1 \frac{\dot{W}}{W} \tag{C46}$$

$$Z_2 = \frac{Y_1(K_1) + Y_2(K_1)}{K_1} \Rightarrow \dot{Z}_2 = \frac{\dot{K}_1}{K_1}\left(\frac{d\left(Y_1(K_1) + Y_2(K_1)\right)}{dK_1} - Z_2\right) \tag{C47}$$

$$\dot{Z}_2 = \Psi_K \left(\frac{d\left(Y_1(K_1) + Y_2(K_1)\right)}{dK_1} - Z_2\right) \tag{C48}$$

Originally time differentiating the transformed variables will produce a dynamical system given by (C46) and (C47), this system is not satisfactory because we do not have a functional form for the growth rate of aggregate formal capital. In order to obtain a system dependent only of functional forms we have to assume that capital grows at a constant exogenous growth rate given by Ψ_K. This assumption will mean that transitional dynamics only impact the holdings of foreign assets/debt, while capital grows at a constant positive rate. We need to further assume that both labour/leisure and labour allocation decisions are constant parameters in this framework. This assumption may be

[17] Again we refer to Barro and Sala-i-Martin (1999), pages 161 to 167, for the specific framework presented in this sub-section.

relaxed if we assume (C20) and (C23) as part of this system and varying in the short to medium run also. For reasons of simplification we will not extend our analysis in that direction and we will assume that both parameters are exogenous constants. As the build up of a dynamical system in $\dot{Z}_1$ and $\dot{Z}_2$ involves some rather messy algebra, we will follow a step by step approach in order to be possible for the reader to track back our steps easily. First we obtain the functions needed to complete (C46) and (C48) in order of Z_1 and Z_2:

$$\frac{dY}{dK_1} = Z_2(\beta+\eta-\mu) + \frac{Y_1(K_1)}{K_1}(1-\beta-\eta+\mu) \tag{C49}$$

$$\frac{\dot{C}}{W} = \left[\frac{\rho+\delta-\Theta_2^{-1}\left(Z_2(\beta+\eta-\mu)+\frac{Y_1(K_1)}{K_1}(1-g-\beta-\eta+\mu)\right)}{\gamma-1}\right]Z_1 \tag{C50}$$

$$\frac{\dot{W}}{W} = \Theta_2^{-1}\left[Z_2\left(1-(\beta+\eta-\mu-1)\omega_{for}\right)+\left(-g+(\beta+\eta-\mu-1)\omega_{for}\right)\frac{Y_1(K_1)}{K_1}\right]-\delta-Z_1 \tag{C51}$$

Now we can substitute (C49), (C50) and (C51) in (C46) and (C48) in order to obtain the two dimensional system of transformed variables that describes transitional dynamics for this economy:

$$\begin{cases} \dot{Z}_1 = Z_1\left\{\Theta_2^{-1}\left[\frac{Z_2\left(\beta+\eta-\mu-(1-\gamma)(1-\Theta_3)\right)}{1-\gamma}+\right.\right. \\ \left.\left.+\frac{\frac{Y_1(K_1)}{K_1}\left(1-g-\beta-\eta+\mu-(1-\gamma)(-g+\Theta_3)\right)}{1-\gamma}\right]-\frac{\rho+\delta\gamma}{1-\gamma}+Z_1\right\} & \text{(C52)} \\ \dot{Z}_2 = \Psi_K\left(Z_2(\beta+\eta-\mu)+\frac{Y_1(K_1)}{K_1}(1-\beta-\eta+\mu)-Z_2\right) & \text{(C53)} \end{cases}$$

where $\Theta_3 = (\beta+\eta-\mu-1)\omega_{for}$

Although rather messy at first sight the equilibrium in this dimensional system is easily derived:

$$\begin{aligned}\bar{Z}_1 = {} & -\Theta_2^{-1}\frac{\left(\beta+\eta-\mu-(1-\gamma)(1-\Theta_3)\right)}{1-\gamma}\bar{Z}_2+\frac{\rho+\delta\gamma}{1-\gamma}- \\ & -\Theta_2^{-1}\frac{\frac{Y_1(K_1)}{K_1}\left(1-g-\beta-\eta+\mu-(1-\gamma)(-g+\Theta_3)\right)}{1-\gamma}\end{aligned} \tag{C54}$$

$$\bar{Z}_2 = \frac{Y_1(K_1)}{K_1} \tag{C55}$$

substituting $\bar{Z}_2$ in (C54), $\bar{Z}_1 = \dfrac{\left[\Theta_2^{-1}(1-g)\frac{Y_1(K_1)}{K_1}-\delta\right]\gamma-\rho}{\gamma-1}$ **(C56)**

The steady state for this system is composed by two well known functions described by (C55) and (C56). Its analysis is also straightforward, the equilibrium for $\dot{\boldsymbol{Z}}_2$, the state-like variable, just defines that the growth engine in the long run is the aggregate marginal productivity of capital, whereas $\overline{\boldsymbol{Z}}_1$ defines the long run equilibrium for the consumption to net wealth ratio. Both these results were obtained previously and its implications discussed in previous sections. We can begin to describe the dynamics of this system by performing a qualitative analysis of the linearized system in the phase plane.

The linearized nullclines for this system are given by expressions (C54) and (C55). The $\dot{\boldsymbol{Z}}_2 = 0$ locus poses no additional difficulty, while the $\dot{\boldsymbol{Z}}_1 = 0$ locus has to be simplified in order to determine the possible sign of the slope and intercept. Expressions (C57) and (C58) simplify these two expressions for the slope and intercept, respectively:

$$-\Theta_2^{-1}\frac{\left[(\beta+\eta-\mu-1)\left(\omega_{\boldsymbol{for}}(1-\gamma)+1\right)\right]+\gamma}{1-\gamma} \tag{C57}$$

$$\frac{\rho+\delta\gamma}{1-\gamma}-\Theta_2^{-1}\frac{\dfrac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1}\left[(\beta+\eta-\mu-1)\left(-1-\omega_{\boldsymbol{for}}-\gamma\right)+\gamma\boldsymbol{g}\right]}{1-\gamma} \tag{C58}$$

From expression (C57) it is straightforward to assume that the slope of (C54) will be positive, unless we assume a very high degree of intertemporal substitution, γ. The sign of the intercept is not as obvious as the sign of the slope because the second term will almost probably assume a positive value. We will assume that it follows the closed economy hypothesis and probably assumes a negative value.

In order to deliver the dynamics portrayed in the phase diagram we have to evaluate the *Jacobian* matrix of the linearized system near equilibrium. These results follow bellow:

$$\boldsymbol{J} = \begin{vmatrix} \boldsymbol{J}_{1,1} & \boldsymbol{J}_{1,2} \\ \boldsymbol{J}_{2,1} & \boldsymbol{J}_{2,2} \end{vmatrix} \Rightarrow \boldsymbol{Tr}(\boldsymbol{J}) = \boldsymbol{J}_{1,1} + \boldsymbol{J}_{2,2},\ \boldsymbol{Det}(\boldsymbol{J}) = \boldsymbol{J}_{1,1}\cdot\boldsymbol{J}_{2,2} \prec 0,$$

$$\lambda_{1,2} = \frac{\boldsymbol{Tr}(\boldsymbol{J})}{2} \pm \left[\left(\frac{\boldsymbol{Tr}(\boldsymbol{J})}{2}\right)^2 - \boldsymbol{Det}(\boldsymbol{J})\right]^{\frac{1}{2}},$$

$$\boldsymbol{J}_{1,1} = \overline{\boldsymbol{Z}}_1 \succ 0,\ \boldsymbol{J}_{1,2} = \Theta_2^{-1}\frac{\left[(\beta+\eta-\mu-1)\left(\omega_{\boldsymbol{for}}(1-\gamma)+1\right)\right]+\gamma}{1-\gamma}\overline{\boldsymbol{Z}}_1 \prec 0,$$

$$\boldsymbol{J}_{2,1} = 0,\ \boldsymbol{J}_{2,2} = \Psi_{\boldsymbol{K}}\left(\beta+\eta-\mu-1\right) \prec 0$$

The sign of the *Jacobian* matrix determinant is negative, so we are in the presence of a converging saddle path. We do not need to determine the sign of the trace, neither the expression for the roots of the characteristic equation in order to describe the system dynamics and obtain our generic optimum policy functions. All we need is to determine the position of the system's stable arm and deliver some general intuition about transitional dynamics for this first best optimum. By assuming that λ_1 is the negative characteristic root associated with the stable eigenvector, it is straightforward to obtain its slope, ν_1, using the *Jacobian* matrix. The slope of ν_1 is given by $\frac{-\boldsymbol{J}_{1,2}}{\boldsymbol{J}_{1,1}-\lambda_1}$ and has a positive sign. We can compare this result with the slope of $\dot{Z}_1 = 0$ locus, given by (C57). Transforming (C57) in terms of the *Jacobian* matrix elements we obtain $\frac{-\boldsymbol{J}_{1,2}}{\boldsymbol{J}_{1,1}}$, which is always steeper than the slope of ν_1. This result is also in accordance with the phase diagram associated to closed economy models.

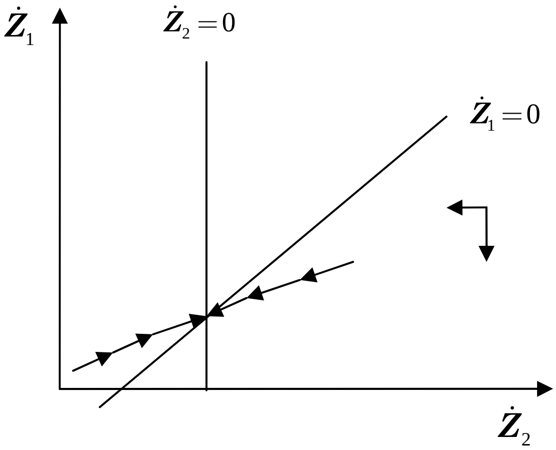


Fig. 6- Phase diagram for open economy transitional dynamics

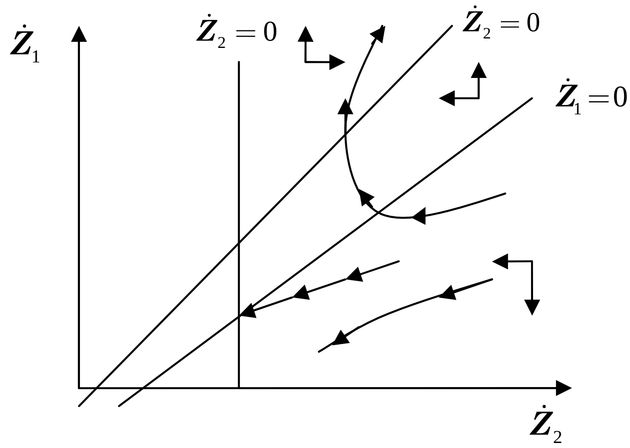


Fig. 7- Phase diagram for closed economy transitional dynamics

We can compare these results with the usual phase diagram for closed economy models portrayed in figure 7[18]. The main difference arising from both models is the existence of another nullcline for $\dot{\boldsymbol{Z}}_2$ in the closed economy case. This is a consequence of our assumption about the growth rate of formal aggregate capital, $\Psi_{\boldsymbol{K}}$, that we had to perform in order to solve the transformed system. In the *Ramsey* class of closed economy models no assumptions are needed to transform the system as $\Psi_{\boldsymbol{K}} = \boldsymbol{f}\left(\boldsymbol{Z}_1, \boldsymbol{Z}_2\right)$, which assuming standard assumptions is just a straight line with slope equal to one, steeper than the $\dot{\boldsymbol{Z}}_1 = 0$ locus. Despite this difference the main conclusions that apply for the closed economy case also apply for a small open economy when assuming these set of initial hypotheses.

Using the results obtained so far we can derive a set of generic functions for the first best optimum paths of transition arising from exogenous shocks. This issue was never considered in our previous models because the set of policy rules defined applied both to short and long run dynamics. Now this

[18] This phase diagram is taken from Barro and Sala-i-Martin (1999) page 163 closed economy transitions model. Different formulations may slightly alter this phase diagram but its structure and conclusions remain unchanged overall.

is no longer true. There will be a set of long run and short run multipliers that will define both the new long run equilibrium and the optimal path followed. These functions are straightforward to obtain through the use of the variational system. Generically, we will have long run multipliers defined by $\overline{Z}_{1,\Delta}$ and $\overline{Z}_{2,\Delta}$, where Δ represents a given parameter variation:

$$\begin{cases} \overline{Z}_{1,\Delta} = -\dfrac{J_{2,2}}{Det(J)} \cdot \left.\dfrac{d\dot{Z}_1}{d\Delta}\right|_{\overline{Z}_1,\overline{Z}_2} + \dfrac{J_{1,2}}{Det(J)} \cdot \left.\dfrac{d\dot{Z}_2}{d\Delta}\right|_{\overline{Z}_1,\overline{Z}_2} & \textbf{(C59)} \\ \overline{Z}_{2,\Delta} = -\dfrac{J_{1,1}}{Det(J)} \cdot \left.\dfrac{d\dot{Z}_2}{d\Delta}\right|_{\overline{Z}_1,\overline{Z}_2} & \textbf{(C60)} \end{cases}$$

Long run multipliers are influenced by the whole structure of the variational system and also depend on the structure of the initial transformed system, which in turn is obtained from the initial endogenous dynamical system. As short run multipliers depend upon long run multipliers and in the saddle path defined by the systems stable eigenvalue and eigenvector, we can conclude that the simplified set of optimal policy rules defined in section *5.3.3.1.* might not be sufficient to attain a first best optimum in the short to medium run. However, by definition, long run policy rules are generally the ones that maximize welfare in an intertemporal framework, if this is the case, policy is only optimal in the long run and second optimum policies might arise in the short to medium run.

Assuming that one of the transformed variables adjusts discontinuously we can obtain the generic expressions for the short run multipliers. When $\dot{Z}_2$ adjusts discontinuously the short run multiplier for $\dot{Z}_1$ comes:

$$\overline{Z}_{1,\Delta(t)} = \overline{Z}_{1,\Delta} - \nu_1 \cdot \overline{Z}_{2,\Delta} \cdot e^{\lambda_1 t} \qquad \textbf{(C61)}$$

When we admit that $\dot{Z}_1$ adjusts discontinuously we obtain:

$$\overline{Z}_{2,\Delta(t)} = \overline{Z}_{2,\Delta} - \nu_1 \cdot \overline{Z}_{1,\Delta} \cdot e^{\lambda_1 t} \qquad \textbf{(C62)}$$

6.3. Analytical Results for the Decentralized Economy

In this final section we will present the analytical framework for the decentralized economy with an informal sector. This presentation will be based on the results obtained and discussed in sections *6.2.* and 2.3. of the appendix for the centralized economy. The main objective of this section is to deliver some intuition on welfare and policy, taking into account the social optimum derived from the centralized economy and the methodological issues discussed in the appendix. The structure of this presentation will follow closely the one from section *6.2.*, though we will resume all mathematical hypothesis and results already described previously.

6.3.1. Labour and Leisure Choices and Labour Allocation Decisions in the Decentralized Economy

We will base this presentation on the general two state optimum control problem presented on section 2.2. of the appendix. We know from previous results that neither the number of states considered, neither its relation will affect optimality conditions, except of course, for the optimal arbitrage investment conditions. The optimal condition for consumption remains equal to the equations presented for the previous economies. Extending the results from section 2.2. of the appendix for labour/leisure and labour allocation decisions we obtain:

$$\theta \boldsymbol{c}^{\gamma} \boldsymbol{l}^{\theta\gamma-1} + \lambda \left[(1-\tau_{\boldsymbol{w}}) \boldsymbol{w}_1 \ell_1 + \boldsymbol{w}_2 (1-\ell_1) \right] = 0 \qquad \textbf{(C63)}$$

$$\lambda \left[-(1-\tau_{\boldsymbol{w}}) \boldsymbol{w}_1 (1-\boldsymbol{l}) + \boldsymbol{w}_2 (1-\boldsymbol{l}) \right] = 0 \qquad \textbf{(C64)}$$

Applying market clearing and aggregate conditions to these expressions and rearranging in the usual form we obtain:

$$\frac{\boldsymbol{l}}{(1-\boldsymbol{l})} = \frac{\theta (1+\tau_{\boldsymbol{c}}) \boldsymbol{C}}{(1-\tau_{\boldsymbol{w}}) \phi \boldsymbol{Y}_1 (\boldsymbol{K}_1) \ell_1 + \xi \boldsymbol{Y}_2 (\boldsymbol{K}_1)(1-\ell_1)} \qquad \textbf{(C65)}$$

$$(1-\tau_{\boldsymbol{w}}) \phi \boldsymbol{Y}_1 (\boldsymbol{K}_1) = \xi \boldsymbol{Y}_2 (\boldsymbol{K}_1) \qquad \textbf{(C66)}$$

Decentralized labour/leisure decisions described by equation (C65) mimic the dynamics already described for the centralized economy. The only difference arises in parameters that affect the representative agent decisions. If we consider the previous hypothesis that asymptotically labour will allocate exclusively in the formal sector, this equation is reduced to the constant ratio that we obtain in the decentralized economy of section *5.3.2.*, (B50). However, we cannot derive the long run hypothesis for labour allocation, as clearly as we did in the centralized case. We know that in a growing economy, the equation given by (C66) does not hold due to the neoclassic assumption for the informal sector production function, unless we considered that aggregate formal capital converged to equilibrium. This, of course, would only happen if we had considered a neoclassical technology for the formal sector. As this is not the case, we have to rearrange equation (C66) in order to deliver the intuition, already discussed, for labour allocation between sectors. We know that informal production technology depends both on ℓ_1 and $(1-\ell_1)$, so it is straightforward to derive an expression similar to the one given for the centralized economy, (C23), without the denominator term related to the informal sector. Main conclusions will not change for labour allocation decisions, though our optimality condition reduces to:

$$\frac{(1-\ell_1)}{\ell_1} = \frac{\xi \boldsymbol{Y}_2 (\boldsymbol{K}_1)(1-\ell_1)}{(1-\tau_{\boldsymbol{w}}) \phi \boldsymbol{Y}_1 (\boldsymbol{K}_1) \ell_1} \qquad \textbf{(C67)}$$

Recall from the optimal agents decision about labour allocation, (C64), we obtain the same market clearing condition for wages that we had considered theoretically from the marginal productivity assumption, (C11). This means that optimal control conditions also guarantee at the micro level equilibrium in labour allocation between sectors, as both sectors have neoclassical technologies. On the other hand this equilibrium does not hold when we consider market clearing and aggregate conditions, as a consequence of positive externalities arising from public spending that lead to endogenous growth in the formal sector.

