Optimal Beliefs and Self-Confirming Equilibrium in a Class of Differential Games with Economic Applications☆

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Abstract

We propose a class of non-cooperative differential games, where players have optimal, but naive, beliefs about the evolution of the state of the game. This result requires that a specific state-separability property is fulfilled for players choosing open loop strategies. The existence of optimal beliefs consistent with this property, provides, in our opinion, a unique framework to study the implications of the self-confirming equilibrium (SCE) hypothesis in a dynamic game setup. State-separability in this setup is a consequence of the existence of optimal beliefs. We argue that solutions to this class of games are optimal if they coincide with the individual state-separable solutions. Only SCE outcomes are optimal solutions for this class of games. We propose to answer the following question. Are players' able to concur on SCE, where their expectations are self-fulfilling? The answer is yes, but it will depend on players asymmetries and on the degree of nonlinearity of the game. When complex nonlinear interactions are considered, SCE solutions are only achievable if learning agents converge to an unique equilibrium. Our main objective is to evaluate the properties of SCE solutions in this class of games. For that purpose we consider the following conjecture. If beliefs bound the state-space of the game asymptotically and strategies are Lipschitz continuous, then it is possible to describe the conditions for the existence of SCE solutions and use standard methods to evaluate the qualitative properties of equilibrium. If strategies are not smooth, which is likely in complex learning environments, then optimal solutions can only be evaluated numerically as a Hidden Markov Model (HMM). We illustrate these conjectures in two multi-player differential games, populated by investors that seek to maximize their consumption utility in a growing environment.

Keywords: Nonlinear Differential Games, Optimal Beliefs, Self-Confirming Equilibrium

1. Introduction

In this paper we introduce a conjecture regarding the existence of a class of nonlinear multi-player general sum differential games with optimal subjective beliefs. Our framework starts from the hypothesis first put forward by Dockner et al. [10], describing the conditions for the existence of qualitative and explicit solutions to differential games with state-separability properties. Dockner et al. [10] suggested that under some conditions it might be possible to solve qualitatively differential games that have state-separability properties, if optimal control conditions are consistent with non interacting dynamics with respect to state and controls. The literature on differential games with state-separability properties is a vast growing field in economics literature. Research in this field is not limited to Dockner et al. [10] results. Caputo [8], for example, discusses further state-separability properties, for a class of discounted infinite horizon optimal control models similar to the one discussed in this paper. The papers by Ling and Caputo [26] and Bacchiega et al. [2] are recent examples of the literature dealing with Caputo [8] state-separability hypothesis and its applications in economics theory. We extend Dockner et al. [10] conjectures on differential games with the state-separability property, and propose a setup, where player controls depend on beliefs defined by optimality conditions. In this context, it is our opinion that the optimal solution to the game has to be consistent with a Self-Confirming Equilibrium (SCE) that satisfies the state-separability property of the optimal individual solution. The notion of strategic SCE in non-cooperative incomplete information games, was first

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proposed by Fudenberg and Levine [16][2]. According to Fudenberg and Levine [16] the existence of stable belief based solutions in games, implies that player decisions under uncertainty are rational and equilibrium is SCE[3]. In the setup we propose, only SCE solutions can qualify as optimal solutions asymptotically. It is our opinion that the class of games we propose provides a unique framework to study the existence and stability of SCE solutions in nonlinear multi-player dynamic games. Our intuition on this matter is straightforward. non-cooperative differential games[4], as defined in the seminal paper of Isaacs [24], are a natural generalization of optimal control theory. Therefore, open loop no feedback differential games with the state-separability property, as defined by Dockner et al. [10], have optimal solutions given by the set of independent individual solutions. These solutions are defined independently from the state evolution of the game, by each player optimal control dynamics. On the other hand, state-separability in our class of games arises from an optimum belief condition regarding the evolution of the state of the game. In this framework, we consider that the optimal solution to this class of differential games has to be simultaneously consistent with self-fulfilling beliefs and with the optimal solution to the individual state-separable problem.

The initial motivation for this proposal departs from the recent focus on the study of nonlinear economies as multi-player games. In economics literature this topic is framed by the seminal proposal of Grandmont [20], on the stability analysis of equilibrium in large socio-economic systems. In this paper, the author puts forward an extensive discussion on the vast implications and mathematical challenges of undertaking stability analysis in large systems with decentralized decision dynamics under incomplete information. More recently, the topic of SCE has been gaining ground in the broader field of macroeconomic dynamics literature as an interesting hypothesis for evaluating macroeconomic policy models. Hansen [21] describes *"...the concept of SCE, a type of rational expectations that seems natural for macroeconomics."*. Fershtman and Pakes [12] propose the adoption of SCE, given that it is *"...an equilibrium notion for dynamic games with asymmetric information which does not require a specification for players' beliefs about their opponents' types. This enables us to determine equilibrium conditions which, at least in principle, are testable and can be computed using a simple reinforcement learning algorithm."*. Fudenberg and Levine [19] relates the concept of SCE with the macroeconomics paradigm known as the *Lucas Critique*[5].

To demonstrate our state-separability conjecture, we propose two differential games that are set up as growing economies driven by agents' financial strategic decisions. We consider an endogenous growth environment, following the seminal proposal by Romer [30], where linear productive capital dynamics are the growth engine of the economy. The papers by Clemhout and Wan [9], Vencatachellum [32], Bethamann [6] and Hori and Shibata [22] are some examples of the literature dealing with growing economies defined as multi-player non-cooperative dynamic games. However, given that our specific setup departs from a well known theoretical financial framework, which has its roots in the Merton [28] intertemporal consumption-investment problem, one can interpret our specific proposals as a simplified version of multi-player non-cooperative portfolio games. One may also interpret these models as a foreign exchange game, where agents invest in domestic risk-free deposits, and may leverage their domestic investments by selling bonds to foreign investors, or invest part of their capital in foreign deposits and face exchange rate risk. In both of the games proposed, we consider that players' decisions are coupled by the evolution of aggregate risk premium. The foreign bond market measures aggregate risk premium as the ratio of net aggregate financial assets to domestic capital. Optimal beliefs in this context are a condition imposed by first order *Pontryagin* maximum conditions. This outcomes guarantees the state-separability of the game in a perverse fashion. In the one hand, the existence of optimal solutions to these games must be consistent with individual optimal belief solution, given by the solution to the state-separable problem. On the other hand, optimal beliefs are only consistent with the true state of the game when agents concur on a solution that fits their individual beliefs. Given these two contradictory results, we propose to answer the following relevant questions in this paper. Are players with optimal naive beliefs able to converge to a SCE, where their expectations are self-fulfilling? If game dynamics allow the existence of a SCE, what is the qualitative nature of this solution? Is it a stable equilibrium? Is there history dependence resulting from multiple SCE solutions, thus making game solutions a conjectural SCE[6]? The answer to the first question is yes, but it will depend on the degree of asymmetry among players and/or the degree of nonlinearity considered. In the former case, it is possible to evaluate qualitatively the dynamics in