One form of interpreting this subject is to consider that the long run is a sum of short and medium run specific periods and that in all of these shorter periods, opportunities for entrepreneurial informal activities arise[19]. Therefore, it would be reasonable to consider transitional dynamics analysis, as we presented in section *6.2.4.2.*, where labour allocation is given by a constant parameter and informal activities play a role in short to medium run dynamics. One generic example are firms that take advantage of new technologies, which have no regulation from government authorities[20]. By exploring these gaps, these firms, although lacking scale and organization, can obtain market shares and profits, which will able them to attract both workers and investors. This situation will last until government regulates and controls formal use of this new technology. Then formal activities arise and through competition, economies of scale and government fiscalization force informal firms market share to become marginal or even inexistent in the long run. Although fiscal policy is not a vital element for these specific circumstances, it will be fundamental at the micro level for individual decision, by providing a financial incentive for labour allocation in the informal sector, and also at the macro level by affecting the transition process. This result can be extended to investment decisions also, as we will discuss in the next section.

6.3.2. Incomplete Information about Informal Investment Decisions in the Decentralized Economy with no Adjustment Costs

When discussing the optimal control problem for the centralized economy in the appendix, the main topic that brought on different but comparable formulations, was the extension of information that the central planner had about investment and capital in both sectors. We dismissed the incomplete information hypothesis described in section *2.2.3.* of the appendix because macro equilibrium does not arise when considering a dominant formal technology with increasing returns to scale. In the previous section we showed how micro dynamic equilibrium arises for optimal individual decisions of labour allocation. This equilibrium does not hold when we consider it at a macro level. The same issue

[19] Various recent examples of these phenomena are related to internet and online technology.

[20] Peer-to-Peer is an example of a technology that was initially developed to promote copyright piracy online, by taking advantage of legal gaps in most developed countries, which did not considered this activity to be illegal. Nowadays most developed countries have considered this activity to be illegal and regulated formal activities related to this technology. Informal firms were consequently reduced to a marginal expression due to government fiscalization and competition from formal sector firms. The most famous case was the Napster vs. music industry legal dispute about copyright abuse, which ended up with Napster technology and legal use of its denomination being sold by its founders to companies that later on settled all legal disputes financially. Napster was basically just an internet site that linked users through servers and the dispute was between the music industry and a group of individuals that ran this business informally, even though the technology used was one major technological innovation. Napster still operated as a formal business between the years 2000 and 2001 until a court order illegalized it, but that only happened after the legal disputes had already started. Today its technology is widespread, the service as reappeared as a legal activity, with the same denomination, is successful and its founders are famous for their achievements and audacity.

is true for capital returns, when information about the relation of investment and capital between sectors is incomplete. At the micro level, considering that all shadow prices equalize, we obtain the microeconomic indifference theoretical relation, (C9), from the co-state conditions of the three states optimization problem from section *2.2.* of the appendix. This condition will not hold at the macro level also. Therefore, we have to assume that the representative agent has some information about the relation between the formal and informal sectors, in order to develop an optimization problem that may be comparable to the full information hypothesis considered for the centralized economy. In this section we will discuss the hypothesis that agents have information about the capital relation between sectors but lack information about investment relations.

We have already discussed the modelling implications of this assumption for the centralized economy in sections *2.3.1.* and *2.3.2.* of the appendix, as a result, we focus this analysis on the policy implications that arise in this framework for the case with no adjustment costs, rather than on the mathematical implications that were discussed in the appendix. The main assumptions for this case are the equality between shadow prices obtained from the optimal investment arbitrage conditions, the co-state condition for formal capital accumulation and the co-state condition for informal capital accumulation that lacks any meaningful information. Although this was discussed as a three state optimization problem in the appendix, the exact same results could be obtained if we had considered just one aggregate state for capital accumulation. This would eliminate the redundant co-state condition for informal capital and leave everything else unchanged, while only one optimal arbitrage investment condition for formal investment is considered in this specific case.

Substituting the relation for capital between sectors, (C13), in the optimization problem described in section 2.2. of the appendix, we can rewrite the representative agent co-state condition for capital accumulation easily and obtain the aggregate expression that is fundamental for indifference in accumulation in this economy[21]:

$$\dot{\boldsymbol{q}}_1 = (\rho + \Theta_2 \delta)\boldsymbol{q}_1 + \lambda\left((1-\tau_{\boldsymbol{k}})\boldsymbol{r}_{\boldsymbol{k}_1} + \frac{\eta(1-\tau_{\boldsymbol{w}})\phi}{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi}\boldsymbol{r}_{\boldsymbol{k}_2}\right) \quad \textbf{(C68)}$$

$$\dot{\boldsymbol{q}}_1 = (\rho + \Theta_2 \delta)\boldsymbol{q}_1 + \lambda\left((1-\tau_{\boldsymbol{k}})(1-\beta)\frac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1} + \eta\frac{\eta(1-\tau_{\boldsymbol{w}})\phi}{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi}\frac{\boldsymbol{Y}_2(\boldsymbol{K}_1)}{\boldsymbol{K}_1}\right) \quad \textbf{(C69)}$$

Using the usual methodology we can derive the steady state condition for indifference in accumulation in this economy using the co-state aggregate expression (C69):

$$\boldsymbol{r} = (1-\tau_{\boldsymbol{k}})(1-\beta)\frac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1} + \eta\frac{\eta(1-\tau_{\boldsymbol{w}})\phi}{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi}\frac{\boldsymbol{Y}_2(\boldsymbol{K}_1)}{\boldsymbol{K}_1} - \Theta_2\delta \quad \textbf{(C70)}$$

[21] This assumption will imply that from now on we will consider (C14) to be the individual firm informal technology. This assumption is necessary in order to consider just one state of capital accumulation for individual agents.

$$\lim_{t\to\infty} \boldsymbol{r} = \left(1-\tau_{\boldsymbol{k}}\right)\left(1-\beta\right)\frac{\boldsymbol{Y}_1\left(\boldsymbol{K}_1\right)}{\boldsymbol{K}_1} - \Theta_2\delta \qquad \textbf{(C71)}$$

We know from section *5.3.3.1*. that the steady state for indifference in accumulation has important implications for both fiscal policy and growth in the decentralized equilibrium. In that section, we specified expression (B59) as the rule for optimal spending and capital taxing policy in an economy without short to medium run transitions. We can now extend this result to long run optimal policy in an endogenous economy with transitions using the asymptotic steady state equation for indifference in accumulation described by (C71), for the decentralized economy, and the correspondent result for the centralized economy, described by (C30). The maximizing growth rule for long run fiscal policy in this economy is easily obtained from the equality described by (C30) and (C71):

$$\frac{\boldsymbol{Y}_1\left(\boldsymbol{K}_1\right)}{\boldsymbol{K}_1}\left[\Theta_2^{-1}\left(1-\boldsymbol{g}\right)-\left(1-\tau_{\boldsymbol{k}}\right)\left(1-\beta\right)\right] = \delta\left(1-\Theta_2\right) \qquad \textbf{(C72)}$$

As expected, the fiscal policy growth rule does not simplify to a manageable expression when agents lack information about informal investment. Optimal capital taxing and spending policy may not hold in this framework and if there is a first best optimal growth rule then it will probably differ from the one derived in section *5.3.3.1*.. This has strong implications for labour and consumption taxation. Recall that in the long run, both endogenous equilibrium expressions for labour/leisure decisions in this economy are exactly the same to the previous expression for the economy of chapter 5., therefore the Ramsey optimal taxation principle, defined by $-\tau_{\boldsymbol{w}} = \tau_{\boldsymbol{c}}$, for this class of models might be different in this case or even not exist, as a deterministic rule, as it is dependent of the optimal growth policy rules in order to hold.

If we assume that there is no optimal rules of fiscal policy in this framework, or at least that there must exist a choice between optimal grow or non-distorting labour markets and utility enhancing fiscal policy, no longer first best optimum policy is an option as a long run rule. In this framework different choices between sets of second best optimal fiscal policies are available and the second best choice will depend on the central planner preferences or be a subject of public choice and preferences. We can extend this issue to possible trade-offs between short and long run policies. Decision makers or voters might also have preferences about intertemporal outcomes and decide that it is best to adjust policy in the short run according to those preferences. This issue is relevant if we consider that in the absence of long run deterministic policy rules, short run policies are also defined by a set of available second best optimal rules and preferences might determine trade-offs between short and long run outcomes.

Fiscal policy in this framework is no longer an issue of welfare maximization but of preferences and choices. Different preferences will determine different sets of policies that maximize welfare according to the specific decision making process that may be considered for this economy. We will not extend this analysis further as our main goal is to focus in the link between fiscal policy and growth and not in public choice or policy options outcomes. Nevertheless, this section served the purpose of showing how the existence of an informal sector may give rise to policies that are no longer a first best optimum in the long run, in the presence of asymmetric information about informal investment, between the central planner and the representative agent.

6.3.3. Analytical Results for the Complete Information Decentralized Economy

Assuming that information about the formal and informal sectors is completely available to the representative agents we can develop a model that is fully comparable to the centrally planned economy of section *6.1.*. The issues discussed will be similar to the incomplete information hypothesis. First, we will discuss if a set of rules for optimal fiscal policy are feasible in the long run for the simplified economy with no adjustment costs. As this is the case, we will then compare the dynamics for the transitional dynamics models, using numerical simulations, for both the centrally planned and decentralized economies and discuss some qualitative results.

We start as usual by defining the optimal arbitrage condition for investment and the co-state condition for capital for the representative agent, assuming that (C24) holds at the micro level also:

$$\boldsymbol{q}_1 = -\lambda \Theta_2 \left(1 + \boldsymbol{h}\frac{\boldsymbol{i}_1}{\boldsymbol{k}_1}\right) \tag{C73}$$

$$\dot{\boldsymbol{q}}_1 = (\rho + \delta)\boldsymbol{q}_1 + \lambda \left[\Theta_2 \frac{\boldsymbol{h}}{2}\left(\frac{\boldsymbol{i}_1}{\boldsymbol{k}_1}\right)^2 + (1-\tau_{\boldsymbol{k}})\boldsymbol{r}_{\boldsymbol{k}_1} + \frac{\eta(1-\tau_{\boldsymbol{w}})\phi}{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi}\boldsymbol{r}_{\boldsymbol{k}_2}\right] \tag{C74}$$

Applying market clearing and aggregate condition we obtain:

$$\boldsymbol{q}_1 = -\lambda \Theta_2 \left(1 + \boldsymbol{h}\frac{\boldsymbol{I}_1}{\boldsymbol{K}_1}\right) \tag{C75}$$

$$\begin{aligned}\dot{\boldsymbol{q}}_1 &= (\rho + \delta)\boldsymbol{q}_1 + \\ &+\lambda \left[\Theta_2 \frac{\boldsymbol{h}}{2}\left(\frac{\boldsymbol{I}_1}{\boldsymbol{K}_1}\right)^2 + (1-\tau_{\boldsymbol{k}})(1-\beta)\frac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1} + \eta\frac{\eta(1-\tau_{\boldsymbol{w}})\phi}{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi}\frac{\boldsymbol{Y}_2(\boldsymbol{K}_1)}{\boldsymbol{K}_1}\right]\end{aligned} \tag{C76}$$

We can now obtain the indifference in accumulation equations for both adjustment costs cases:

$$\boldsymbol{r} = \Theta_2^{-1}\left[(1-\tau_{\boldsymbol{k}})(1-\beta)\frac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1} + \eta\frac{\eta(1-\tau_{\boldsymbol{w}})\phi}{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi}\frac{\boldsymbol{Y}_2(\boldsymbol{K}_1)}{\boldsymbol{K}_1}\right] - \delta \tag{C77}$$

$$\lim_{\boldsymbol{t}\to\infty} \boldsymbol{r} = \Theta_2^{-1}(1-\tau_{\boldsymbol{k}})(1-\beta)\frac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1} - \delta \tag{C78}$$

$$\dot{I}_1 = \frac{I_1^{\,2}}{2K_1} + rI_1 +$$

$$+\left[\delta + r - \Theta_2^{-1}\left((1-\tau_k)(1-\beta)\frac{Y_1(K_1)}{K_1} + \eta\frac{\eta(1-\tau_w)\phi}{(1-\tau_k)(1-\beta)\xi}\frac{Y_2(K_1)}{K_1}\right)\right]\frac{K_1}{h} \quad \textbf{(C79)}$$

From the asymptotic steady state rule for indifference in accumulation, (C78), it becomes clear that a first best optimal fiscal policy can be achieved in the simplified case with no adjustment costs. This optimal policy will mimic the rules discussed in section *5.3.3.1.*, but will only apply to the long run outcome. In the next sections we will discuss long run outcomes for both adjustment costs assumptions and evaluate the effects on convergence and transitional dynamics that arise as a consequence of the existence of an informal sector.

6.3.3.1. Long Run Dynamics

Before describing the long run dynamics for the decentralized economy we shall present the equations that compose the simplified dynamical system. Substituting as usual the steady state condition, defined by (C77), in the differential equation for consumption we obtain:

$$\dot{C} = \left(\frac{\rho + \delta - \Theta_2^{-1}\left[(1-\tau_k)(1-\beta)\frac{Y_1(K_1)}{K_1} + \eta\frac{\eta(1-\tau_w)\phi}{(1-\tau_k)(1-\beta)\xi}\frac{Y_2(K_1)}{K_1}\right]}{\gamma - 1}\right)C \quad \textbf{(C80)}$$

Using results from section *2.2.* of the appendix the intertemporal aggregate budget constraint for the decentralized economy comes:

$$\dot{B} = (1+\tau_c)C + I_1\Theta_2 + rB + T -$$

$$-\left[(1-\tau_k)(1-\beta) + \phi(1-\tau_w)\ell_1\right]Y_1(K_1) - \left[\xi(1-\ell_1) + \eta\frac{\eta(1-\tau_w)\phi}{(1-\tau_k)(1-\beta)\xi}\right]Y_2(K_1) \quad \textbf{(C81)}$$

Substituting again the exogenous international interest rate by the steady state rule of indifference in accumulation, (C77), investment by the capital accumulation equation and rearranging in the usual form, the wealth differential equation for this economy comes:

$$\dot{W} = W\Theta_2^{-1}\left[\frac{Y_1(K_1)}{K_1}\left[(1-\tau_k)(1-\beta) + \phi(1-\tau_w)\ell_1\omega_{dom}\Theta_2\right] - \Theta_2\delta\right] +$$

$$+\left[\frac{Y_2(K_1)}{K_1}\left(\eta\frac{\eta(1-\tau_w)\phi}{(1-\tau_k)(1-\beta)\xi} + \xi(1-\ell_1)\omega_{dom}\Theta_2\right)\right]W\Theta_2^{-1} - (1+\tau_c)C - T \quad \textbf{(C82)}$$

As usual we have considered the parameter ω_{dom} for aggregate labour incomes in order to simplify our system. We had already used this assumption in previous decentralized economies for reasons of convenience. In the centralized economy with an informal sector we opted to use the parameter ω_{for}, related to aggregate informal output, also for reasons of convenience. Although both formulations were based on reasons of mathematical convenience, presenting the system with this structure makes it possible to distinguish between the different sources that generated the wealth related parameters.

We can now describe the long run endogenous equilibrium for this economy by applying the standard asymptotic assumptions, about labour allocation and leisure, to the system defined by (C81) and (C82). The long run dynamical system comes:

$$\begin{cases} \dot{\boldsymbol{C}}_{l/r} = \left(\dfrac{\rho + \delta - \Theta_2^{-1}(1-\tau_k)(1-\beta)\dfrac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1}}{\gamma - 1} \right) \boldsymbol{C}_{l/r} & \textbf{(C83)} \\ \dot{\boldsymbol{W}}_{l/r} = \left[\dfrac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1} \left[\Theta_2^{-1}(1-\tau_k)(1-\beta) + \phi(1-\tau_w)\omega_{dom} \right] - \delta \right] \boldsymbol{W}_{l/r} - (1+\tau_c)\boldsymbol{C}_{l/r} - \boldsymbol{T} & \textbf{(C84)} \end{cases}$$

Solving for a common growth rate as usual and assuming that lump-sum taxation decays asymptotically to zero in the long run, the endogenous equilibrium expressions are given by:

$$\begin{cases} \bar{\Psi}_{l/r} = \dfrac{\rho + \delta - \Theta_2^{-1}(1-\tau_k)(1-\beta)\dfrac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1}}{\gamma - 1} & \textbf{(C85)} \\ \dfrac{\bar{\tilde{\boldsymbol{c}}}_{l/r}}{\bar{\tilde{\boldsymbol{w}}}_{l/r}} = \dfrac{\dfrac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1}\left[\Theta_2^{-1}(1-\tau_k)(1-\beta)\gamma + \phi(1-\tau_w)\omega_{dom}(\gamma-1)\right] - \delta\gamma - \rho}{(\gamma-1)(1+\tau_c)} & \textbf{(C86)} \end{cases}$$

Linearising the system around equilibrium we obtain the *Jacobian* matrix as usual:

$$\boldsymbol{J} = \begin{bmatrix} 0 & 0 \\ -(1+\tau_c) & \dfrac{\dfrac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1}\left[\Theta_2^{-1}(1-\tau_k)(1-\beta)\gamma + \phi(1-\tau_w)\omega_{dom}(\gamma-1)\right] - \delta\gamma - \rho}{(\gamma-1)} \end{bmatrix},$$

where $\det(\boldsymbol{J}) = 0$, $\boldsymbol{tr}(\boldsymbol{J}) = \dfrac{\dfrac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1}\left[\Theta_2^{-1}(1-\tau_k)(1-\beta)\gamma + \phi(1-\tau_w)\omega_{dom}(\gamma-1)\right] - \delta\gamma - \rho}{(\gamma-1)}$

and the roots of the characteristic equation are $\lambda_1 = \boldsymbol{tr}(\boldsymbol{J})$ and $\lambda_2 = 0$

Final restrictions for long run endogenous equilibrium in this economy come:

$$\begin{cases} \bar{\Psi}_{l/r} \succ 0 \Rightarrow \rho + \delta - \Theta_2^{-1}\left(1-\tau_k\right)\left(1-\beta\right)\dfrac{Y_1\left(K_1\right)}{K_1} \prec 0 \\ \bar{\tilde{c}}_{l/r}, \bar{\tilde{w}}_{l/r} \succ 0 \Rightarrow \dfrac{Y_1\left(K_1\right)}{K_1}\left[\Theta_2^{-1}\left(1-\tau_k\right)\left(1-\beta\right)\gamma + \phi\left(1-\tau_w\right)\omega_{dom}\left(\gamma-1\right)\right] - \delta\gamma - \rho \prec 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \frac{\left(\rho+\delta\right)\Theta_2}{\left(1-\tau_k\right)\left(1-\beta\right)} \prec \frac{Y_1\left(K_1\right)}{K_1} \prec \frac{\rho + \delta\gamma}{\Theta_2^{-1}\left(1-\beta\right)\left(1-\tau_k\right)\gamma + \phi\left(1-\tau_w\right)\omega_{dom}\left(\gamma-1\right)}$$

As we discussed in the previous section, the dynamics of the decentralized long run equilibrium do not present any innovation, either to fiscal policy issues or endogenous long run growth dynamics. Section *5.3.3.1.* rules for fiscal policy will hold in this framework and assure that a first best optimum is achieved in the long run. System dynamics, in the long run, will also follow closely those from the decentralized economy of section *5.3.3.*, except for differences in parameters that arise from the inclusion of the additional informal productive sector.

6.3.3.2. Numerical Analysis of Transitional Dynamics

We can now use the results obtained for the long-run equilibrium to obtain a system that describes transitional dynamics in the decentralized economy. Again, we will use the Jones and Manuelli (1990) formulation, discussed in section *6.2.4.2.*, in order to obtain a system that portraits transitional dynamics for this economy. Resuming results already discussed in section *6.2.4.2.*, the system of transitional dynamics is given by[22]:

$$\begin{cases} \dot{Z}_1 = Z_1\left\{\Theta_2^{-1}\left[\dfrac{Z_2\left(\eta\dfrac{\eta\left(1-\tau_w\right)\phi}{\left(1-\tau_k\right)\left(1-\beta\right)\xi}\gamma - \xi\left(1-\ell_1\right)\omega_{dom}\Theta_2\left(1-\gamma\right)\right)}{1-\gamma} + \right.\right. \\ \left. + \dfrac{\dfrac{Y_1(K_1)}{K_1}\left[\left(\left(1-\tau_k\right)\left(1-\beta\right) - \eta\dfrac{\eta\left(1-\tau_w\right)\phi}{\left(1-\tau_k\right)\left(1-\beta\right)\xi}\right)\gamma - \left(\phi\left(1-\tau_w\right)\ell_1 - \xi\left(1-\ell_1\right)\right)\omega_{dom}\Theta_2\left(1-\gamma\right)\right]}{1-\gamma}\right] - \\ \left. -\dfrac{\rho+\delta\gamma-\omega_T\left(1-\gamma\right)}{1-\gamma} + \left(1+\tau_c\right)Z_1\right\} \qquad \textbf{(C87)} \\ \dot{Z}_2 = \Psi_K\left(\beta+\eta-\mu-1\right)\left(Z_2 - \dfrac{Y_1(K_1)}{K_1}\right) \qquad \textbf{(C88)} \end{cases}$$

[22] The new parameter, ω_T, is obtained after simplifying the ratio of aggregate lump sum taxes to total net wealth. As usual we assume that this parameter is an exogenous constant. In this specific case we will consider ω_T to be either very small and positive or even equal to zero.

We will not describe the dynamics of this system analytically as we did for the centralized economy. Instead, we will discuss the saddle path dynamics already described in the centralized economy and the convergence paths associated with that trajectory numerically, for both the centralized and decentralized economies. For this purpose we will use we use the Matlab software *pplane7*[23] developed by John Polking for evaluating two dimensional ODE systems. Details for this specific program can be obtained from Polking and Arnold (2004)[24].

We start this analysis by evaluating numerically the centralized transitions model from section *6.2.4.2.*, described by equations (C52) and (C53). The choice for numerical parameters for this case is given in table 5:

Parameter	ρ	γ	τ_c	T	ϕ	τ_w	β	η	μ	ℓ_1
Value	0,02	0,3	0,2	0	0,5	0,2	0,3	0,2	0,1	0,8
Parameter	A	g	N	l	δ	τ_k	θ	ξ	ω_{for}	Ψ_K
Value	0,1	0,4	400	0,25	0,06	0,2	0,2	0,2	-0,2	0,05

Table 5- Base parameters for phase plane and steady state evaluation of the centralized economy

One of the advantages of using *pplane7* is that it able us to obtain easily a set of trajectories in the phase plain that are useful to describe initial value dynamics. In figure 8, the saddle path trajectory is portrayed by the green line while all other meaningful trajectories show up in blue. Nullclines are depicted as the orange and rose straight lines. Direction fields and paths show that the behaviour of this system is similar to the one presented usually for closed economy models. There are two types of possible trajectories when an initial value problem is considered, in addition to the stable saddle trajectory. Those trajectories above the saddle path trajectory are completely divergent. Those that are bellow the saddle path trajectory converge to the other possible dynamic equilibrium for this system, where $Z_1 = 0$ and $Z_2 = \dfrac{Y_1(K_1)}{K_1}$, which has no economic interpretation.

[23] This software is available at http://math.rice.edu/~dfield/.

[24] An online version of this book can be obtained at http://www.owlnet.rice.edu/~math211/manual3.html. A brief description of *pplane7* is included in chapters 7 and 13 of this online version.

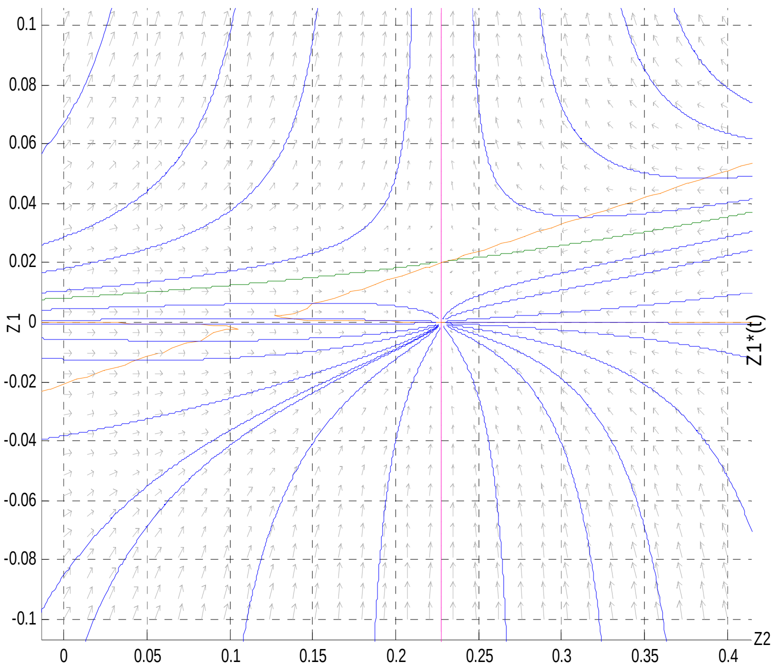


Fig. 8- Phase portrait and directional field with various trajectories simulated using the *Dormand Prince* algorithm

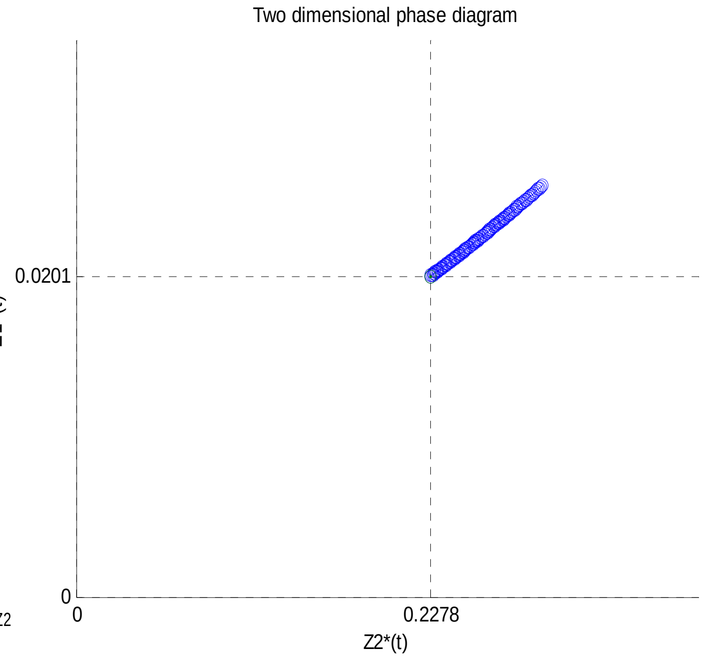


Fig. 9- Convergence in the phase plane from initial value to computed steady state

Using the information provided by *pplane7* about the numerically determined saddle path trajectory we can use one of the computed values, in order to show Z_1 and Z_2 time convergence to equilibrium. We can use the generic initial value problem solver from section *4.1.* of the appendix for this purpose and choose the initial computed value $\left(Z_{1,0}, Z_{2,0}\right) = \left(0.02584414;0,3\right)$. The phase diagram convergence dynamics is portrayed in figure 9 and the time paths for the variables convergence from the initial values to computed steady state are given by figures 10 and 11:

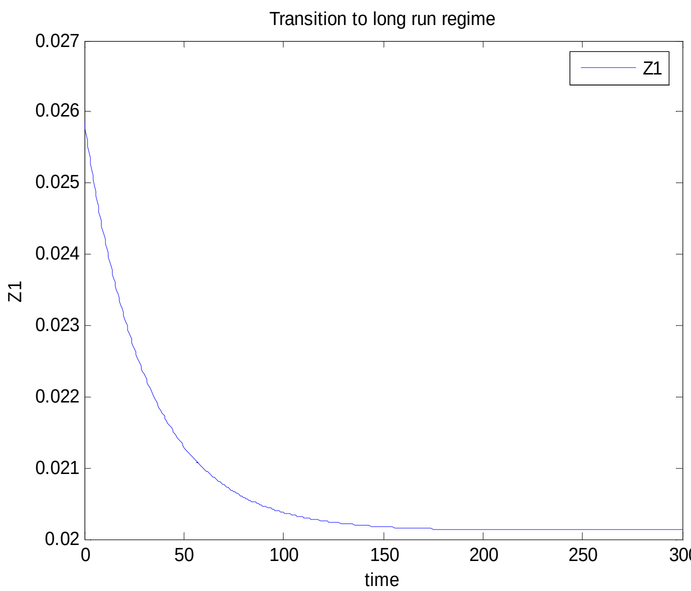


Fig. 10- Time transition to steady state for Z_1

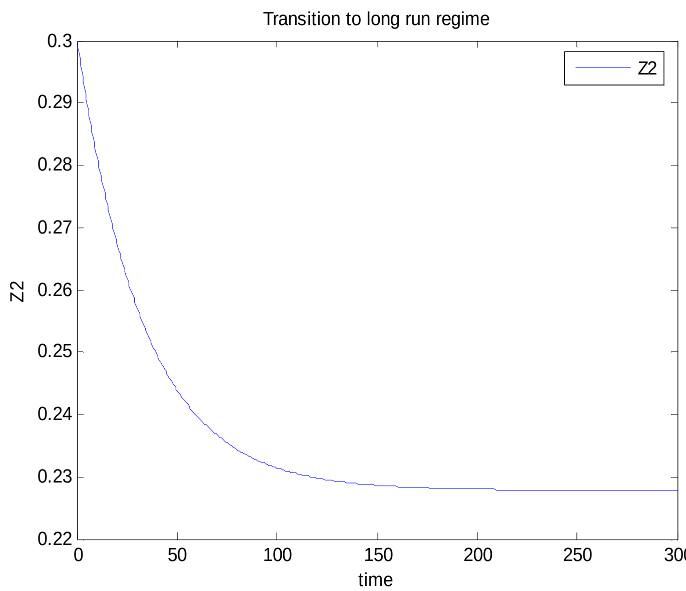


Fig. 11- Time transition to steady state for Z_2

The results obtained in this set of simulations confirm the analytical results already described in section *6.2.4.2.* and served the purpose of obtaining a complete comprehension of the existent dynamics in the phase plane framework. Additionally, we have confirmed that transitions to steady state equilibrium in this class of models are extremely long, as we can observe in figures 10 and 11. This is also a result that is similar to results obtained in closed economy models.

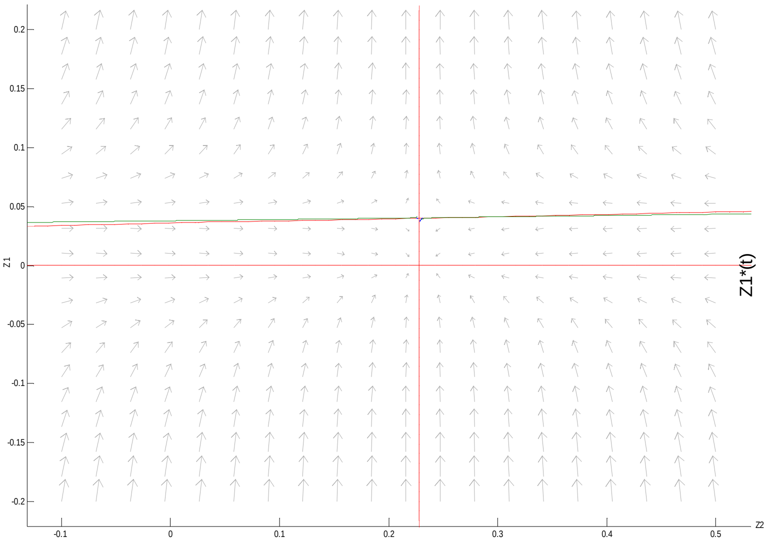
As we referred at the beginning of this section we have not performed any analytical analysis for the decentralized economy transitions model, as it would involve a duplication of the results already obtained for the centralized model, except for the specific parameter expressions involved. In order to determine the dynamics of this economy we will again use *pplane7*, with the aim of studying the various possible trajectories in the phase plane for the system given by (C87) and (C88).

Due to the wage income effect that exists in the decentralized intertemporal budget constraint we have to adjust some parameters, ξ, ℓ_1 and η, so that both the $\dot{Z}_1$ nullcline and the stable eigenvector slope remain positive. This choice of parameters is therefore mainly illustrative, as the results will vary with different sets of feasible parameters, between an almost horizontal $\dot{Z}_1$ nullcline and stable eigenvector, which can have either a positive or negative slope. This specific choice of parameters follows bellow in table 6.

Parameter	ρ	γ	τ_c	ω_T	ϕ	τ_w	β	η	μ	ℓ_1
Value	0,02	0,3	0,2	0	0,5	0,2	0,3	0,15	0,1	0,7
Parameter	$\boldsymbol{A}$	$\boldsymbol{g}$	$\boldsymbol{N}$	$\boldsymbol{l}$	δ	τ_k	θ	ξ	ω_{dom}	Ψ_K
Value	0,1	0,4	400	0,25	0,06	0,2	0,2	0,3	0,5	0,05

Table 6- Base parameters for phase plane and steady state evaluation of the decentralized economy

We can observe in figure 12 that although both slopes have flattened considerably, due to the wage effect that occurs in the decentralized informal economy, the dynamics of this system remain equal to the dynamics of the centralized economy for this specific set of parameters. Both slopes remain positive and the slope of the $\dot{Z}_1$ nullcline is steeper than the one of the stable eigenvector.