[2]The original concept was introduced in the working paper by Fudenberg and Kreps [14]. See also the paper by Fudenberg and Levine [17] on steady-state learning and *Nash* equilibrium. A broad overview on the topic of learning in incomplete information non-cooperative games can be found in the book by Fudenberg and Levine [18]. Recently, Battigalli and Siniscalchi [5] proposed an extension of the SCE notion to signalling games. Finally, Kamada [25] proposes a definition for strongly consistent SCE.

[3]The original proposal on SCE deals with solutions to extensive form games, where players have incomplete information and subjective beliefs. See Fudenberg and Kreps [15] for a discussion on this topic and learning in extensive form games.

[4]The book by Başar and Olsder [3] provides a modern detailed overview on the topic of dynamic non-cooperative game theory.

[5]Following the famous proposal by Lucas [27].

[6]See Wellman et al. [33] on the implications and characterization of conjectural equilibrium solutions in the field of multi-agent learning.

the vicinity of a SCE, but existence and stability of solutions involves considering a smaller or higher degree of asymmetry among players. In the latter case, players are required to learn SCE. We can confirm the existence of optimal solutions by geometrically defining SCE solutions as intersections between feasible conjectural solutions and actual outcomes in a bounded interval. However, dynamics in the vicinity of SCE solutions now depend on the gradients of the individual learning functions, which are correlated and depend on higher order moments. In the likely scenario that learning dynamics lead to non smooth strategic dynamics, then it is not possible to use standard qualitative techniques to evaluate SCE. Both these issues are discussed in detail in Grandmont [20]. Finally, when multiple SCE solutions and learning dynamics coexist, then it is not possible to describe game dynamics in a well defined mathematical fashion. However, the individual player dynamics defined by the state-separate solution, provide some insight on what are the most likely outcomes.

To demonstrate our main hypotheses, we depart from a simple conjecture regarding the solution and analysis of dynamic games under incomplete information. We argue that if beliefs are consistent with the existence of asymptotic equilibrium solutions, then it is possible to evaluate strategic equilibrium outcomes. If the belief function is known and fulfils *Lipchitz* continuity conditions, then it is possible to obtain a full qualitative description of the game dynamics and the stability of SCE can be evaluated at least locally. Equilibrium in this class of games can be fully described as a *Cauchy* boundary value problem, under this set of conditions, as long as we have knowledge of the value and gradient of the belief function, evaluated in the vicinity of the asymptotic equilibrium region. We portray this hypothesis in the game proposed in section 3. In this game, player beliefs impose a unique equilibrium. The existence of a SCE solution requires the existence of constraints on individual parameter distributions. If agents' strategies are a result of both beliefs and learning, then asymptotic game dynamics can always be evaluated numerically as a Hidden Markov Model (HMM)[7] outcomes of a static version of the game. A static version of the game should describe the most faithfully possible the game asymptotic solution as a multi-objective optimization problem under uncertainty.

We organize our presentation in the following fashion. In 2 we put forward the general framework for a class of exponentially discounted differential games with concave pay-offs, and introduce our main conjecture regarding the existence of state-separable games with optimal beliefs. We then put forward the general conditions for the existence of an optimal SCE solution for this class of games. In 3 and 4, we demonstrate our main conjecture and evaluate the implications of our hypotheses in two non-cooperative consumption/investment differential games, where agents seek to maximize consumption utility, choosing open loop consumption and investment strategies. In both games, players accumulate productive assets linearly and may choose to invest in bonds or leverage their productive asset portfolio. Player dynamics are coupled by risk premium dynamics, which is driven by the aggregate ratio of net financial assets to productive capital. In the first example, discussed in 3, we consider that players face an individual institutional measure of market risk premium. We show that asymptotic SCE solutions are not consistent with the existence of asymmetries in institutional risk premium valuations or, on the other hand, further asymmetries have to be considered to define a SCE solution. In this section, we assume the existence of a perfect capital market with an efficient regulator, which guarantees that institutional risk premium is defined optimally. Given some further simplifications, we are able to perform an extended numerical qualitative analysis of SCE solutions for a wide range of institutional scenarios. In the second example, we drop the institutional risk premium assumption and consider that individual risk premium market valuations are now a function of the players' individual ratio of net assets to productive capital. We show that a SCE solution in this framework is only possible when complex learning dynamics are considered. The complexity of this solution is a direct outcome of considering additional nonlinearities in our initial setup. We propose a static version of this game and discuss the implementation of a *Markov* switching regime model under incomplete information to evaluate learning outcomes. Given the *Bayesian* nature of this problem, we discard the numerical evaluation of the game asymptotic dynamics on the ground that the computational costs of implementing a HMM are not justified in this specific environment. Instead, we focus our analysis on a geometrical description of conditions guaranteeing the existence of SCE solutions, and evaluate the robustness of these outcomes for a reasonable range of state outcomes. We finish this discussion with a description of likely outcomes in the vicinity of SCE solutions for two distinct institutional scenarios, based on the qualitative analysis results of the solution to the individual fully state-separable problem.