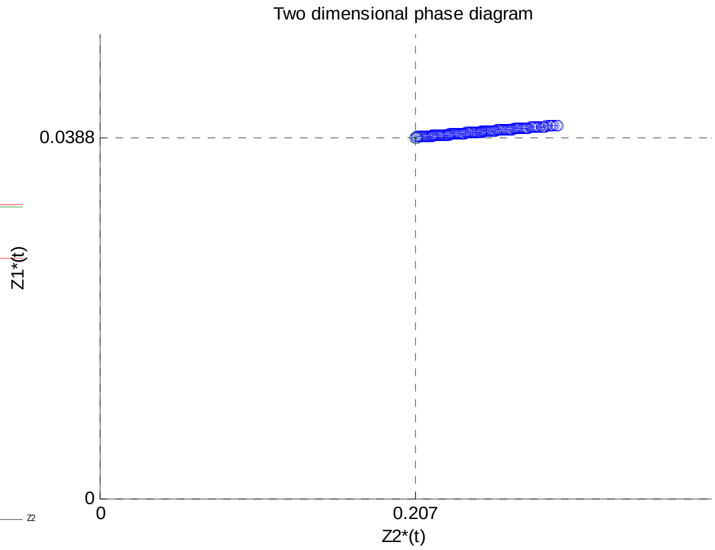
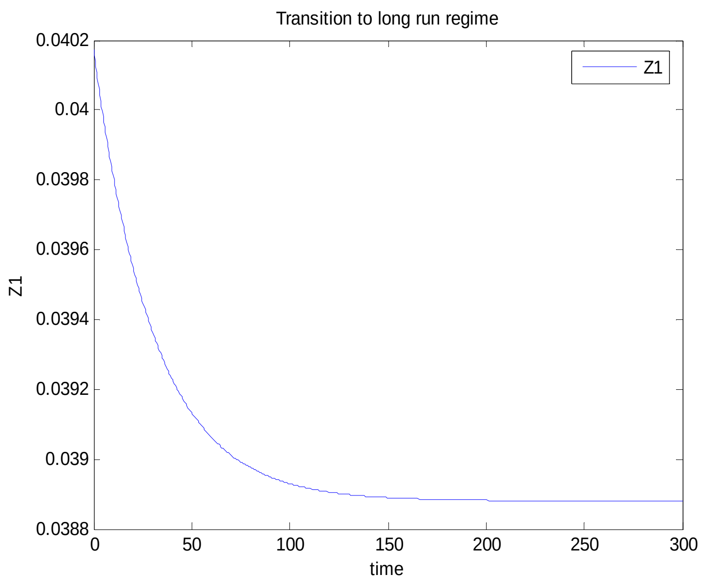


Fig. 12- Phase portrait and directional field simulated using the *Dormand Prince* algorithm

Fig. 13- Convergence in the phase plane from initial value to computed steady state

Using the information provided by *pplane7* about the numerically determined saddle trajectory we can use one of the computed values, in order to show Z_1 and Z_2 time convergence to equilibrium. Again, we use the generic initial value problem solver from section *4.1.* of the appendix for this purpose and choose the initial computed value $(Z_{1,0}, Z_{2,0}) = (0{,}04017376857; 0{,}3)$. The two dimension phase diagram of figure 13 confirms that the stable trajectory has a positive slope in this specific case.

Figures 14 and 15 confirm that we are again in the presence of very long transitions to long run steady state, which comes in line with our expectations.



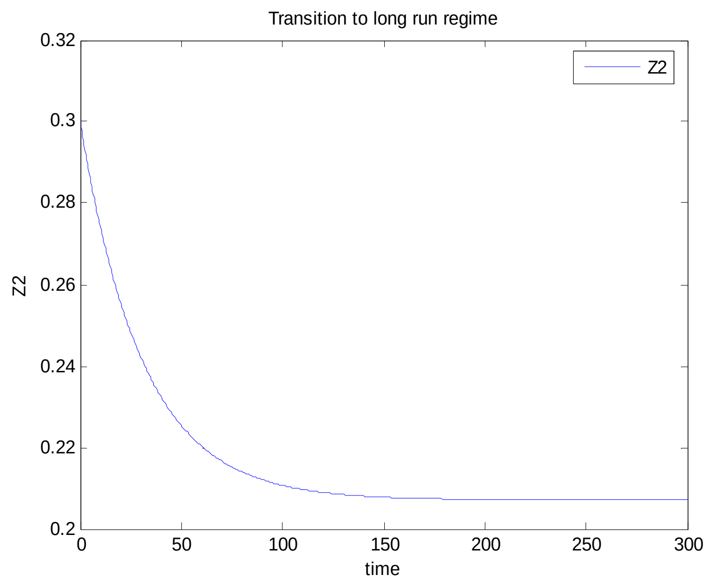


Fig. 14- Time transition to steady state for Z_1

Fig. 15- Time transition to steady state for Z_2

The main importance of this section was to confirm the analytical results obtained in section *6.2.4.2.*, which follow closely those of closed economy models, and to present one feasible dynamic portrait of the decentralized economy transitions model. Additional simulations have showed that the structure of the decentralized case can vary depending on the choice of parameters that we make. This result is a consequence of the additional wage income effect from both sectors that is introduced in this model, which causes both the $\dot{Z}_1$ nullcline and stable eigenvector slope to flatten considerably.

6.3.4. Numerical Simulation Example for Comparative Transitional Dynamics

One interesting feature about the simplified models with transitions is that we can analyse welfare outcomes during transitions to the long run. For this purpose we will use a routine[25] for solving boundary value problems, BVP4C from Matlab, in order to describe some qualitative results for comparative dynamics. The specific routine that we will use for simulations is described in section *4.2.* of the appendix. Unfortunately, as we choose a set of standard transformations, (C46) and specifically (C47), for obtaining the transformed model of transitions, we ended up by not differentiating completely between each specific engine of growth. The result of this option is portrayed by the dynamic system equilibrium for Z_2, the state-like variable, that describes the engine of growth, in the centralized and decentralized economies, which is equal to $\frac{Y_1(K_1)}{K_1}$ in both systems equilibrium. Consequently, the systems of transitions for both economies do not retain all features that differentiate one from another, specifically when describing the long run regime dynamics between growth and policy decisions. This does not mean that transitional dynamics are not described correctly by our choices for transformed variables, however, we should define a rule of policy, in order to guarantee that there are no differences between both engines of growth. The other option would be to assume a transformation for Z_2 that retained the specific differences described by both accumulation indifference equations, (C29) and (C77). This problem can be also overcome with our routine, by manipulating computed steady states, with the objective of assuming the long run differences between the growth engines of the two economies.

The rule of policy that guarantees both engines of growth are comparable is the rule for optimal spending policy without commitment to any specific rule for taxes on capital returns, $(1-g)=(1-\beta)(1-\tau_k)$. We will use this straightforward assumption to analyse the comparative dynamics of the original transitions systems sharing an equivalent long run growth equilibria. The choice of parameters will follow those from table 7, for the initial value simulation of the decentralized economy, except for the spending rule already described.

[25] This routine was based upon an initial routine provided from my thesis adviser Miguel St. Aubyn. We just transformed the initial routine slightly in a way that it is able to simulate comparative dynamics with multiple exogenous shocks for two systems simultaneously.

Parameter	ρ	γ	τ_c	ω_T	ϕ	τ_w	β	η	μ	ℓ_1	ω_{for}
Value	0,02	0,3	0,2	0	0,5	0,2	0,3	0,15	0,1	0,7	$\omega_{dom}-1$
Parameter	A	g	N	l	δ	τ_k	θ	ξ	ω_{dom}	Ψ_K	
Value	0,1	$1-(1-\beta)(1-\tau_k)$	400	0,25	0,06	0,2	0,2	0,3	0,5	0,05	

Table 7- Base parameters for comparative dynamics simulation

Our first example will use the original systems of transitions to describe the dynamics of a positive and permanent shock in exogenous technology, A, to 0,11. As both models share the exact rate of growth no specific issue occurs in this case, except for the scale in which Z_2 is given. Figures 16 and 17 depict the permanent technology shock:

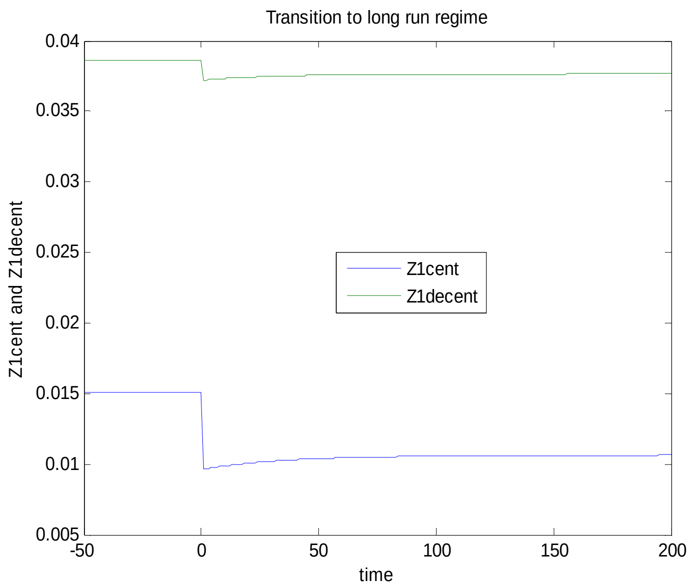


Fig. 16- Transitions for Z_1 after technology shock

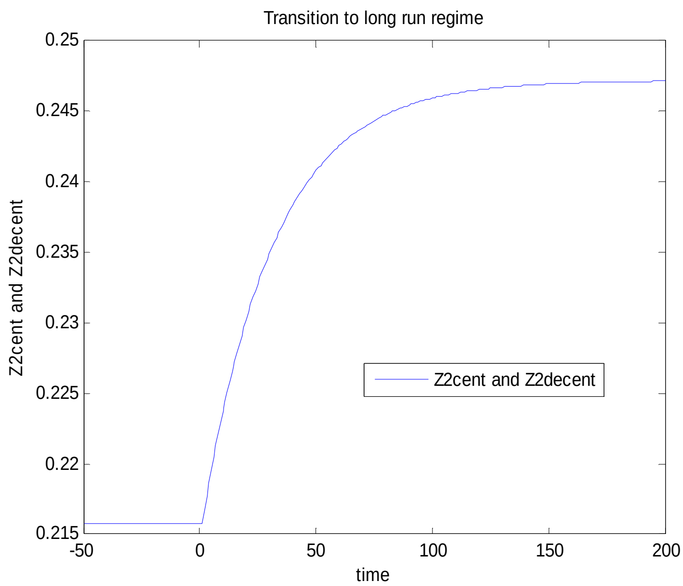
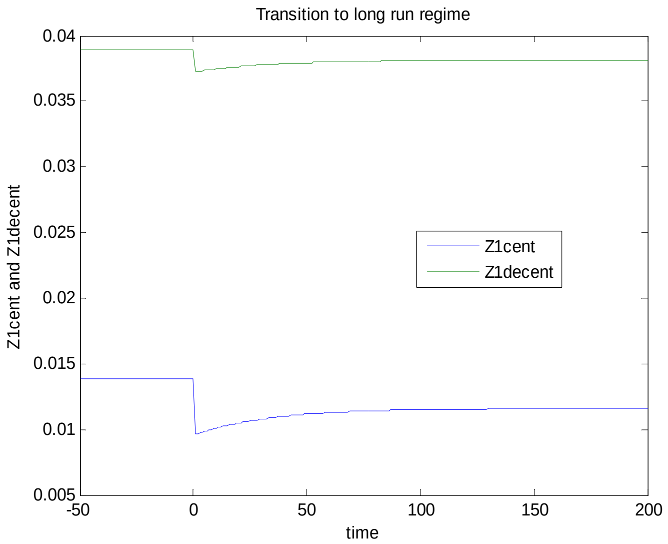


Fig. 17- Transitions for Z_2 after technology shock

As expected, the rate of growth increases monotonically to the new steady state and the effect on the consumption to net wealth ratio is negative in the long run. This effect is relatively larger for the centrally planned economy, than it is for the decentralized economy, meaning that the central planner will have an increased preference to accommodate the intertemporal financial position of the economy even further, in the presence of positive technology shocks. Agents follow this behaviour but will not be as orthodox as the central planner.

Our second example will deal with a shift on fiscal policy. In order to do so, we will manipulate computed steady state results and trajectories, to assume specific policy outcomes on growth, and relax the optimal spending rule with no commitment to be an exogenous parameter equal to 0,4. We will assume the innovation faced by these economies, to be a simultaneous policy shift to optimal growth policy, where capital income taxes, τ_k , are reduced to zero, and public spending, g, accommodates the fiscal shock by reducing overall spending to be equal to the optimal spending rule. Results for this simulation follow bellow in figures 18 and 19:



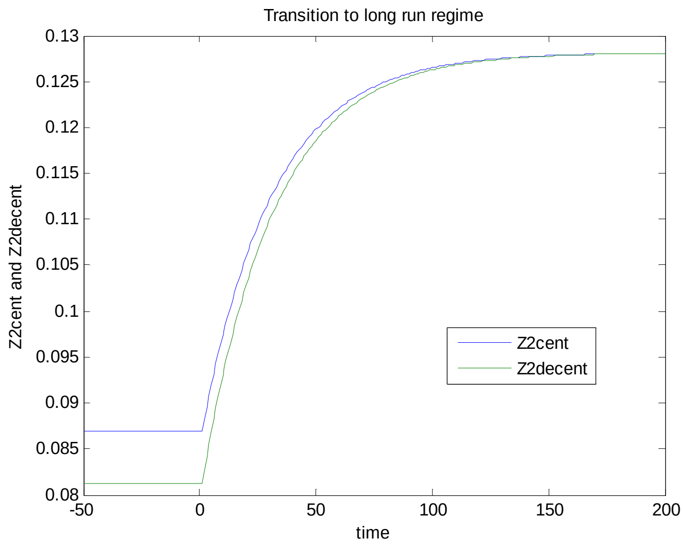


Fig. 18- Transitions for Z_1 after fiscal shock

Fig. 19- Transitions for Z_2 after fiscal shock

The results obtained for this simulation are very similar to the ones obtained for the technological shock. The qualitative analysis of both examples reveals that both the transitions and the long run outcomes of these innovations follow very similar patterns, although now, only long run growth outcomes match, as initial conditions and scale differ for Z_2 between both economies. Fig. 19 shows a loss of welfare during the transition to optimal growth, despite in the long run the welfare outcome is optimal. This effect, although relatively small, is aggravated by the very long transitions that we discussed in section 6.3.3.2.. We could have considered a different value for initial government expenditure, where both intertemporal welfare outcomes would be positive. This would involve considering an initial long run growth rate assumption for the centrally planned economy that would be smaller, than the initial conditions faced by the decentralized economy, which does not seem to be a reasonable assumption. Nevertheless, we have showed in this section how to deal with different short to medium run welfare outcomes, when transitions are introduced by an informal sector, using a simple boundary value problem routine. This analysis can be extended to a wide range of possible innovations and fiscal policy decisions that we have not discussed here. Further simulations may also consider long run convergence of leisure and labour allocation decisions that we have not yet considered in our analytical and numerical analysis.

6.3.5. Simulation and Results for the Decentralized Economy with Adjustment Costs

In this section we extend the convergence analysis that we had already performed in section 5.3.4. to the full information decentralized economy with an informal sector and adjustment costs. The main assumptions for leisure and government taxing and spending already considered in section 5.3.4.

remain unchanged. In addition, we will impose the same exogenous assumptions for labour allocation decisions, in order to limit our analysis to the four dimensional dynamical system described bellow:

$$\begin{cases} \dot{\boldsymbol{C}} = \left(\dfrac{\rho-\boldsymbol{r}}{\gamma-1}\right)\boldsymbol{C} & \textbf{(C89)} \\ \dot{\boldsymbol{B}} = (1+\tau_{\boldsymbol{c}})\boldsymbol{C} + \boldsymbol{I}_1\left(1+\boldsymbol{h}\dfrac{\boldsymbol{I}_1}{\boldsymbol{K}_1}\right)\Theta_2 + \boldsymbol{rB} + \boldsymbol{T} - & \\ -\left[(1-\tau_{\boldsymbol{k}})(1-\beta)+\phi(1-\tau_{\boldsymbol{w}})\ell_1\right]\boldsymbol{Y}_1(\boldsymbol{K}_1) - \left[\xi(1-\ell_1)+\eta\dfrac{\eta(1-\tau_{\boldsymbol{w}})\phi}{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi}\right]\boldsymbol{Y}_2(\boldsymbol{K}_1) & \textbf{(C90)} \\ \dot{\boldsymbol{K}}_1 = \boldsymbol{I}_1 - \delta\boldsymbol{K}_1 & \textbf{(C91)} \\ \dot{\boldsymbol{I}}_1 = \dfrac{\boldsymbol{I}_1^{\;2}}{2\boldsymbol{K}_1} + \boldsymbol{rI}_1 + & \\ +\left[\delta+\boldsymbol{r}-\Theta_2^{\;-1}\left((1-\tau_{\boldsymbol{k}})(1-\beta)\dfrac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1}+\eta\dfrac{\eta(1-\tau_{\boldsymbol{w}})\phi}{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi}\dfrac{\boldsymbol{Y}_2(\boldsymbol{K}_1)}{\boldsymbol{K}_1}\right)\right]\dfrac{\boldsymbol{K}_1}{\boldsymbol{h}} & \textbf{(C92)} \end{cases}$$