2. General setup

In this section, we define the general setup for the class of differential games we wish to consider, and put forward the main conjectures for the existence of solutions driven by optimal beliefs. Consider the following

[7] A HMM is a *Markov* decision process where agents lack information about the state evolution.

general N-Player non-cooperative differential game[8] faced by player $i \in N$:

$$\begin{aligned} & \underset{u_i(t)}{MAX} \int_0^T \beta_i(t)\,\pi_i(u_i(t))\,dt \\ & \text{subject to the solution of:} \\ & \dot{x}_i(t) = g_i(u_i(t), X(t)); \\ & x_i(0) = x_{i,0}. \end{aligned} \tag{1}$$

where:

- $N = \{1, \ldots, n\}$- Discrete set of players;
- $\beta_i(t)$- Discount function for player i;
- $u_i(t) = \left\{u_i^1, \ldots, u_i^k\right\} \in \Re^k$- Finite dimensional vector of player i controls;
- $\pi_i(u_i(t)) \in \Re^{k'}$- Instantaneous pay-off for player i, where $k' \leq k$;
- $x_i(t) = \left\{x_i^1, \ldots, x_i^w\right\} \in \Re^w$-Finite dimensional vector of player i states;
- $x_i(0) = \left\{x_i^1, \ldots, x_i^w\right\} \in \Re^w$- Finite dimensional vector of initial conditions on player i states
- $X(t) = [x_1(t), \ldots, x_n(t)] \in \Re^{w'}$- Finite dimensional vector of state variables, where $w' \leq w$.

We consider solutions to (1) consistent with players choosing open loop no feedback strategies, $\eta(t) = \{X(0)\}, \forall t \in [0\ \ T]$, where $\eta(t)$ is the information set available to players at period t. Players seek to maximize a concave pay-off function, $\pi_i(u_i(t))$. Pay-offs in this class of games are discounted at an individual constant exponential rate, $\beta_i(t) = e^{-\rho_i t}$, where ρ_i is player i discount rate. These conditions lead to solutions that can be defined as initial value problems, given by *Pontryagin* first order conditions. In this framework, *Pontryagin* maximum conditions are sufficient for the existence of an optimal solution to (1), provided that transversality conditions, following Arrow and Kurtz [1], are fulfilled[9], thus guaranteeing that $x_i(t)$ does not grow too fast. We assume the following conjecture for the game given in (1). The conjecture is that player i optimal control solutions to (1) can be defined generally in the following fashion,

$$\dot{u}_i(t) = f_i(u_i(t), X(t)). \tag{2}$$

Following the conjecture put forward in (2), we consider that a game has solutions consistent with the existence of individual optimal subjective beliefs, $X_{opt}^{(i)}(t)$, if optimality conditions also impose the existence of an unique belief function, $X_{opt}^{(i)}(t) = v_i(t)$. We further assume that this belief function is consistent with a game with the state-separability property. This requires that beliefs are at most a function of the player controls and individual state evolution[10]. Optimal beliefs for this class of games are thus generally defined in the following fashion:

$$X_{opt}^{(i)}(t) = v_i(u_i(t), x_i(t)). \tag{3}$$

The assumption forwarded in (3) is crucial to our proposal. Our first intuition is that if the existence of optimal beliefs is consistent with a differential game with the state-separability property, then optimal solutions to (1) have to be consistent with solutions to the individual state-separable solution, that is obtained by substituting (3) in (2), and in the state condition of the general game defined in (1). The optimal control solution to (1) is thus defined by individual solutions to the state-separable system, after considering the optimal belief condition defined in(3):

$$\dot{u}_i(t) = f_i(u_i(t), v_i(t)); \tag{4}$$

$$\dot{x}_i(t) = g_i(u_i(t), v_i(t)). \tag{5}$$

[8]This specific class of games is contained in the broad general framework for exponentially discounted differential games with concave utility discussed in Dragone et al. [11].

[9]For this class of control problems, Arrow and Kurtz [1] have shown that first order conditions are sufficient, provided that transversality conditions, defined generally by $\lim_{t\to\infty} e^{-\rho_i t}\Gamma_i(t)\,x_i(t) = 0$, are fulfilled. $\Gamma_i(t)$ are the co-state variables of the optimal control problem defined in (1), describing the marginal adjustment of the players control to the individual state evolution, when no feedback strategies are considered.

[10]This assumption is rather general. In the game discussed in 3, optimal beliefs are a function of game parameters, while in 4, optimal beliefs are a function of the evolution of the player state variables. However, we maintain this general assumption throughout this section, because this specific conjecture is consistent with Dockner et al. [10] state-separability definition.

On the other hand, differential game theory does not allow us to tamper with the state condition of the game. The intuition on this matter is straightforward. The state condition of a dynamic game defines the evolution of the state of the game, so, even if players have optimal beliefs, the game evolution takes into account the true state outcomes and not the ones resulting from belief based decisions. Thus, by definition, the solution to the game defined in (1) should be given by a system, where players follow strategies that are in accordance with their optimal beliefs, as defined in (4), but where the game evolution, $X(t)$, is defined by the original state condition of the game described in (1). The game solution to (1) is thus correctly defined generally by the following system:

$$\dot{u}_i(t) = f_i(u_i(t), v_i(t)); \tag{6}$$

$$\dot{x}_i(t) = g_i(u_i(t), X(t)). \tag{7}$$

This contradictory result has, in our opinion, a straightforward interpretation. Optimal solutions for games with the state-separability property defined in (3), must have solutions consistent with the SCE hypothesis forwarded by Fudenberg and Levine [16]. In layman terms, we mean that solutions to the system defined in (6) and (7), are only consistent with the optimality condition defined in (3), if they are also solutions to the optimal control solution defined in (4) and (5). We define this argument in the following fashion. First, we assume that $X(t)$ and $X_{opt}^{(i)}(t)$ asymptotically converge to an equilibrium solution, and thus can be generally defined in the long run by:

$$\lim_{t\to\infty} X(t) = \bar{X}; \tag{8}$$

$$\lim_{t\to\infty} X_{opt}^{(i)}(t) = \bar{X}_{opt}^{(i)}. \tag{9}$$

If (8) and (9) bound the state-space of the systems, (4)-(5) and (6)-(7), in equilibrium, then we argue that a game solution is optimal when it coincides with the optimal control solution to the state-separable problem. This implies assuming the existence of SCE solutions where players beliefs are self-fulfilling,

$$|X_{opt}^{(i)}(t) - X(t)| \to 0, \quad \text{as} \quad t \to \infty. \tag{10}$$