As usual we detrend this dynamical system assuming that there exists a common growth rate. We obtain the following scaled system that we will use for simulation purposes:

$$\begin{cases} \dot{\tilde{\boldsymbol{c}}} = \left(\dfrac{(\rho-\boldsymbol{r})}{(\gamma-1)}-\Psi\right)\tilde{\boldsymbol{c}} & \textbf{(C93)} \\ \dot{\tilde{\boldsymbol{b}}} = (1+\tau_{\boldsymbol{c}})\tilde{\boldsymbol{c}}+\tilde{\boldsymbol{i}}_1\left(1+\dfrac{\boldsymbol{h}\tilde{\boldsymbol{i}}_1}{2\tilde{\boldsymbol{k}}_1}\right)+(\boldsymbol{r}-\Psi)\tilde{\boldsymbol{b}}+\boldsymbol{Te}^{-\Psi\boldsymbol{t}}-\left[\phi(1-\tau_{\boldsymbol{w}})+(1-\beta)(1-\tau_{\boldsymbol{k}})\right]\dfrac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1}\tilde{\boldsymbol{k}}_1 - & \\ -\left[\xi(1-\ell_1)+\eta\dfrac{\eta(1-\tau_{\boldsymbol{w}})\phi}{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi}\right]\boldsymbol{D}\Omega\left(\dfrac{\eta(1-\tau_{\boldsymbol{w}})\phi}{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi}\right)^{\eta}\tilde{\boldsymbol{k}}_1^{\;\beta+\eta-\mu}\boldsymbol{e}^{\Psi\boldsymbol{t}(\beta+\eta-\mu-1)} & \textbf{(C94)} \\ \dot{\tilde{\boldsymbol{k}}}_1 = \tilde{\boldsymbol{i}}_1-(\delta+\Psi)\tilde{\boldsymbol{k}}_1 & \textbf{(C95)} \\ \dot{\tilde{\boldsymbol{i}}}_1 = \dfrac{\tilde{\boldsymbol{i}}^2}{2\tilde{\boldsymbol{k}}_1}+(\boldsymbol{r}-\Psi)\tilde{\boldsymbol{i}}_1 + & \\ +\left(\delta+\boldsymbol{r}-\Theta_2^{\;-1}\left[(1-\tau_{\boldsymbol{k}})(1-\beta)\dfrac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1}+\eta\boldsymbol{D}\Omega\left(\dfrac{\eta(1-\tau_{\boldsymbol{w}})\phi}{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi}\right)^{\eta+1}\tilde{\boldsymbol{k}}_1^{\;\beta+\eta-\mu-1}\boldsymbol{e}^{\Psi\boldsymbol{t}(\beta+\eta-\mu-1)}\right]\right)\dfrac{\tilde{\boldsymbol{k}}_1}{\boldsymbol{h}} & \textbf{(C96)} \end{cases}$$

The same balanced growth path hypotheses obtained in section *5.3.4.* satisfy this system dynamics for the existence of a common growth rate. Again we choose the rule $\Psi = \dfrac{\tilde{\boldsymbol{i}}}{\tilde{\boldsymbol{k}}} + \boldsymbol{r}$ as the most feasible one in this adjustment costs framework. The scaled investment to capital ratio that guarantees equilibrium in this dynamical system is obtained bellow:

$$\frac{\tilde{\boldsymbol{i}}}{\tilde{\boldsymbol{k}}} = \left[\left(\delta+\boldsymbol{r}-\Theta_2^{\;-1}\left[(1-\tau_{\boldsymbol{k}})(1-\beta)\frac{\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{K}_1}+\eta\boldsymbol{D}\Omega\left(\frac{\eta(1-\tau_{\boldsymbol{w}})\phi}{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi}\right)^{\eta+1}\tilde{\boldsymbol{k}}_1^{\;\beta+\eta-\mu-1}\boldsymbol{e}^{\Psi\boldsymbol{t}(\beta+\eta-\mu-1)}\right]\right)\frac{2}{\boldsymbol{h}}\right]^{\frac{1}{2}} \quad \textbf{(C97)}$$

This expression can be reduced to an endogenous expression when we consider its asymptotical properties. In the long run equilibrium the effect of the marginal productivity of capital in the informal sector will decay to zero, due to the neo-classical assumptions of the informal production function, and the expression obtained will be very similar to the one obtained in section *5.3.4.*.

$$\lim_{t\to\infty}\frac{\tilde{\boldsymbol{i}}}{\tilde{\boldsymbol{k}}}=\left[\left(\delta+\boldsymbol{r}-\Theta_2^{-1}\left[(1-\tau_k)(1-\beta)\left(\boldsymbol{A}\left[\boldsymbol{g}\boldsymbol{N}\right]^{\beta}\right)^{\frac{1}{1-\beta}}\left[\ell_1\left(1-\boldsymbol{l}\right)\right]^{\frac{\phi}{1-\beta}}\right]\right)\frac{2}{\boldsymbol{h}}\right]^{\frac{1}{2}} \qquad \textbf{(C98)}$$

We can now use the generic initial value problem solver from section *4.1.* of the appendix in order to present some convergence hypotheses, as we did in section *5.3.4.*. For this purpose we choose a set of numerical values that are similar to the ones already used for tackling the previous initial value problems.

Parameter	ρ	γ	τ_c	$\boldsymbol{T}$	ϕ	τ_w	β	η	μ	ℓ_1
Value	0,02	0,3	0,2	0,1	0,5	0,2	0,3	0,2	0,1	0,8
Parameter	$\boldsymbol{A}$	$\boldsymbol{g}$	$\boldsymbol{N}$	$\boldsymbol{l}$	δ	τ_k	θ	ξ	$\boldsymbol{D}$	
Value	0,1	0,4	400	0,25	0,06	0,2	0,2	0,2	0,05	

Table 8- Numerical values for base parameters

As we did in section *5.3.4.*, we will assume an economy where consumption and capital are growing initially bellow their long run regime. Foreign debt/wealth is growing in initial state above its long run regime. The world interest rate faced by this small open economy is 5%, making it now more profitable to hold domestic assets than foreign assets. Investment, however, has recovered making the scaled investment to capital ratio to be negative at an initial value. Simulated convergence results when adjustment costs are equal to -5 are presented bellow:

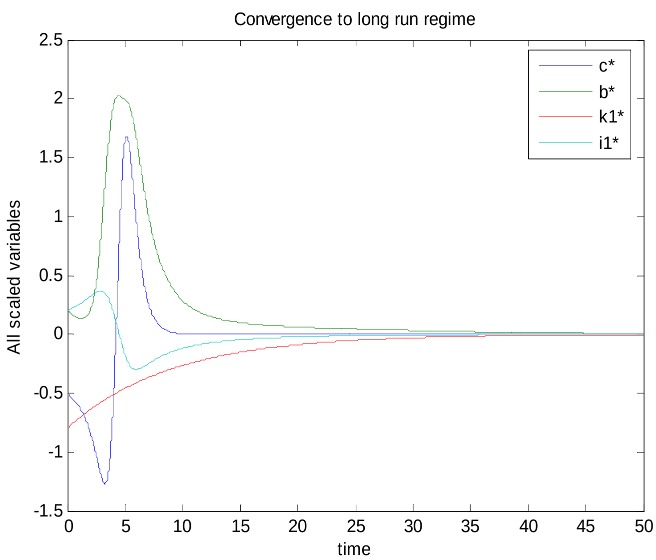


Fig. 20- Convergence to long run regime

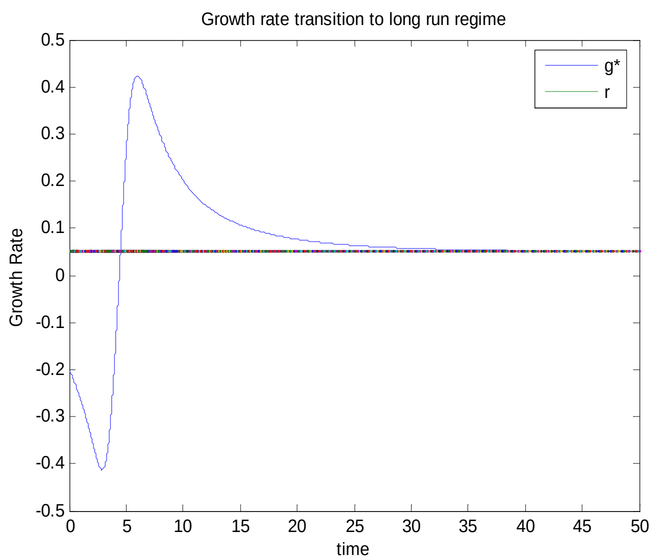


Fig. 21- Growth rate transition

The double over-shoot effect or oscillatory convergence behaviour on scaled consumption, investment and foreign debt/assets, suggest that our choice for adjustment costs was not correct. All simulations using negative values for adjustment costs that we performed, following similar initial macroeconomic assumptions, seem to bear the same problem and different initial values for scaled investment does not seem to alter this situation. We can either increase the world exchange rate or change the sign of adjustment costs parameter, in order to obtain a more feasible behaviour from our variables that is similar to the one in section *5.3.4.*, for this specific simulation.

We choose to change adjustment costs to be equal to 2 and initial values for scaled investment and capital that are closer to zero. This seems enough to correct the oscillatory behaviour of our variables as it is shown in figures 22 and 23:

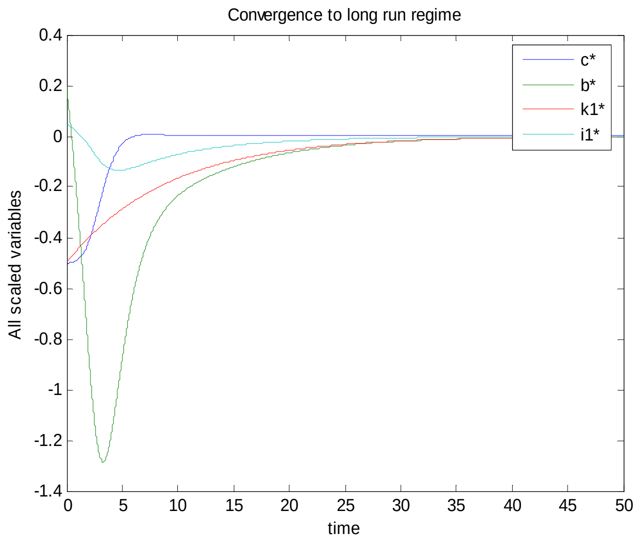


Fig. 22- Convergence to long run regime

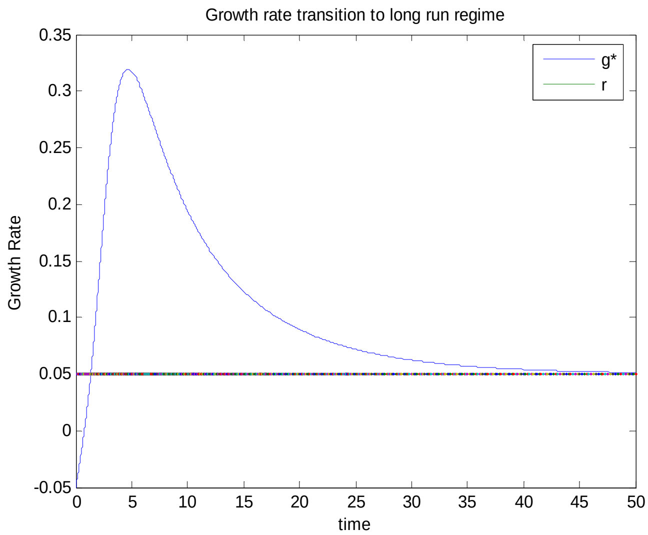


Fig. 23- Growth rate transition

Although this choice of initial values and adjustment costs is sufficient to obtain a more feasible set of time paths, this result is sensitive to changes on initial values of both scaled capital and investment and double over-shoot paths of convergence are likely to arise in many situations that are also plausible to be considered.

The introduction of an informal sector has brought on two innovations regarding the results discussed in section *5.3.4.*. Firstly, the additional income effect caused by the informal productive sector seems to be responsible for both the oscillatory behaviour, that we have discussed in this section, and also for the reduction of the over-shoot effect on scaled foreign debt convergence path, in the case of a economy growing above its long run regime that we presented in section *5.3.4.*. This specific case happens only in specific sets of parameters and initial values and in the majority of simulations, the convergence behaviour, already discussed in section *5.3.4.*, is more likely to occur. We decided not to present these results in this section because, with both exceptions discussed for foreign debt/assets, all other qualitative analyses remain similar. Secondly, the introduction of an informal sector has widened the range of admissible initial values that guarantee convergence to a long run steady state growth equilibrium. This is a clear result, as almost all performed simulations have converged to the expected steady state growth equilibrium, which is not the case when we consider the model presented in section *5.3.4.* with just a formal productive sector.

6.4. Concluding Remarks

As we suggested in this chapter, one form of interpreting the existence of an informal sector in a developed economy is to consider that the long run is just a sum of short and medium run specific periods and that in all of these shorter periods, opportunities for entrepreneurial informal activities arise. Using this hypothesis, we developed a framework based on the ideas from Jones and Manuelli (1990) and on the assumption that not all the information about the neo-classical relation between the formal and informal sectors of the economy is incomplete. These assumptions allowed us to develop an economy, which is not only comparable with the class of economies that we have discussed in chapters *4.* and *5.*, but also allowed us to discuss both policy implications and relevant solution strategies for this dynamic problem. Therefore, we can conclude that the extension we proposed is not only feasible to tackle our specific case of an informal sector but can also be used as a benchmark for further extensions of small open economy models.

On the other hand, we have to assume that the reduction from a three state dynamic optimal control problem to an economy with two sectors expressed in formal sector variables and the strict additional hypotheses that we have imposed, weakens our assumptions about policy and limits our insight to the limited range of alternatives that we have discussed throughout this chapter and section *2.* of the appendix. Furthermore, all the dynamic analysis that we have performed in the transitions models suggests that not only we did not solve the problems of long and rigid transitions but, in fact, have caused this problem to worsen, when compared to the simpler case of a centralized closed economy model. So a question remains, are the possible factors for transitions to long run regimes persistent and therefore transitions will be long and rigid, as this methodology suggests, or is this extension of endogenous growth models not entirely correct or at least still incomplete or misused in this context? The answer to this question lies on both the empirical evidence and the related paradigms about economic social systems and long run growth dynamics. If we consider that even in a developed economy, dealing with the informal economy is a long process, because old habits die hard and new opportunities for illegal activities arise, than we may consider the hypotheses described in this chapter as a reasonable introduction to this phenomena in endogenous growth models. If we consider otherwise, than we must redefine our assumptions about this issue in order to deliver a better understanding of this subject in the future.

7. Conclusions

Whenever we decide to undertake a vast research project on mathematical economics that involves dealing with a large subject in literature, with the purpose of improving the insight of existing proposals and specific applications, and propose additional extensions, we tend to end up in a path where there are more doors opened than the ones we closed. As we initially expected, this straightforward metaphor is a good broad description for the research hypotheses that we proposed and the particular outcomes we obtained. However, there were still a number of issues addressed, where we were able to deliver several conclusive results and advance some interesting proposals. We showed that the optimal policy outcome of the Turnovsky (1999) open economy model with public infrastructure is equivalent to the decentralized outcome of our open economy extension of the Romer (1986) class of increasing returns endogenous growth models, with positive externalities resulting from capital agglomeration. We also proposed a set of feasible hypotheses for introducing an informal sector in this class of increasing returns models, based upon micro foundations. Finally, we showed how to solve for the simplified comparable case and described an analytical solution to the specific models of transitions in the Jones and Manuelli (1990) fashion, although in this specific case, we had to rely on a set of restrictive assumptions, in order to extend this method to an open economy model. On the other hand, we were not able to extend the numerical simulations as extensively as we desired for the large scale systems that we proposed. This has limited our simulations to a set of restricted specifications and qualitative examples. In the future, this is clearly an issue to tackle, taking advantage of new and recent methods to deal with higher order differential-algebraic systems, in order to obtain more flexible and reliable simulations for this specific class of models.

As it is usual in academic and scientific research documents, we end up this dissertation proposing a set of future directions for related research. We have already discussed thoroughly the set of opportunities that are still available for the development of improved specifications for simulation, with the purpose of delivering a better understanding of this class of economies and the extensions we proposed. Therefore, we will conclude by advancing a set of theoretical proposals that are already present in contemporary literature and can be used to extend this class of multiple fiscal instruments endogenous growth models. We discussed most of these extensions in section *1.1.*, when we introduced the subject of optimal fiscal policy in models of endogenous growth. In that section, we discussed a set of existing research proposals, which extend other features of fiscal policy and government spending in a framework very similar to ours. Those proposals included assuming debt financed expenditures by government authorities, existence of non-productive public spending and introduction of human capital, in addition to public productive investment, as a source of increasing returns and endogenous growth dynamics. In this specific hypothesis, it can be considered that human capital is a function of public services, physical capital and existing human capital, following Agénor

(2005) proposal. All of these assumptions will undoubtedly approach this class of models to real world conditions faced by developed economies, although increasing the complexity of dealing with additional sets of variables and assumptions in a dynamic mathematical framework.

The introduction of an informal sector in an endogenous growth framework for a developed economy is still a literature subject open to a wide discussion. Our proposal was based on a specific set of hypotheses, primarily consistent with the existence of arbitrage conditions for informal activities, arising from fiscal policy distortions at a micro level during transitions. Future extensions may include barriers to entry, such as legal barriers and other bureaucratic costs, which have financial impacts in firm's decisions and, as a result, act as an additional incentive for informal activities. The inclusion of specific government spending, for fiscalization of these activities, is an obvious candidate for the non-productive government expenditures hypothesis that we already discussed. All these proposals come in line with recent research about the different sources for informality, in both developed and developing economies. Therefore, they represent interesting candidates to consider, in addition to government taxation, for future proposals about this subject in a long run economic growth environment.

Appendix

1. Analytical Framework for the Centralized Economy with no Government Sector and no Adjustment Costs [26]:

The present value Hamiltonian for this optimization problem is:

$$\boldsymbol{H}^{*}=\frac{1}{\gamma}\left(\left[\boldsymbol{C}\;/\;N\;\right]\boldsymbol{l}^{\theta}\right)^{\gamma}+\lambda\left[\boldsymbol{C}+\boldsymbol{I}+\boldsymbol{r}B\text{ - }AK\,(1\text{ - }\boldsymbol{l})^{\phi}N^{\beta}\right]+\boldsymbol{q}\left[\boldsymbol{I}\text{ - }\delta K\right]$$

The Pontryagin maximum conditions for this optimal control problem are:

Optimality Conditions

$$\boldsymbol{C}^{\gamma-1}\boldsymbol{N}^{-\gamma}\boldsymbol{l}^{\theta\gamma}+\lambda=0 \tag{D1}$$

$$\theta\boldsymbol{C}^{\gamma}\boldsymbol{N}^{-\gamma}\boldsymbol{l}^{\theta\gamma-1}+\lambda\phi\boldsymbol{A}\boldsymbol{K}(1-\boldsymbol{l})^{\phi-1}\boldsymbol{N}^{\beta}=0 \tag{D2}$$

$$\lambda+\boldsymbol{q}=0 \tag{D3}$$

Admissibility Conditions

$\boldsymbol{B}_0=\boldsymbol{B}_{(0)}$ and $\boldsymbol{K}_0=\boldsymbol{K}_{(0)}$

Multipliers Conditions

$$\dot{\lambda}=\lambda(\rho-\boldsymbol{r}) \tag{D4}$$

$$\dot{\boldsymbol{q}}=\boldsymbol{q}\left(\rho+\delta\right)+\lambda\left(\boldsymbol{A}\left(1-\boldsymbol{l}\right)^{\phi}\boldsymbol{N}^{\beta}\right) \tag{D5}$$

State Conditions

$$\dot{\boldsymbol{B}}=\boldsymbol{C}+\boldsymbol{I}+\boldsymbol{r}\boldsymbol{B}-\boldsymbol{A}\boldsymbol{K}\left(1-\boldsymbol{l}\right)^{\phi}\boldsymbol{N}^{\beta} \tag{D6}$$

$$\dot{\boldsymbol{K}}=\boldsymbol{I}-\delta\boldsymbol{K} \tag{D7}$$

Transversality Conditions

$$\lim_{\boldsymbol{t}\to\infty}\lambda\boldsymbol{B}\boldsymbol{e}^{-\rho\boldsymbol{t}}=0 \tag{D8}$$

$$\lim_{\boldsymbol{t}\to\infty}\boldsymbol{q}\boldsymbol{K}\boldsymbol{e}^{-\rho\boldsymbol{t}}=0 \tag{D9}$$

Using the optimality condition in (D1) and substituting in (D2) we can derive the expressions that will determine the investment and leisure decisions in this economy:

$$\frac{\boldsymbol{C}}{\boldsymbol{Y}}=\frac{\phi}{\theta}\frac{\boldsymbol{l}}{\left(1-\boldsymbol{l}\right)} \tag{D10}$$

$$\boldsymbol{q}=-\lambda \tag{D11}$$

The relation expressed in (D10) determines optimal leisure and labour decisions and it is easy to show that it renders a constant relation in an endogenous framework. When dealing with this class of endogenous growth models, time varying variables must share the same rate of growth, therefore for all periods, the rate of growth of consumption and output must be equal. It is clear then, that the

[26] The objective of this section of the appendix is to develop a simpler model for the small open economy than the one considered with investment adjustment costs, in order to produce some valuable analytical results.

output-consumption ratio will be determined solely by initial values of output and consumption. However, if we want to obtain an exact condition for these optimum relations we must solve the following polynomial expression:

$$\boldsymbol{l}^{\phi} - \boldsymbol{l} + \frac{\boldsymbol{C}\,\theta}{\boldsymbol{A}\boldsymbol{K}\boldsymbol{N}^{\beta}\phi} = 0 \qquad \textbf{(D12)}$$

Equation (D11) states that optimal investment decisions must satisfy the equality between the shadow price of capital and the marginal value of foreign bonds (in this case, λ, is the marginal value of debt) must be equal for every period. This condition arises when assuming that there are no adjustment costs on investment and is needed to assure that individuals will not choose to hold only foreign bonds and no domestic capital, or hold domestic capital at the cost of an ever increasing debt. The condition needed for this not to happen can be obtained by two simple strategies.

First, we can obtain two motion equations for $\boldsymbol{C}$ using the optimal conditions of (D1) and (D11), taking time derivatives and then substituting it in the multipliers conditions (D4) and (D5). The following equations of motion for $\boldsymbol{C}$ are obtained:

$$\dot{\boldsymbol{C}} = \frac{\rho - \boldsymbol{r}}{\gamma - 1}\boldsymbol{C} \qquad \textbf{(D13)}$$

$$\dot{\boldsymbol{C}} = \frac{\rho + \delta - \boldsymbol{A}\left(1 - \boldsymbol{l}\right)^{\phi} \boldsymbol{N}^{\beta}}{\gamma - 1}\boldsymbol{C} \qquad \textbf{(D14)}$$

As both differential equations for $\boldsymbol{C}$ were obtained using the optimal conditions for consumption and investment, and each of the co-state equations, it is straightforward to assume that (D13) must be equal to (D14). This will only happen if the following condition is imposed:

$$\boldsymbol{r} = \boldsymbol{A}\left(1 - \boldsymbol{l}\right)^{\phi} \boldsymbol{N}^{\beta} - \delta \qquad \textbf{(D15)}$$

This condition implies that aggregate marginal productivity of capital minus the rate of depreciation must be equal to the exogenous given world interest rate. The same condition can also be obtained by using optimal condition (D11), substituting it in the multiplier condition (D5) and then solving the dynamical system for the two co-state variables. The system obtained will be the special degenerate case that arises in endogenous growth models, where both variables must share the same rate of growth, which only occurs when condition (D15) is imposed.

This special case that arises when considering no adjustment investment costs is interesting, not only to develop strategies that impose conditions in domestic and foreign capital accumulation, but also to understand some of the problems that arise when dealing with economies that are widely exposed to international markets, in a long run endogenous growth framework. The questions addressed here will have other important implications when considering the decentralized equilibrium, where there is incomplete information in national capital markets.

2. The Maximization Problem in the Two Sector Economy

In this section of the appendix a two sector dynamic optimization problem is presented. This formulation adds an informal sector to the maximum problems presented in section *5.2.* and *5.3.* by considering that now there are two capital accumulation equations and two optimal control conditions for investment.

2.1. The Centrally Planned Economy Maximum Problem

The maximum problem for the centrally planned economy with an informal sector is:

$$\underset{C,l,\ell_1,I_1,I_2}{MAX} \; U = \int_0^{\infty} U\left(C,l\right)e^{-\rho t}dt \qquad \textbf{(D16)}$$

subject to :

$$\begin{cases} \dot{B} = C + \Phi\left(I,K\right) + rB - (1-g)Y_1\left(K_1,\ell_1,l\right) - Y_2\left(K_2,\ell_1,l\right) & \textbf{(D17)} \\ \dot{K}_1 = I_1 - \delta K_1 & \textbf{(D18)} \\ \dot{K}_2 = I_2 - \delta K_2 & \textbf{(D19)} \end{cases}$$

For reasons of simplification we will not use functional forms in this section of the appendix. We consider that the rate of depreciation is the same for both the formal and the informal sector and that each specific technology depends only in the capital aggregates, which are specific to each sector. We also consider investment faces a convex cost function, which depends on formal and informal capital and investment. We will not assume if this function is given separately for each sector or in terms of an aggregate for both sectors.

The present value Hamiltonian for this optimization problem is:

$$H^{*} = U\left(C,l\right) + q_1\left[I_1 - \delta K_1\right] + q_2\left[I_2 - \delta K_2\right] +$$
$$+ \lambda\left[C + \Phi\left(I,K\right) + rB - (1-g)Y_1\left(K_1,\ell_1,l\right) - Y_2\left(K_1,K_2,\ell_1,l\right)\right]$$

The Pontryagin maximum conditions for this optimal control problem are:

Optimality Conditions

$$U_c'\left(C,l\right) + \lambda = 0 \qquad \textbf{(D20)}$$

$$U_l'\left(C,l\right) + \lambda\left[(1-g)Y_{1,l}'\left(K_1,\ell_1,l\right) + Y_{2,l}'\left(K_1,K_2,\ell_1,l\right)\right] = 0 \qquad \textbf{(D21)}$$

$$\lambda\left[(1-g)Y_{1,\ell_1}'\left(K_1,\ell_1,l\right) + Y_{2,\ell_1}'\left(K_1,K_2,\ell_1,l\right)\right] = \mathbf{0} \qquad \textbf{(D22)}$$

$$\lambda\Phi_{I_1}'\left(I,K\right) + q_1 = 0 \qquad \textbf{(D23)}$$

$$\lambda\Phi_{I_2}'\left(I,K\right) + q_2 = 0 \qquad \textbf{(D24)}$$

Admissibility Conditions

$\boldsymbol{B}_0 = \boldsymbol{B}_{(0)}$, $\boldsymbol{K}_{1,0} = \boldsymbol{K}_{1,(0)}$ and $\boldsymbol{K}_{2,0} = \boldsymbol{K}_{2,(0)}$

Multipliers Conditions

$$\dot{\lambda} = \lambda(\rho - \boldsymbol{r}) \quad \textbf{(D25)}$$

$$\dot{\boldsymbol{q}}_1 = \boldsymbol{q}_1(\rho+\delta) - \lambda\left(\Phi'_{\boldsymbol{K}_1}(\boldsymbol{I},\boldsymbol{K}) - (1-\boldsymbol{g})\boldsymbol{Y}_{1,\boldsymbol{K}_1}{}'(\boldsymbol{K}_1,\ell_1,\boldsymbol{l}) - \boldsymbol{Y}_{2,\boldsymbol{K}_1}{}'(\boldsymbol{K}_1,\boldsymbol{K}_2,\ell_1,\boldsymbol{l})\right) \quad \textbf{(D26)}$$

$$\dot{\boldsymbol{q}}_2 = \boldsymbol{q}_2(\rho+\delta) - \lambda\left(\Phi'_{\boldsymbol{K}_2}(\boldsymbol{I},\boldsymbol{K}) - \boldsymbol{Y}_{2,\boldsymbol{K}_2}{}'(\boldsymbol{K}_2,\ell_1,\boldsymbol{l})\right) \quad \textbf{(D27)}$$

State Conditions

$$\dot{\boldsymbol{B}} = \boldsymbol{C} + \Phi(\boldsymbol{I},\boldsymbol{K}) + \boldsymbol{rB} - (1-\boldsymbol{g})\boldsymbol{Y}_1(\boldsymbol{K}_1,\ell_1,\boldsymbol{l}) - \boldsymbol{Y}_2(\boldsymbol{K}_1,\boldsymbol{K}_2,\ell_1,\boldsymbol{l}) \quad \textbf{(D28)}$$

$$\dot{\boldsymbol{K}}_1 = \boldsymbol{I}_1 - \delta\boldsymbol{K}_1 \quad \textbf{(D29)}$$

$$\dot{\boldsymbol{K}}_2 = \boldsymbol{I}_2 - \delta\boldsymbol{K}_2 \quad \textbf{(D30)}$$

Transversality Conditions

$$\lim_{t\to\infty} \lambda \boldsymbol{B} \boldsymbol{e}^{-\rho t} = 0 \quad \textbf{(D31)}$$

$$\lim_{t\to\infty} \boldsymbol{q}_1 \boldsymbol{K}_1 \boldsymbol{e}^{-\rho t} = 0 \quad \textbf{(D32)}$$

$$\lim_{t\to\infty} \boldsymbol{q}_2 \boldsymbol{K}_2 \boldsymbol{e}^{-\rho t} = 0 \quad \textbf{(D33)}$$

2.2. The Decentralized Economy Maximum Problem

The maximum problem for the decentralised economy with an informal sector is:

$$\underset{C,l,\ell_1,i_1,i_2}{\boldsymbol{MAX}}\ \boldsymbol{U} = \int_0^{\infty} \boldsymbol{U}(\boldsymbol{c},\boldsymbol{l})\boldsymbol{e}^{-\rho t}\boldsymbol{dt} \quad \textbf{(D34)}$$

$\boldsymbol{subject\ to}$:

$$\begin{cases} \dot{\boldsymbol{b}} = (1+\tau_{\boldsymbol{c}})\boldsymbol{c} + \Phi(\boldsymbol{i},\boldsymbol{k}) + \boldsymbol{rb} + \tau - (1-\tau_{\boldsymbol{w}})\boldsymbol{w}_1(1-\boldsymbol{l})\ell_1 - \\ -(1-\tau_{\boldsymbol{k}})\boldsymbol{r}_{\boldsymbol{k}_1}\boldsymbol{k}_1 - \boldsymbol{w}_2(1-\boldsymbol{l})(1-\ell_1) - \boldsymbol{r}_{\boldsymbol{k}_2}\boldsymbol{k}_2 & \textbf{(D35)} \\ \dot{\boldsymbol{k}}_1 = \boldsymbol{i}_1 - \delta\boldsymbol{k}_1 & \textbf{(D36)} \\ \dot{\boldsymbol{k}}_2 = \boldsymbol{i}_2 - \delta\boldsymbol{k}_2 & \textbf{(D37)} \end{cases}$$

The present value Hamiltonian for this optimization problem is:

$$\boldsymbol{H}^* = \boldsymbol{U}(\boldsymbol{c},\boldsymbol{l}) + \boldsymbol{q}_1[\boldsymbol{i}_1 - \delta\boldsymbol{k}_1] + \boldsymbol{q}_2[\boldsymbol{i}_2 - \delta\boldsymbol{k}_2] +$$

$$+\lambda\left[(1+\tau_{\boldsymbol{c}})\boldsymbol{c} + \Phi(\boldsymbol{i},\boldsymbol{k}) + \boldsymbol{rb} + \tau - (1-\tau_{\boldsymbol{w}})\boldsymbol{w}_1(1-\boldsymbol{l})\ell_1 - (1-\tau_{\boldsymbol{k}})\boldsymbol{r}_{\boldsymbol{k}_1}\boldsymbol{k}_1 - \boldsymbol{w}_2(1-\boldsymbol{l})(1-\ell_1) - \boldsymbol{r}_{\boldsymbol{k}_2}\boldsymbol{k}_2\right]$$

The Pontryagin maximum conditions for this optimal control problem are:

Optimality Conditions

$$\boldsymbol{U}'_{\boldsymbol{c}}(\boldsymbol{c},\boldsymbol{l}) + \lambda(1+\tau_{\boldsymbol{c}}) = 0 \quad \textbf{(D38)}$$

$$\boldsymbol{U}'_{\boldsymbol{l}}(\boldsymbol{c},\boldsymbol{l}) + \lambda\left[(1-\tau_{\boldsymbol{w}})\boldsymbol{w}_1\ell_1 + \boldsymbol{w}_2(1-\ell_1)\right] = 0 \quad \textbf{(D39)}$$

$$\lambda\left[-\left(1-\tau_{w}\right)\boldsymbol{w}_{1}\left(1-\boldsymbol{l}\right)+\boldsymbol{w}_{2}\left(1-\boldsymbol{l}\right)\right]=0 \qquad \textbf{(D40)}$$

$$\lambda\Phi'_{i_1}\left(\boldsymbol{i},\boldsymbol{k}\right)+\boldsymbol{q}_{1}=0 \qquad \textbf{(D41)}$$

$$\lambda\Phi'_{i_2}\left(\boldsymbol{i},\boldsymbol{k}\right)+\boldsymbol{q}_{2}=0 \qquad \textbf{(D42)}$$

Admissibility Conditions

$\boldsymbol{b}_0=\boldsymbol{b}_{(0)}$, $\boldsymbol{k}_{1,0}=\boldsymbol{k}_{1,(0)}$ and $\boldsymbol{k}_{2,0}=\boldsymbol{k}_{2,(0)}$

Multipliers Conditions

$$\dot{\lambda}=\lambda(\rho-\boldsymbol{r}) \qquad \textbf{(D43)}$$

$$\dot{\boldsymbol{q}}_{1}=\boldsymbol{q}_{1}\left(\rho+\delta\right)-\lambda\left(\Phi'_{k_1}\left(\boldsymbol{i},\boldsymbol{k}\right)-\left(1-\tau_{k}\right)\boldsymbol{r}_{k_1}\right) \qquad \textbf{(D44)}$$

$$\dot{\boldsymbol{q}}_{2}=\boldsymbol{q}_{2}\left(\rho+\delta\right)-\lambda\left(\Phi'_{k_2}\left(\boldsymbol{i},\boldsymbol{k}\right)-\boldsymbol{r}_{k_2}\right) \qquad \textbf{(D45)}$$

State Conditions

$$\begin{aligned}\dot{\boldsymbol{b}}&=\left(1+\tau_{c}\right)\boldsymbol{c}+\Phi\left(\boldsymbol{i},\boldsymbol{k}\right)+\boldsymbol{r}\boldsymbol{b}+\tau-\left(1-\tau_{w}\right)\boldsymbol{w}_{1}\left(1-\boldsymbol{l}\right)\ell_{1}-\\&-\left(1-\tau_{k}\right)\boldsymbol{r}_{k_1}\boldsymbol{k}_{1}-\boldsymbol{w}_{2}\left(1-\boldsymbol{l}\right)\left(1-\ell_{1}\right)-\boldsymbol{r}_{k_2}\boldsymbol{k}_{2}\end{aligned} \qquad \textbf{(D46)}$$

$$\dot{\boldsymbol{k}}_{1}=\boldsymbol{i}_{1}-\delta\boldsymbol{k}_{1} \qquad \textbf{(D47)}$$

$$\dot{\boldsymbol{k}}_{2}=\boldsymbol{i}_{2}-\delta\boldsymbol{k}_{2} \qquad \textbf{(D48)}$$

Transversality Conditions

$$\lim_{t\to\infty}\lambda\boldsymbol{b}\boldsymbol{e}^{-\rho t}=0 \qquad \textbf{(D49)}$$

$$\lim_{t\to\infty}\boldsymbol{q}_{1}\boldsymbol{k}_{1}\boldsymbol{e}^{-\rho t}=0 \qquad \textbf{(D50)}$$

$$\lim_{t\to\infty}\boldsymbol{q}_{2}\boldsymbol{k}_{2}\boldsymbol{e}^{-\rho t}=0 \qquad \textbf{(D51)}$$

2.3. Alternative Hypotheses for the Centralized Optimization Problem using Functional Forms

This section will deal with the consequences of assuming different hypotheses about functional forms for adjustment costs and central planner information, regarding the relation of capital between sectors, when extending the results obtained in section *2.1.* of the appendix for the centralized economy maximum conditions.