Following the result in (10), we argue that we can evaluate qualitatively SCE solutions, if the belief function ensures that the players' strategic dynamics are *Lipschitz* continuous. For games fulfilling this property, SCE outcomes can be qualitatively evaluated using standard analysis techniques for hyperbolic dynamical systems. Our intuition goes as follows. SCE solutions to incomplete information differential games can always be evaluated qualitatively, if asymptotic dynamics can be described as a *Cauchy* boundary problem. In games where players follow smooth strategies, described as initial value solutions, we require knowledge of the value and gradient of the belief function in the game asymptotic frontier, in order to be able to perform qualitative analysis of self-fulfilling solutions. If strategies are not smooth, which is likely to occur in games where players have to learn a SCE, but asymptotic solutions are still consistent with a game with a bounded state-space, then asymptotic dynamics can always be evaluated numerically as a HMM. In such scenarios, the qualitative analysis of the game asymptotic outcomes is limited to statistical analysis of different hypotheses for the learning process driving players decisions. This approach is computationally costly and analysis of equilibrium is limited in scope. To overcome this issue, we suggest that SCE solutions can be revealed geometrically and some considerations about game dynamics can be obtained trough qualitative analysis of the state-separable solution. Recall that given the result in (8), state-separable dynamics are always consistent with a well defined *Cauchy* boundary problem, if (4)-(5) has an equilibrium solution.

In the next two sections, we illustrate the two hypotheses discussed above in two nonlinear multi-player consumption/investment differential games. In the first example, we show that SCE solutions can be obtained by imposing parameter constraints and that under simple assumptions, it is possible to describe fully the qualitative dynamics in equilibrium. In the second example, we show that the existence of SCE solutions is only achievable if we consider the existence of complex learning dynamics.

3. A consumption/investment game with coupled institutional risk premium

To demonstrate these conjectures, we consider two nonlinear extensions of the general investor problem and set it up as a non-cooperative differential game under incomplete information. In this section, we consider an economy populated by a discrete set of players, $N = \{1, \ldots, n\}$, that seeks to maximize their intertemporal pay-offs, given by a consumption utility function, $U_i(C_i)$, subject to the evolution of individual net financial assets, $B_i(t) \in \Re$, describing the budget constraint of each player, and productive capital accumulation, $K_i(t) \in \Re^+$, where $i \in N$. In order to pursue this objective, agents choose open loop, $\eta(t) = \{X(0)\}$, no feedback consumption, $C_i(t) \in \Re^+$ and investment strategies, $I_i(t) \in \Re^+$, and discount future consumption exponentially at a constant rate $\rho_i \in \Re^+$, in a game of infinite duration, $\forall t \in [0\ T]$ and $T = \infty$. Player decisions are coupled by a risk premium mechanism that depends on the overall evolution of the state of the game, defined by $X(t) = \{B(t), K(t)\}$, where $B(t) = \sum_{i \in N} B_i(t)$ and $K(t) = \sum_{i \in N} K_i(t)$, following the proposal by Bardhan [4] on convex risk premium dynamics. The objective of each players is to maximize the flow of discounted consumption pay-offs,

$$U_i(C_i) = \int_0^T u_i(C_i(t)) e^{-\rho t} dt, \text{ with } u_i(C_i(t)) = C_i(t)^{\gamma_i}, \tag{11}$$

where γ_i is the intertemporal substitution elasticity between consumption in any two periods, measuring the willingness to substitute consumption between different periods. We impose the usual constraint on the intertemporal substitution parameter, $0 < \gamma_i < 1$, such that $u_i'(C_i(t)) > 0$. This specification for utility belongs to the family of constant relative risk aversion (CRRA) utility functions and is widely used in economic optimization setups, where savings behaviour is crucial, such as economic growth problems. This setup also guarantees the concavity of the utility function, $u_i''(C_i(t)) < 0$. This is a necessary condition to define optimal solutions to our open loop differential game as an initial value problem.

Each player faces a budget constraint describing the evolution of net financial assets, $\dot{B}_i(t)$. We consider that players are bond buyers when $B_i(t) < 0$ and bond sellers when $B_i(t) > 0$. Each player uses their financial resources to finance consumption and investment activities, and to repay interest on their outstanding bonds or reinvest in financial assets. Players have revenues arising from productive capital investments, $r_k K_i(t)$, where $r_k \in \Re^+$ is the marginal revenue of investment in productive capital, and receive interest payments on holdings of financial assets, if they are bond investors. Interest payments follow a convex specification defined by, $rB_i(t)[1 + d_i B(t)/K(t)]$, where $r \in \Re^+$ stands as usual for the real market interest rate and $d_i \in \Re$ is an institutional measure of risk premium, resulting from capital markets' sentiments on the quality of the bonds issued by a specific player.

This assumption on risk premium is justified by bias arising from historical and psychological perceptions. A higher value of d_i means that holding bonds yields a higher risk for investors, but investment in financial assets pays a greater premium. A smaller value of d_i means that holding debt bonds yields a smaller risk for investors, but investment in financial assets pays a smaller premium. Such outcomes are reinforced if agents portfolios match the aggregate financial situation of the economy. However, bond investors are rewarded with smaller interest premiums, when the aggregate economy is a net seller of bond contracts, and debt issuing agents benefit from smaller interest premiums when the aggregate economy is net buyer of bonds. This setup can be interpreted in terms of the degree of financial development of a given economy and on distortions arising from the complex link between microeconomic and macroeconomic outcomes. This phenomenon can also be explained by historical, political and economic factors, which bias investors' sentiments towards successful players, disregarding real economic information. It can also result from information costs, which deter investors from acquiring relevant information on the state of a specific economy and rely on individual or collective market beliefs[11]. Finally, players accumulate productive capital exponentially following their strategic investment decisions, $\dot{K}_i(t) = I_i(t) - \delta K_i(t)$, and productive capital accumulation is subject to depreciation, which is defined by the common capital depreciation rate, $\delta \in \Re^+$. We assume that players playing open loop strategies do not commit to a common investment strategy, but their decisions will be such that they always fulfil growth, $\liminf_{t \to \infty} I_i(t) K_i(t)^{-1} > \delta$, and optimality conditions.