Basically we will stress out the differences, when considering specific functional forms for adjustment costs in sections *2.3.1.* and *2.3.2.*, for a central planner with full information which takes into account the aggregate capital market clearing relation derived from equation (C13), in section *2.3.2.*, and a central planner that lacks that specific information, in section *2.3.1.*.

In these sections we will assume that adjustment costs could be either given by an aggregate investment cost function or by specific to sectors investment cost functions. Functional forms are described below, respectively:

$$\Phi(\boldsymbol{I},\boldsymbol{K}) = (\boldsymbol{I}_1 + \boldsymbol{I}_2)\left(1 + \frac{\boldsymbol{h}}{2}\frac{(\boldsymbol{I}_1 + \boldsymbol{I}_2)}{(\boldsymbol{K}_1 + \boldsymbol{K}_2)}\right) \tag{D52}$$

$$\Phi(\boldsymbol{I},\boldsymbol{K}) = \boldsymbol{I}_1\left(1 + \frac{\boldsymbol{h}}{2}\frac{\boldsymbol{I}_1}{\boldsymbol{K}_1}\right) + \boldsymbol{I}_2\left(1 + \frac{\boldsymbol{h}}{2}\frac{\boldsymbol{I}_2}{\boldsymbol{K}_2}\right) \tag{D53}$$

In sub-section *2.3.1.* we will assume that our central planner does not have information about the relation between formal and informal aggregate capital and will assume that production in the informal sector will be given by equation (C8). We will call this function $\boldsymbol{Y}_2(\boldsymbol{K}_1,\boldsymbol{K}_2)$. In sub-section *2.3.2.* we will assume that the central planner has full information about the linear relation between formal and informal capital and will assume that informal sector production is given by expression (C16) and that informal aggregate production will depend only on formal aggregate capital. We will call this function $\boldsymbol{Y}_2(\boldsymbol{K}_1)$.

2.3.1. Aggregate versus Specific to Sector Adjustment Costs Functions not Assuming a Linear Relation between Formal and Informal Capital

We can now obtain a specific formulation for maximum conditions of a central planner that assumes aggregate production to be given by $\boldsymbol{Y} = \boldsymbol{Y}_1(\boldsymbol{K}_1) + \boldsymbol{Y}_2(\boldsymbol{K}_1,\boldsymbol{K}_2)$ using results from section *2.1.* of the appendix. As usual we will use the utility function given by equation (A5) to complete our optimization problem. We can now substitute these functions by the results obtained in section *2.1.* of the appendix, to obtain our optimal control conditions for the case with an aggregate adjustment cost function:

$$\boldsymbol{C}^{\gamma-1}\boldsymbol{N}^{-\gamma}\boldsymbol{l}^{\theta\gamma} + \lambda = 0 \tag{D54}$$

$$\theta\boldsymbol{C}^{\gamma}\boldsymbol{N}^{-\gamma}\boldsymbol{l}^{\theta\gamma-1} - \lambda\left[(1-\boldsymbol{g})\frac{\boldsymbol{d}\,\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{d}\,\boldsymbol{l}} + \frac{\boldsymbol{d}\,\boldsymbol{Y}_2(\boldsymbol{K}_1,\boldsymbol{K}_2)}{\boldsymbol{d}\,\boldsymbol{l}}\right] = 0 \tag{D55}$$

$$-\lambda\left[(1-\boldsymbol{g})\frac{\boldsymbol{d}\,\boldsymbol{Y}_1(\boldsymbol{K}_1)}{\boldsymbol{d}\,\ell_1} + \frac{\boldsymbol{d}\,\boldsymbol{Y}_2(\boldsymbol{K}_1,\boldsymbol{K}_2)}{\boldsymbol{d}\,\ell_1}\right] = 0 \tag{D56}$$

$$\lambda\left(1 + \boldsymbol{h}\,\frac{(\boldsymbol{I}_1 + \boldsymbol{I}_2)}{(\boldsymbol{K}_1 + \boldsymbol{K}_2)}\right) + \boldsymbol{q}_i = 0,\ \ \boldsymbol{i} = 1,2 \tag{D57}$$

A straightforward result is obtained from the hypothesis considered for the case with aggregate adjustment costs, arbitrage conditions obtained in equation (D57) imply that the shadow price of formal and informal capital must be equal in all periods. If we use a specific to sector adjustment costs function all optimality conditions remain the same except for the arbitrage condition that becomes:

$$\lambda\left(1 + \boldsymbol{h}\,\frac{\boldsymbol{I}_i}{\boldsymbol{K}_i}\right) + \boldsymbol{q}_i = 0,\ \ \boldsymbol{i} = 1,2 \tag{D58}$$

Now if we assume the shadow price of each type of capital to be equal in all periods for (D58), we will obtain the following relation between formal and informal investment and capital:

$$\frac{I_1}{K_1} = \frac{I_2}{K_2} \tag{D59}$$

State conditions are obtained substituting our different hypotheses in the constraints of our optimization problem obtained in section *2.1.* of the appendix and co-state conditions for the aggregate adjustment costs case come as usual:

$$\dot{\lambda} = \lambda(\rho - r) \tag{D60}$$

$$\dot{q}_1 = q_1(\rho + \delta) + \lambda\left[\frac{h}{2}\frac{(I_1 + I_2)^2}{(K_1 + K_2)^2} + (1 - g)\frac{dY_1(K_1)}{dK_1} + \frac{dY_2(K_1, K_2)}{dK_1}\right] \tag{D61}$$

$$\dot{q}_2 = q_2(\rho + \delta) + \lambda\left[\frac{h}{2}\frac{(I_1 + I_2)^2}{(K_1 + K_2)^2} + \frac{dY_2(K_1, K_2)}{dK_2}\right] \tag{D62}$$

In the case of a specific to sector adjustment costs functions only the co-state conditions of formal and informal capital come differently. In this case, the shadow price of formal and informal capital will be independent of investment and capital from the opposing sectors, but the relation implied by the conditions will be very similar to the one obtained above, when we considered the relations obtained from the optimal arbitrage investment conditions.

$$\dot{q}_1 = q_1(\rho + \delta) + \lambda\left[\frac{h}{2}\left(\frac{I_1}{K_1}\right)^2 + (1 - g)\frac{dY_1(K_1)}{dK_1} + \frac{dY_2(K_1, K_2)}{dK_1}\right] \tag{D63}$$

$$\dot{q}_2 = q_2(\rho + \delta) + \lambda\left[\frac{h}{2}\left(\frac{I_2}{K_2}\right)^2 + \frac{dY_2(K_1, K_2)}{dK_2}\right] \tag{D64}$$

2.3.2. Aggregate versus Specific to Sector Adjustment Costs Functions Assuming a Linear Relation between Formal and Informal Capital

When we assume the linear relation described by equation (C13) and aggregate it, we have an optimization problem where the central planner observes all production as a function of aggregate formal sector capital, $Y = Y_1(K_1) + Y_2(K_1)$. This will have further implications in our maximization problem as it will affect all maximum conditions described, except for our optimal consumption equation. Optimal control conditions for the aggregate adjustment cost function assuming this hypothesis are now given by:

$$\theta C^{\gamma} N^{-\gamma} l^{\theta\gamma - 1} - \lambda\left[(1 - g)\frac{dY_1}{dl} + \frac{dY_2(K_1)}{dl}\right] = 0 \tag{D65}$$

$$-\lambda\left[(1 - g)\frac{dY_1(K_1)}{d\ell_1} + \frac{dY_2(K_1)}{d\ell_1}\right] = 0 \tag{D66}$$

$$\lambda\left(1+\boldsymbol{h}\,\frac{(\boldsymbol{I}_1+\boldsymbol{I}_2)}{(\boldsymbol{K}_1)}\frac{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi}{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi+\eta(1-\tau_{\boldsymbol{w}})\phi}\right)+\boldsymbol{q}_{\boldsymbol{i}}=0,\ \boldsymbol{i}=1,2 \qquad \textbf{(D67)}$$

Again, if we consider the specific to sector adjustment costs functions, only our optimal arbitrage conditions for investment decisions change:

$$\lambda\left(1+\boldsymbol{h}\,\frac{\boldsymbol{I}_1}{\boldsymbol{K}_1}\right)+\boldsymbol{q}_1=0 \qquad \textbf{(D68)}$$

$$\lambda\left(1+\boldsymbol{h}\,\frac{\boldsymbol{I}_2}{\boldsymbol{K}_1}\frac{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi}{\eta(1-\tau_{\boldsymbol{w}})\phi}\right)+\boldsymbol{q}_2=0 \qquad \textbf{(D69)}$$

Extending the relation described in equation (D67), which implies that both sector shadow prices of capital must be equal in all periods, we obtain the same relation for investment between sectors that we had obtained for capital, in equation (C13):

$$\boldsymbol{I}_2=\frac{\eta(1-\tau_{\boldsymbol{w}})\phi}{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi}\boldsymbol{I}_1 \qquad \textbf{(D70)}$$

State conditions are obtained as usual by substituting our full information hypotheses and adjustment cost functions in the constraints of our maximization problem of section *2.1.* of the appendix. The co-state condition for the shadow price of foreign bonds remains unchanged with these assumptions and the co-state conditions for the shadow price of capital in the formal sector for our aggregate adjustment costs functions become:

$$\dot{\boldsymbol{q}}_1=\lambda\left[\frac{\boldsymbol{h}}{2}\frac{(\boldsymbol{I}_1+\boldsymbol{I}_2)^2(1-\tau_{\boldsymbol{k}})(1-\beta)\xi}{\boldsymbol{K}_1^{\,2}\left((1-\tau_{\boldsymbol{k}})(1-\beta)\xi+\eta(1-\tau_{\boldsymbol{w}})\phi\right)}+(1-\boldsymbol{g})\frac{\boldsymbol{dY}_1}{\boldsymbol{dK}_1}+\frac{\boldsymbol{dY}_2(\boldsymbol{K}_1)}{\boldsymbol{dK}_1}\right]$$
$$+\boldsymbol{q}_1(\rho+\delta)+\boldsymbol{q}_2\left(\delta\,\frac{\eta(1-\tau_{\boldsymbol{w}})\phi}{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi}\right) \qquad \textbf{(D71)}$$

For the specific to sector adjustment cost functions case the shadow price of capital is given by:

$$\dot{\boldsymbol{q}}_1=\lambda\left[\frac{\boldsymbol{h}}{2}\left(\frac{\boldsymbol{I}_1}{\boldsymbol{K}_1}\right)^2+\frac{\boldsymbol{h}}{2}\left(\frac{\boldsymbol{I}_2}{\boldsymbol{K}_1}\right)^2\frac{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi}{\eta(1-\tau_{\boldsymbol{w}})\phi}+(1-\boldsymbol{g})\frac{\boldsymbol{dY}_1}{\boldsymbol{dK}_1}+\frac{\boldsymbol{dY}_2(\boldsymbol{K}_1)}{\boldsymbol{dK}_1}\right]$$
$$+\boldsymbol{q}_1(\rho+\delta)+\boldsymbol{q}_2\left(\delta\,\frac{\eta(1-\tau_{\boldsymbol{w}})\phi}{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi}\right) \qquad \textbf{(D72)}$$

Assuming the result derived from our optimal arbitrage investment conditions, $\boldsymbol{q}_1=\boldsymbol{q}_2$, we can substitute expressions (D71) and (D72), so that they will depend exclusively on the shadow price of formal sector capital. This result is necessary for the full information hypothesis because both co-state conditions obtained are redundant conditions that arise only because we have considered two state conditions, while considering the market clearing condition, (C13), between formal and informal capital. The co-state conditions retain no useful information about the state and they are reduced to the following differential equation, $\dot{\boldsymbol{q}}_2=\rho\boldsymbol{q}_2$.

2.3.2.1. Accumulation Indifference in the Complete Information about Capital Relations Centrally Planned Economy

In this section, we present the outcomes that arise when considering the three states dynamic optimization problem presented in the appendix, when compared to results from section *6.2.2.* and *6.2.3.*, where we used the linear relation for aggregate capital in order to internalize all informal decisions and reduce our optimization problem to one with only two state variables.

Setting $\boldsymbol{h}$ to zero in maximum conditions for the complete information centralized economy case, expressed in section 2.3.2. of the appendix, we can obtain the usual static expression for equilibrium in accumulation for the no adjustment costs model. From optimal arbitrage investment conditions expressed in (D69) it is straightforward to derive that the shadow price of capital of each specific sector and the marginal value of wealth must equalize. We can use this result to substitute in the full information co-state condition for the shadow price of formal capital, (D71) or (D72), and in the optimal consumption condition (D54). Taking time derivatives on (D54) and substituting both co-state conditions as usual, we can derive our usual motion equations for consumption:

$$\dot{\boldsymbol{C}} = \left(\frac{\rho - \boldsymbol{r}}{\gamma - 1}\right)\boldsymbol{C} \qquad \textbf{(D73)}$$

$$\dot{\boldsymbol{C}} = \left(\frac{\rho + \delta + \delta\dfrac{\eta\left(1-\tau_{\boldsymbol{w}}\right)\phi}{\left(1-\tau_{\boldsymbol{k}}\right)\left(1-\beta\right)\xi} - \left(1-\boldsymbol{g}\right)\dfrac{\boldsymbol{Y}_1\left(\boldsymbol{K}_1\right)}{\boldsymbol{K}_1} - \left(\beta+\eta-\mu\right)\dfrac{\boldsymbol{Y}_2\left(\boldsymbol{K}_1\right)}{\boldsymbol{K}_1}}{\gamma - 1}\right)\boldsymbol{C} \qquad \textbf{(D74)}$$

Applying the usual equality in numerator we can obtain the expression for indifference in accumulation for this economy:

$$\boldsymbol{r} = \left(1-\boldsymbol{g}\right)\frac{\boldsymbol{Y}_1\left(\boldsymbol{K}_1\right)}{\boldsymbol{K}_1} + \left(\beta+\eta-\mu\right)\frac{\boldsymbol{Y}_2\left(\boldsymbol{K}_1\right)}{\boldsymbol{K}_1} - \delta\left(\frac{\left(1-\tau_{\boldsymbol{k}}\right)\left(1-\beta\right)\xi + \eta\left(1-\tau_{\boldsymbol{w}}\right)\phi}{\left(1-\tau_{\boldsymbol{k}}\right)\left(1-\beta\right)\xi}\right) \qquad \textbf{(D75)}$$

Again long run versus short to medium run differences arise in the static equilibrium conditions between foreign and domestic accumulation. Expression (D75) defines equilibrium in the short to medium run but since aggregate marginal productivity of informal capital decays asymptotically in a growing economy, long run accumulation indifference equilibrium will be given by:

$$\lim_{t\to\infty}\boldsymbol{r} = \left(1-\boldsymbol{g}\right)\frac{\boldsymbol{Y}_1\left(\boldsymbol{K}_1\right)}{\boldsymbol{K}_1} - \delta\left(\frac{\left(1-\tau_{\boldsymbol{k}}\right)\left(1-\beta\right)\xi + \eta\left(1-\tau_{\boldsymbol{w}}\right)\phi}{\left(1-\tau_{\boldsymbol{k}}\right)\left(1-\beta\right)\xi}\right) \qquad \textbf{(D76)}$$

Using results from section *2.3.2.1.* of the appendix we can extend the long run result to be almost the same to the accumulation indifference static relation, (B23), obtained in the centrally planned economy of chapter *5.*. Assuming no impacts in labour/leisure decisions and labour allocation from informal sector production, the indifference accumulation condition, (D75), will only be influenced by the informal sector rate of capital depreciation in the long run.

When assuming a specific to sector adjustment cost function it is straightforward to derive our usual differential equation for formal investment using the linear relation for investment between sectors, (D70), obtained from our optimal arbitrage conditions. As usual our differential equation will reflect our maximum conditions and the results obtained for the simpler case described above. The method to obtain this equation follows the framework presented in section *4.2.2.*. Substituting linear relations in the co-state condition, (D72), and taking time derivatives in the formal investment arbitrage condition (D68), we obtain, after some fair amount of algebra, the formal investment differential equation:

$$\dot{\boldsymbol{I}}_1 = \frac{\boldsymbol{I}_1^{\,2}}{2\boldsymbol{K}_1}\left(\frac{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi - \eta(1-\tau_{\boldsymbol{w}})\phi}{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi}\right) + \left(\boldsymbol{r} + \delta\frac{\eta(1-\tau_{\boldsymbol{w}})\phi}{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi}\right)\boldsymbol{I}_1 + \\ + \left[\delta\frac{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi + \eta(1-\tau_{\boldsymbol{w}})\phi}{(1-\tau_{\boldsymbol{k}})(1-\beta)\xi} + \boldsymbol{r} - (1-\boldsymbol{g})\frac{\boldsymbol{Y}_1}{\boldsymbol{K}_1} - (\beta+\eta-\mu)\frac{\boldsymbol{Y}_2(\boldsymbol{K}_1)}{\boldsymbol{K}_1}\right]\frac{\boldsymbol{K}_1}{\boldsymbol{h}} \quad \textbf{(D77)}$$

The relevance of this differential equation is not in its structure, which remains similar to our other quadratic differential investment equations, but on the form that the linear relation for informal capital and investment enters the problem, when a three state dynamic optimization problem is considered and the linear investment relation is obtained from maximum conditions. This result reflects a central planner that does not internalize investment informal decisions, when considering optimal conditions, as opposed to the central planner from sections *6.2.2.* and *6.2.3.*.