Following the description of the decision problem faced by each member of this economy, the non-cooperative game faced by player $i \in N$, is defined by the following dynamic optimization problem:

$$\begin{aligned} &\underset{C_i(t), I_i(t)}{MAX} \int_0^\infty e^{-\rho_i t} C_i(t)^{\gamma_i} dt \\ &\text{subject to the solution of:} \\ &\dot{B}_i(t) = C_i(t) + I_i(t) + rB_i(t)\left(1 + d_i \tfrac{B(t)}{K(t)}\right) - r_k K_i(t); \\ &\dot{K}_i(t) = I_i(t) - \delta K_i(t); \end{aligned} \tag{12}$$

satisfying the transversality conditions, (A.8) and (A.9), guaranteeing that solutions to (12) do not grow too fast.

Due to its simplicity, the framework proposed in (12) can have different interpretations. These interpretations

[11] Stiglitz and Weiss [31] have shown that even in cases of individual borrowing, because of informational asymmetries or problems associated with moral hazard, risk premium or credit constraints, or both, are known to exist.

depend on the economic context we choose to consider[12]. This game can be interpreted as an economy populated by investors that seek to finance their intertemporal consumption, given the returns of their portfolios. Investors may finance consumption by investing their initial capital in a portfolio composed of a risk-free asset and a risky asset that is linked to other players investment decisions and market institutional conditions. This type of investor chooses to diversify their portfolios to finance present and future consumption. Otherwise, investors may choose to hold only risk-free assets in their portfolio and finance their investment and consumption decisions through the accumulation of financial debt. In this particular case, investors use risk-free asset returns to roll on existing debt contracts. Present and future accumulation of risk-free assets guarantee the sustainability of their leveraged position. Another interesting interpretation is given by an economy populated by exchange rate speculators that can deposit their capital domestically and borrow capital from abroad, or invest in foreign currency and face exchange risk in foreign currency deposits. Throughout the remainder of this paper, we shall follow a twofold approach regarding the interpretation of solutions to the games discussed in section 3 and section 4. When convenient, we shall consider that these setups define growing economies populated by agents that invest in productive/domestic assets and choose to be either net borrowers or net lenders. However, the alternative interpretation as a portfolio investor game is also reasonable and provides interesting insight on some of the results arising from these specific proposals.

Following this introduction, we now focus on the the solution to the open loop case, described by the *Pontryagin* maximum conditions given in Appendix A.1. We start by deriving the optimal *Keynes-Ramsey* consumption strategies[13]. The first optimal consumption strategy is defined in the following fashion. We start by taking the time derivative from the optimality condition in consumption given by (A.2), which is given by,

$$\dot{\lambda}_i(t) = -(\gamma_i - 1)\gamma_i C_i(t)^{\gamma_i - 2}\dot{C}_i(t) \Longleftrightarrow \dot{\lambda}_i(t) = \lambda_i(\gamma_i - 1)C_i(t)^{-1}\dot{C}_i(t). \tag{13}$$

Substituting (13) and (A.2), in the co-state condition (A.4), we obtain the optimal *Keynes-Ramsey* consumption open loop strategy, describing optimal consumption strategies driven by net financial assets accumulation:

$$\dot{C}_i(t) = \frac{C_i(t)}{\gamma_i - 1}\left(\rho_i - r - rd_i\frac{B(t)}{K(t)}\right). \tag{14}$$

In order to obtain the second *Keynes-Ramsey* consumption strategy, we follow the same procedure as above. First we take the time derivative of the optimality condition for investment decisions, given by equation (A.3). We obtain:

$$\dot{q}_i(t) = -\dot{\lambda}_i(t) \Longleftrightarrow \dot{q}_i(t) = \lambda_i(\gamma_i - 1)C_i(t)^{-1}\dot{C}_i(t). \tag{15}$$

Substituting (15) and (A.2), after considering the result in (A.3), in the co-state condition describing the shadow price of productive capital, (A.5), the second optimal consumption strategy[14] is given by:

$$\dot{C}_i(t) = \frac{C_i(t)}{\gamma_i - 1}(\rho_i + \delta - r_k). \tag{16}$$

[12]The simplicity of these proposals have both advantages and drawbacks. The main advantages of the simplified framework proposed in this section and the next is, in our opinion, its mathematical tractability and the flexibility it allows in terms of economic interpretation. On the other hand, the extreme simplicity of these proposals does not take into account the diversity and complexity of real economic phenomena. Moreover, these setups are loosely related to neoclassical economic theory, but they do not not take into account all theoretical assumptions usually required in mainstream economics. We consider that this trade-off eventually arises when one considers nonlinear economies in a differential game framework. The economic modeller must take into account this natural trade-off, and seek a reasonable balance between the mathematical tractability of the proposed problem, and its economic interpretation, in the context of related theoretical fundamentals.

[13]By *Keynes-Ramsey* consumption rules, we mean the intertemporal dynamic consumption decisions that are obtained for this control variable in an optimal control problem with a constant intertemporal discount rate. In macroeconomics literature these dynamic equations are known by *Keynes-Ramsey* consumption rules, following the work by the two famous Cambridge scholars, which related intertemporal consumption decisions with the discounted value of expected future incomes and optimal savings for capital accumulation. It is our opinion that in economy optimization problems with two capital accumulation state variables, this rule is not unique, since state defined income accumulation can vary in its source. Therefore it is reasonable to impose two possible consumption paths that satisfy the optimal investment condition. In this model, optimal investment decisions impose an indifference rule on the intertemporal marginal adjustment between different assets, in order to allow for distinct capital accumulation decisions. This mechanism has the following interpretation, investors will always choose to accumulate assets that adjust faster to optimum outcomes rather than invest in assets that yield longer adjustment rates. In economics jargon the co-state variables represent the shadow price (or marginal value) of a specific asset.

[14]Condition (16) defines optimal consumption paths assuming income arising from the accumulation of productive capital while condition (14) defines consumption financed by financial net assets accumulation.

Now we need to impose the optimal accumulation rule that guarantees indifference between consumption strategies[15] for player i. Setting (14) equal to (16), and substituting the capital accumulation equation, (A.7), we obtain the following result defined in terms of the aggregate net financial balances ratio, $B(t)/K(t)$,

$$\frac{B(t)}{K(t)} = \frac{r + \delta - r_k}{rd_i}. \tag{17}$$

Given that in an open loop setup, agents do not have information on the evolution of the state of the system, this outcome can be interpreted as an individual belief regarding the true outcome of aggregate market risk premium. In the absence of information on the evolution of the state of the system, players base their decisions on individual beliefs. Since beliefs depend on d_i, we shall have asymmetric beliefs arising from asymmetries on market determined institutional risk premium. We deal with this feature of the game later on and focus now on the description of asymptotic conditions guaranteeing the existence of SCE solutions consistent with (17). The open loop solution to this game is defined by consumption, (14), net financial assets, (A.6), and productive capital dynamics, (A.7), assuming the existence of optimal beliefs, as defined by (17), in (14). This system defines a solution described by a set of non-stationary variables. It is necessary to define a scaling rule consistent with the existence of stationary dynamics. We define a stationary dynamical system by taking advantage of the scaled invariance of the dynamics, and redefine the variables, $X_{m,i}(t)$, in terms of domestic capital units:

$$Z_{m,i}(t) = \frac{X_{m,i}(t)}{K_i(t)} \Rightarrow \dot{Z}_{m,i}(t) = \frac{\dot{X}_{m,i}(t)}{K_i(t)} - \frac{X_{m,i}(t)}{K_i(t)}\frac{\dot{K}_i(t)}{K_i(t)}, \tag{18}$$

where $m \in \{1, 2, 4\}$ and $X_{m,i}(t)$ defines consumption, $C_i(t)$, net foreign assets, $B_i(t)$, investment, $I_i(t)$, and $Z_{m,i}(t)$ each corresponding scaled variable for each player $i \in N$, respectively. Following the rule given in (18) the stationary solution to (12) comes out as:

$$\dot{Z}_{1,i}(t) = Z_{1,i}(t)\left[\frac{\rho_i + \delta - r_k - (Z_{4,i}(t) - \delta)(\gamma_i - 1);}{\gamma_i - 1}\right]; \tag{19}$$

$$\dot{Z}_{2,i}(t) = Z_{1,i}(t) + Z_{4,i}(t) + Z_{2,i}(t)\left[r + rd_i Z_2(t) - Z_{4,i}(t) + \delta\right] - r_k; \tag{20}$$

where:

$$K_i(t) = K_i(0)\, e^{\int_0^t (Z_{4,i}(s) - \delta) ds} \Rightarrow K(t) = \sum_{i \in N} K_i(0)\, e^{\int_0^t (Z_{4,i}(s) - \delta) ds}; \tag{21}$$

$$Z_2(t) = \frac{\sum_{i \in N} Z_{2,i}(t) K_i(t)}{\sum_{i \in N} K_i(t)}. \tag{22}$$

The proposed solution to (12) does not provide any information regarding scaled investment strategies. We solve this issue by assuming that in the long run agents commit to linear strategies on investment per unit of capital that are consistent with the existence of equilibrium, $\lim_{t\to\infty} Z_{4,i}(t) = \bar{Z}_{4,i} \Rightarrow \lim_{t\to\infty} \dot{Z}_{1,i}(t) = 0$. These strategies are optimal if they solve the stationary differential system, given in (19) and (20), and the solution is consistent with the tranversality conditions given in (A.8) and (A.9). If such strategies are asymptotically consistent with players' optimal beliefs, $Z_2^{b,i}(t)$, following condition (17), we consider that equilibrium is a SCE, when $\lim_{t\to\infty} Z_2^{b,i}(t) = \bar{Z}_2$. We start by defining scaled investment strategy equilibrium, and assume these strategies guarantee asymptotic convergence to a feasible scaled consumption strategic equilibrium, $Z_1(t) > 0, \forall t \in T$. Setting $\dot{Z}_{1,i}(t) = 0$ we obtain:

$$\lim_{t\to\infty} Z_{4,i}(t) = \bar{Z}_{4,i} = \frac{\rho_i + \delta - r_k}{\gamma_i - 1} + \delta. \tag{23}$$

Following (23), long run capital dynamics can be defined in the following fashion:

$$\lim_{t\to\infty} t^{-1} \log K_i(t) = \bar{Z}_{4,i} - \delta > 0. \tag{24}$$

Given (23), we can redefine consumption dynamics as a function of scaled investment strategies:

$$Z_{1,i}(t) = Z_{1,i}(0)\, e^{\int_0^t (\bar{Z}_{4,i} - Z_{4,i}(s)) ds} \tag{25}$$

[15] We would like to stress that this result is independent of our interpretation of indifference between optimal consumption strategies. The same condition defining optimal beliefs is obtained when substituting directly (14) while deriving (17).

Given the result in (25), we know that it is optimal for agents to undergo transitions that improve their initial scaled consumption strategies, such that in equilibrium $\bar{Z}_{1,i} > Z_{1,i}(0)$[16] . This result is sufficient to study the qualitative dynamics of this game SCE, when we assume that (23) is a globally stable solution to $Z_{4,i}(t)$ dynamics. We can now derive the conditions for the existence of game outcomes consistent with an asymptotic optimal SCE. We start by defining aggregate state dynamics, as given in (22), asymptotically. First, we assume that there is an unique equilibrium solution for individual state dynamics, $\bar{Z}_{2,i}$, obtained by solving $\dot{Z}_{2,i} = 0$. The long run outcome of $\bar{Z}_2$ is given by the asymptotic limit of (22), following the result in (23) for productive capital dynamics in the long run. Aggregate risk premium in the long run is thus given by:

$$\lim_{t\to\infty} Z_2(t) = \bar{Z}_2 = \frac{\sum_{j\in L} \bar{Z}_{2,j} K_j(0)}{\sum_{j\in L} K_j(0)}, \tag{26}$$