The same methodology can be applied to the case with an aggregate adjustment costs investment function, but in this case we cannot obtain a specific relation between formal and informal investment from maximum conditions. The method to tackle this issue is to consider informal investment as a function of time also and then obtain, in the usual form, the differential equation for aggregate investment. To build our usual four dimensional dynamical system we would have to consider capital accumulation as an aggregate function of aggregate investment and formal capital, while in the specific to sector adjustment cost functions case we consider only formal capital accumulation. Although appealing, this hypothesis will not be considered because we cannot compare results with the hypothesis presented in sections *6.2.2.* and *6.2.3.*, where our analysis is restricted to formal sector investment and accumulation activities. For that reason we will restrict our results to the ones presented already and leave just this short intuition

2.3.3. Dynamic Optimum Problem for the Central Planner with Incomplete Information

In this section we will extend results from section *2.3.1.* of the appendix, in order to define a strategy for solving the intertemporal optimization problem faced by a central planner that lacks information about market equilibrium conditions, between formal and informal sector activities. In order to deliver some intuition on this subject, we will assume that if it is possible to obtain a relation between formal and informal aggregate capital, from our optimal control maximum conditions, such a relation will allow us to simplify our model and apply our usual strategies regarding only a two states instead of three states dynamic optimization problem.

Setting $\boldsymbol{h}=0$ in maximum conditions from section *2.3.1.* of the appendix, we can use the usual shadow price of factors equality obtained from optimal arbitrage investment conditions and substitute it either in co-state conditions (D61) and (D62) or (D63) and (D64), in order to obtain a relation between aggregate formal and informal capital for our usual simplified problem:

$$\left(1-\boldsymbol{g}\right)\frac{\boldsymbol{Y}_1\left(\boldsymbol{K}_1\right)}{\boldsymbol{K}_1}+\left(\beta-\mu\right)\frac{\boldsymbol{Y}_2\left(\boldsymbol{K}_1,\boldsymbol{K}_2\right)}{\boldsymbol{K}_1}=\eta\frac{\boldsymbol{Y}_2\left(\boldsymbol{K}_1,\boldsymbol{K}_2\right)}{\boldsymbol{K}_2} \qquad \textbf{(D78)}$$

This expression obtained is a polynomial equation in $\boldsymbol{K}_2$ which is not straightforward to deal with. Solving this expression in terms of the root solutions of this polynomial would obscure our analysis and modelling even further, in an analysis that could be considered somewhat controversial in terms of the welfare economics mainstream theory and that would only hold when formal aggregate output follows a neoclassical technology. It is questionable to consider a central planner that faces asymmetric information about non-formal activities, although appealing when we define informal sector activities as underground, hidden or shadow activities. Moreover we show that it is possible to obtain the same static market clearing conditions between sectors already presented in section *6.1.*, using the representative agent maximum conditions expressed in section 2.2. of the appendix.

In light of these considerations and of the possibility of deriving market clearing conditions between sectors, both from economic theory and optimal control maximum conditions, we will dismiss this issue as one to tackle and keep our focus on the complete information hypothesis.

3. Non Simulated Dynamical Systems

3.1. Dynamical Systems for the Centralized and Decentralized Economies with no Government Sector

The dynamical system that describes the centralized economy with adjustment costs is given by the four dimensional system in aggregate consumption, capital, investment and the aggregate open economy budget constraint:

$$\begin{cases}\dot{\boldsymbol{C}}=\dfrac{\left(\rho-\boldsymbol{r}\right)}{\left(\gamma-1\right)}\boldsymbol{C} & \textbf{(D79)}\\ \dot{\boldsymbol{B}}=\boldsymbol{C}+\boldsymbol{I}\left(1+\dfrac{\boldsymbol{h}\boldsymbol{I}}{2\boldsymbol{K}}\right)+\boldsymbol{r}\boldsymbol{B}-\boldsymbol{A}\left(1-\boldsymbol{l}\right)^{\phi}\boldsymbol{N}^{\beta}\boldsymbol{K} & \textbf{(D80)}\\ \dot{\boldsymbol{K}}=\boldsymbol{I}-\delta\boldsymbol{K} & \textbf{(D81)}\\ \dot{\boldsymbol{I}}=\dfrac{\boldsymbol{I}^2}{2\boldsymbol{K}}+\boldsymbol{r}\boldsymbol{I}+\left(\delta+\boldsymbol{r}-\boldsymbol{A}\left(1-\boldsymbol{l}\right)^{\phi}\boldsymbol{N}^{\beta}\right)\dfrac{\boldsymbol{K}}{\boldsymbol{h}} & \textbf{(D82)}\end{cases}$$

The dynamical system for the decentralized economy with adjustment costs comes:

$$\begin{cases} \dot{\boldsymbol{C}} = \dfrac{(\rho - \boldsymbol{r})}{(\gamma - 1)}\boldsymbol{C} & \textbf{(D83)} \\ \dot{\boldsymbol{B}} = \boldsymbol{C} + \boldsymbol{I}\left(1 + \dfrac{\boldsymbol{hI}}{2\boldsymbol{K}}\right) + \boldsymbol{rB} - \left[\phi + (1-\beta)\right]\boldsymbol{A}(1-\boldsymbol{l})^{\phi}\boldsymbol{N}^{\beta}\boldsymbol{K} & \textbf{(D84)} \\ \dot{\boldsymbol{K}} = \boldsymbol{I} - \delta\boldsymbol{K} & \textbf{(D85)} \\ \dot{\boldsymbol{I}} = \dfrac{\boldsymbol{I}^2}{2\boldsymbol{K}} + \boldsymbol{rI} + \left(\delta + \boldsymbol{r} - (1-\beta)\boldsymbol{A}(1-\boldsymbol{l})^{\phi}\boldsymbol{N}^{\beta}\right)\dfrac{\boldsymbol{K}}{\boldsymbol{h}} & \textbf{(D86)} \end{cases}$$

3.2. Dynamical Systems for the Centrally Planned Economy with a Government Sector and for the Centrally Planned Economy with an Informal Sector

The dynamical system for the centralized economy of chapter 5. with adjustment costs comes:

$$\begin{cases} \dot{\boldsymbol{C}} = \dfrac{(\rho - \boldsymbol{r})}{(\gamma - 1)}\boldsymbol{C} & \textbf{(D87)} \\ \dot{\boldsymbol{B}} = \boldsymbol{C} + \boldsymbol{I}\left(1 + \dfrac{\boldsymbol{hI}}{2\boldsymbol{K}}\right) + \boldsymbol{rB} - (1-\boldsymbol{g})\left(\boldsymbol{A}\left[\boldsymbol{gN}\right]^{\beta}\right)^{\frac{1}{1-\beta}}(1-\boldsymbol{l})^{\frac{\phi}{1-\beta}}\boldsymbol{K} & \textbf{(D88)} \\ \dot{\boldsymbol{K}} = \boldsymbol{I} - \delta\boldsymbol{K} & \textbf{(D89)} \\ \dot{\boldsymbol{I}} = \dfrac{\boldsymbol{I}^2}{2\boldsymbol{K}} + \boldsymbol{rI} + \left(\delta + \boldsymbol{r} - (1-\boldsymbol{g})\left(\boldsymbol{A}\left[\boldsymbol{gN}\right]^{\beta}\right)^{\frac{1}{1-\beta}}(1-\boldsymbol{l})^{\frac{\phi}{1-\beta}}\right)\dfrac{\boldsymbol{K}}{\boldsymbol{h}} & \textbf{(D90)} \end{cases}$$

The dynamical system for the centralized economy of chapter 6. with adjustment costs comes:

$$\begin{cases} \dot{\boldsymbol{C}} = \left(\dfrac{\rho - \boldsymbol{r}}{\gamma - 1}\right)\boldsymbol{C} & \textbf{(D91)} \\ \dot{\boldsymbol{B}} = \boldsymbol{C} + \boldsymbol{I}_1\left(1 + \boldsymbol{h}\dfrac{\boldsymbol{I}_1}{\boldsymbol{K}_1}\right)\Theta_2 + \boldsymbol{rB} - (1-\boldsymbol{g})\boldsymbol{Y}_1(\boldsymbol{K}_1) - \boldsymbol{Y}_2(\boldsymbol{K}_1) & \textbf{(D92)} \\ \dot{\boldsymbol{K}}_1 = \boldsymbol{I}_1 - \delta\boldsymbol{K}_1 & \textbf{(D93)} \\ \dot{\boldsymbol{I}}_1 = \dfrac{\boldsymbol{I}_1^{\,2}}{2\boldsymbol{K}_1} + \boldsymbol{rI}_1 + \left[\delta + \boldsymbol{r} - \Theta_2^{\,-1}\left((1-\boldsymbol{g})\dfrac{\boldsymbol{Y}_1}{\boldsymbol{K}_1} + (\beta + \eta - \mu)\dfrac{\boldsymbol{Y}_2(\boldsymbol{K}_1)}{\boldsymbol{K}_1}\right)\right]\dfrac{\boldsymbol{K}_1}{\boldsymbol{h}} & \textbf{(D94)} \end{cases}$$

4. Matlab Routines

In this section of the appendix we reproduce the Matlab routines that were developed for simulation purposes of initial and boundary value problems that were discussed in chapters 5. and 6..

4.1. Generic Matlab Routine for Simulation of Initial Value Problems

```
function genericodeinitvalue

%ODE simulation as a initial value point problem. This file is a generic m file for simulation of initial value problems for
%continuous ODE systems of all dimensions. Solves system numerically as an initial value problem using ODE45 a Runge-
%Kutta explicit method of the 4th and 5th order, this routine is based on Dormand-Prince pair

nperiods=###;              %choose number of periods
tspan =[0 nperiods];       %choose simulation range

y0=[####;####];    %Choose the initial values for simulation.

options = odeset('RelTol',1e-10,'AbsTol',1e-10,'OutputFcn',@odephas2,'OutputSel',[2 1]);

%options for two dimensions ODE solver, if system is of higher order you can use @odephase3 for three dimensional
%phase portraits, Outputsel gives the axis order for graphical purposes

[t,y] = ode45(@f,tspan,y0,options);          %Calls function ODE, can use any simulator as ODE23 and others

%Plots solution
figure(1);                                   %graphical solution for phase diagram
title('Two dimensional phase diagram');
xlabel('Variable 2');
ylabel('Variable 1');

figure(2);                                   %Plots time transition of variable 1
plot(t,y(:,1),'-')
title('Transition to long run regime');
xlabel('time');
ylabel('Variable 1');
legend('Variable 1');

figure(3);                                   %plots transition of variable 2
plot(t,y(:,2),'-')
title('Transition to long run regime');
xlabel('time');
ylabel('Variable2');
legend('Variable2');

%Other plots can be introduced here. One advantage is that the time variable is already considered, though you have
% to load parameters again if the time plot desired is composed by variables and parameters

%Numerical and analytical framework for dynamical system
function dy = f(t,y)   %This is the function that ODE45 calls for simulation, in this part of the syntax you can only load the
%system and the parameters

%Calibration values for parameters, choose numerical values for parameters here
Parameter1=###;            %Example for numerical parameterization syntax
parameter2=###;
parameter3=parameter2*parameter1;      %Example of a possible parameter expression

% Dynamical system
dy= [
      ];
%Put system in here. Just the right hand side of ODE system only. One specific ODE each line. You can put as many
%ODE's you like. Remember that only the right hand side is needed. Time variables are defined as y(#).
```

4.2. Matlab Routine for Simulating Comparative Dynamics as a Boundary Value Problem

```
%Fiscal police with informal sector during transitions
% I – This part initializes variables

clear all;
clc;
nperiods = 500; %number of simulation periods
plotperiods1 = 200; %number of periods after shocks
plotperiods2 = 50; %number of periods before shocks

%Calibration values for parameters. Introduce table values here

pmgK1=((A*(g*N)^beta)^(1/(1-beta)))*((l1*(1-l))^(phi/(1-beta)));    %Marginal productivity of aggregate capital or
%just dY/dK
theta2=((1-tk)*(1-beta)*ksi+eta*(1-tw)*phi)/((1-tk)*(1-beta)*ksi);    %additional parameter expression
theta3=(beta+eta-mu-1)*omegafor;                                     %additional parameter expression
ES=zeros(4,1);                                                       %Exogenous shock vector

% Computing the dynamic system as a boundary value problem- y1(1)=Z1cent, y1(2)=Z2cent, y2(1)=Z1cent,
%y2(2)=Z2decent,

%centralized system

dydx1 = @(x1,y1,ES,g,pmgK1,theta2,theta3,bc1) [
                                                    ];              %centralized system comes here
res1 = @(y1a,y1b,ES,g,pmgK1,theta2,theta3,bc1) [y1b(1) - bc1(1)
                                               y1a(2) - bc1(2)];

 %decentralized system
dydx2 = @(x2,y2,ES,g,pmgK1,theta2,theta3,bc2) [
                                                    ];
res2 = @(y2a,y2b,ES,g,pmgK1,theta2,theta3,bc2) [y2b(1) - bc2(1)
                                               y2a(2) - bc2(2)];

% II- Computing steady states for:

%Initial steady state
starcent = fsolve(@(y1) dydx1(0,y1,ES,g,pmgK1,theta2,theta3), [0.024300470477;0.3]);
stardecent = fsolve(@(y2) dydx2(0,y2,ES,g,pmgK1,theta2,theta3), [0.038728617138;0.3]);

%Steady state after exogenous shock
ES=[0,0,0,0]';              %choose shocks and values
shock1=ES(1);
shock2=ES(2);
shock3=ES(3);
shock4=ES(4);

pmgK1=((A*(g*N)^beta)^(1/(1-beta)))*((l1*(1-l))^(phi/(1-beta)));    %parameter expressions must be computed again
theta2=((1-tk)*(1-beta)*ksi+eta*(1-tw)*phi)/((1-tk)*(1-beta)*ksi);
theta3=(beta+eta-mu-1)*omegafor;

starshock1 = fsolve(@(y1) dydx1(0,y1,ES,g,pmgK1,theta2,theta3), starcent);
starshock2 = fsolve(@(y2) dydx2(0,y2,ES,g,pmgK1,theta2,theta3), stardecent);

%steady state values
disp('Initial steady state')
disp('Z1*cent   Z2*cent');
disp(starcent');
disp('');
disp('Positive exogenous shocksteady state')
disp('Z1*cent   Z2*cent');
disp(starshock1');
disp('');
disp('Initial steady state')
```

```
disp('Z1*decent Z2*decent');
disp(stardecent');
disp('');
disp('Positive exogenous shocksteady state')
disp('Z1*decent Z2*decent');
disp(starshock2');
disp('');

%III- Solving Differential equations as a boundary value problem

mesh=5*nperiods;
solinit1 = bvpinit(linspace(1,nperiods,mesh), [starcent(1) starcent(2)]); %initial values
sol1 = bvp4c(dydx1, res1, solinit1,[],ES,g,pmgK1,theta2,theta3,[starshock1(1);starcent(2)]); %solving routine, two ODES,
%two boundary conditions

solinit2 = bvpinit(linspace(1,nperiods,mesh), [stardecent(1) stardecent(2)]);
sol2 = bvp4c(dydx2, res2, solinit2,[],ES,g,pmgK1,theta2,theta3,[starshock2(1);stardecent(2)]);

x1= linspace(1,nperiods, nperiods);
x2= linspace(1,nperiods, nperiods);
y1 = deval(sol1,x1);
y2 = deval(sol2,x2);

disp('Final simulated values');
disp('Z1centf     Z2centf');
disp(y1(:,end)');
disp('Z1decentf   Z2decentf');
disp(y2(:,end)');

xi1=linspace(-(nperiods-1),0, nperiods); %values before shock, periods
xi2=linspace(-(nperiods-1),0, nperiods);
y1i=[starcent(1)*ones(1,nperiods); starcent(2)*ones(1,nperiods)]; %values before shock centralized
y2i=[stardecent(1)*ones(1,nperiods); stardecent(2)*ones(1,nperiods)];
x1 =[xi1 x1]; %all values, periods
x2 =[xi2 x2]; %all values, periods
y1 =[y1i y1]; %all values, centralized variables
y2 =[y2i y2]; %all values, decentralized variables

%series
Z1cent =y1(1,:);
Z2cent =y1(2,:);
Z1decent =y2(1,:);
Z2decent =y2(2,:);

%plots
figure(1);
plot(x1(nperiods-plotperiods2 : nperiods+plotperiods1),Z1cent(nperiods-plotperiods2 : nperiods+plotperiods1),'-
',x2(nperiods-plotperiods2 : nperiods+plotperiods1),Z1decent(nperiods-plotperiods2 : nperiods+plotperiods1),'-')
title('Transition to long run regime');
xlabel('time');
ylabel('Z1cent and Z1decent');
legend('Z1cent','Z1decent');

figure(2);
plot(x1(nperiods-plotperiods2 : nperiods+plotperiods1),Z2cent(nperiods-plotperiods2 : nperiods+plotperiods1),'-
',x2(nperiods-plotperiods2 : nperiods+plotperiods1),Z2decent(nperiods-plotperiods2 : nperiods+plotperiods1),'-')
title('Transition to long run regime');
xlabel('time');
ylabel('Z2cent and Z2decent');
legend('Z2cent','Z2decent');
```

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