where player $j \in L$ corresponds to the subset of players that have scaled investment strategies consistent with $\bar{Z}_{4,j} = \max\left(\bar{Z}_{4,i}\right)$[17]. This result has a straightforward interpretation. Aggregate risk premium dynamics are driven by the game investment leaders in the long run. The long run risk premium condition given in (26) defines a relative measure that takes into account the financial situation of the ensemble of leaders in this economy and weights it against their initial productive capital endowments. In the long run, market forces price aggregate risk premium following the financial outcomes of the players choosing more aggressive, and therefore riskier investment strategies. Leverage based aggressive investment strategies raise the game bond premium, while aggressive investment strategies from players with a diversified portfolio lower the game bond premium. This long run market risk premium measure can be justified by the existence of information costs that deter investors from acquiring relevant information. Under these circumstances it is a reasonable decision to price aggregate risk based on a sample of aggressive investors and their portfolio decisions. Recall now that we assumed that the existence of a SCE requires optimal beliefs, (17), to be fulfilled at least asymptotically. Substituting (26) in (17) and solving in terms of d_i, we obtain the condition guaranteeing beliefs are consistent with a SCE asymptotically:

$$d_i = \frac{r + \delta - r_k}{r\bar{Z}_2}. \tag{27}$$

The existence of a SCE solution is not consistent with asymmetric individual institutional risk premium measures. This game is only consistent with the existence of a SCE if the asymmetry assumption regarding individual institutional risk premium measures is dropped[18] and we consider that d_i is endogenously determined as a function of $\bar{Z}_2$. We can interpret this measure in the context of a market regulator with perfect information about the state of the game, which sets institutional risk premium in accordance with the economy aggregate outcomes. Following this result we can define the equilibrium solution, $\bar{Z}_{2,i}$, by setting $\dot{Z}_{2,i} = 0$, and substituting the asymptotic condition for optimal beliefs, (27). Individual state dynamics in the long run are given by:

$$\bar{Z}_{2,i} = \frac{r_k - \bar{Z}_{1,i} - \bar{Z}_{4,i}}{r + rd_i\bar{Z}_2 - \bar{Z}_{4,i} + \delta}. \tag{28}$$

The individual and aggregate state outcomes given in (28) and (26) are not unique solutions for investment leaders. To solve this issue we substitute the constraint for SCE, (27), in (28), and obtain:

$$\bar{Z}_{2,i} = \frac{r_k - \bar{Z}_{1,i} - \bar{Z}_{4,i}}{2r + 2\delta - r_k - \bar{Z}_{4,i}}. \tag{29}$$

The result in (29) confirms that a unique SCE exists in this game, when individual institutional risk premium valuations are the same for all players. This solution is optimal if $\bar{Z}_1 > 0$ and the transversality constraints are fulfilled. In order to define conditions for transversality in this differential game, it is convenient to redefine (A.8) and (A.9) in terms of the scaled variables. Substituting the co-state variables by the optimality conditions, (A.2) and (A.3), and imposing the scaling rule, (18), the transversality conditions come out as:

$$\lim_{t\to\infty} \left(-\gamma_i Z_{1,i}(t) K_i(t)\right)^{\gamma_i - 1} Z_{2,i}(t) K_i(t) e^{-\rho_i t} = 0; \tag{30}$$

[16] if $Z_{4,i}(t) = \bar{Z}_{4,i}, \forall t \in T$ then $Z_{1,i}(t) = Z_{1,i}(0), \forall t \in T$.

[17] The asymptotic limit of a sum ratio of equal exponential terms with different coefficients is given by the sum ratio of the coefficients of the fastest growing exponential terms.

[18] One could also consider that SCE with asymmetric risk premium requires the existence of asymmetric productive capital returns. We do not explore this hypothesis because it is of no particular relevance to the qualitative description of equilibrium.

$$\lim_{t\to\infty} \left(\gamma_i Z_{1,i}(t) K_i(t)\right)^{\gamma_i-1} K_i(t) e^{-\rho_i t} = 0. \tag{31}$$

These expressions can be rearranged by taking the scaled limit of the logarithm of (30) and (31) and solving the transversality constraints as an asymptotic inequality[19]. The constraints (30) and (31) are now given by:

$$\lim_{t\to\infty} t^{-1} \log\left[\left(-\gamma_i Z_{1,i}(t) K_i(t)\right)^{\gamma_i-1} Z_{2,i}(t) K_i(t) e^{-\rho_i t}\right] < 0; \tag{32}$$

$$\lim_{t\to\infty} t^{-1} \log\left[\left(\gamma_i Z_{1,i}(t) K_i(t)\right)^{\gamma_i-1} K_i(t) e^{-\rho_i t}\right] < 0. \tag{33}$$

From (32) or (33) it is straightforward to obtain the transversality constraint for the existence of an optimal solution as a function of long run scaled investment decisions. Assuming that capital dynamics grows asymptotically at a constant rate, following the result in (24), and that the scaled variables are consistent with balanced long run growth dynamics $\lim_{t\to\infty} Z_{m,i}(t) \to \bar{Z}_{m,i}$, where $\bar{Z}_{1,i}, \bar{Z}_{4,i} \in \Re^+$ and $\bar{Z}_{2,i} \in \Re$, the final condition for existence of asymptotic optimal open loop investment strategies guaranteeing long run productive capital growth is given by:

$$\delta < \bar{Z}_{4,i} < \delta + \frac{\rho_i}{\gamma_i}. \tag{34}$$

Having described the conditions for the existence of SCE solutions consistent with player's optimal beliefs, for the non-cooperative game given in (12), we now focus on the qualitative description of this solution. We base our approach on a weak argument for asymptotic stability. This argument is based on the results described in (23) to (29), which guarantee that a self-fulfilling equilibrium is always achieved asymptotically and independent of other players decisions, when institutional risk premium is unique and a function of the state of game. Since in the long run there are no longer transitions driven by $Z_{1,i}(t)$ and $Z_2(t)$ dynamics, when we assume $Z_{4,i}(t)$ dynamics always converges to the equilibrium defined in (23), we can evaluate qualitatively the local stability of the SCE strategies by testing the stability of the system describing scaled net assets dynamics, $\left\{\dot{Z}_{2,1}(t), \ldots, \dot{Z}_{2,n}(t)\right\}$, following the individual state dynamic condition defined in (20). We start by defining the n by n *Jacobian* matrix describing individual state dynamics in the vicinity of SCE,

$$J = \left[\begin{array}{cccc} \frac{\partial \dot{Z}_{2,1}(t)}{\partial Z_{2,1}(t)} & \frac{\partial \dot{Z}_{2,1}(t)}{\partial Z_{2,2}(t)} & \cdots & \frac{\partial \dot{Z}_{2,1}(t)}{\partial Z_{2,n}(t)} \\ \frac{\partial \dot{Z}_{2,2}(t)}{\partial Z_{2,1}(t)} & \frac{\partial \dot{Z}_{2,2}(t)}{\partial Z_{2,2}(t)} & \cdots & \frac{\partial \dot{Z}_{2,2}(t)}{\partial Z_{2,n}(t)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \dot{Z}_{2,n}(t)}{\partial Z_{2,1}(t)} & \frac{\partial \dot{Z}_{2,n}(t)}{\partial Z_{2,2}(t)} & \cdots & \frac{\partial \dot{Z}_{2,n}(t)}{\partial Z_{2,n}(t)} \end{array}\right]_{Z_{2,i}(t)=\bar{Z}_{2,i}}, \tag{35}$$

where the partial derivatives of this *Jacobian* are given generally by the following expressions:

$$\frac{\partial \dot{Z}_{2,i}(t)}{\partial Z_{2,i}(t)} = r - Z_{4,i}(t) + \delta + rd_i \left[Z_2(t) + Z_{2,i}(t) \frac{K_i(t)}{\sum_{i\in N} K_i(t)}\right]; \tag{36}$$

$$\frac{\partial \dot{Z}_{2,i}(t)}{\partial Z_{2,w}(t)} = rd_i Z_{2,i}(t) \frac{K_w(t)}{\sum_{i\in N} K_i(t)}, w \neq i \wedge w, i \in N. \tag{37}$$

To evaluate these derivatives in equilibrium, we have to distinguish between investment and non investment leaders. If players $i, w \in N$ are investment leaders, $i, w \in L$, then (36) and (37) evaluated in equilibrium come out as,

$$\left.\frac{\partial \dot{Z}_{2,i}(t)}{\partial Z_{2,i}(t)}\right|_{Z_i(t)=\bar{Z}_i} = r - \bar{Z}_{4,i} + \delta + rd_i \left[\bar{Z}_2 + \bar{Z}_{2,i} \frac{K_i(0)}{\sum_{j\in L} K_j(0)}\right], i \in L, \tag{38}$$

$$\left.\frac{\partial \dot{Z}_{2,i}(t)}{\partial Z_{2,w}(t)}\right|_{Z_i(t)=\bar{Z}_i} = rd_i \bar{Z}_{2,i} \frac{K_w(0)}{\sum_{j\in l} K_j(0)}, w \neq i \wedge w \in L. \tag{39}$$

If players $i, w \in N$ are not investment leaders, $i, w \notin L$, then (37) vanishes and (36) reduces to,

[19] Recall that a dynamic process that scales exponentially, $h(t) \sim e^{\phi t}$, can be defined asymptotically in the following fashion, $\lim_{t\to\infty} t^{-1} \log h(t) = \phi$. If $\alpha > 0 \Rightarrow h(t) \to \infty$. If $\phi < 0 \Rightarrow h(t) \to 0$.

$$\left.\frac{\partial \dot{Z}_{2,i}(t)}{\partial Z_{2,i}(t)}\right|_{Z_i(t)=\bar{Z}_i} = r - \bar{Z}_{4,i} + \delta + r d_i \bar{Z}_2. \tag{40}$$

The local stability of SCE solutions for the game defined in (12) can be easily evaluated numerically. If all the eigenvalue solutions of (35) have a negative real part, then we can state that SCE solutions are at least locally weakly asymptotically stable. A robust argument for local asymptotic stability would have to take into account transitions to equilibrium arising from $Z_{4,i}(t)$ decisions and $Z_2(t)$ non-autonomous dynamic transitions[20].

To demonstrate our conjecture, we require a final assumption. In this section, we analyse the qualitative dynamics assuming that players commit to initial investment equilibrium strategies, $Z_{4,i}(t) = \bar{Z}_{4,i} \Rightarrow Z_{1,i}(t) = Z_{1,i}(0), \forall t \in T$. This simplifying hypothesis has the advantage of not requiring any assumption regarding individual investment transitions to equilibrium, thus reducing greatly the burden required to perform a vast qualitative analysis. It allows for a systematic evaluation of the local stability of SCE solutions for large populations, given different probabilistic assumptions regarding initial conditions and parameter distributions. This assumption has the following economic interpretation. Players choose beliefs such that their consumption outcome relative to their wealth, measured by productive assets accumulation, is stable throughout the duration of the game. This result is in accordance with the *Life Cycle* hypothesis for intertemporal consumption[21]In the end of this section, we evaluate the dynamics of this game assuming two different hypotheses for the initial consumption endowments. We shall consider that $Z_{1,i}(0)$ is given by random outcomes distributed according to $Z_{1,i}(0) \sim U(0,1)$ and $Z_{1,i}(0) \sim \exp(1)$. In these experiments, we consider the existence of a robust population, with initial productive/riskless asset endowments randomly given by an exponential distribution, $K_i(0) \sim \exp(1)$. By robust population, we mean a discrete set of $n = 1000$ agents with uniform randomly drawn individual characteristics, $\rho_i, \gamma_i \sim U(0,1)$, such that $\bar{Z}_{4,i}$ outcomes, defined by (23), fulfil the optimal growth constraint, (34), for the range institutional scenarios, $r_k \in [0.05\ 0.25]$ and $r \in [0.03\ 0.25]$, where $\delta = 0.03$. For simplification reasons, we consider that the state of the game is driven by a fixed pool of investors, which is defined by fixed share of the population. We set this share at 30%, and consider that the aggregate risk faced by investors, (26), is obtained from the share of aggressive players with higher rates of investment per unit of capital. This assumption is consistent with risk setting in real markets. The LIBOR spread for example, is defined by a similar institutional mechanism, where only the average interest rate on credit transactions faced by a pool of the large financial institutions is considered.

Given this final set of assumptions, we start the discussion on the qualitative dynamic outcomes of this system with the analysis of a game where $Z_{1,i}(0) \sim U(0,1)$. In Figure 1, below, we portray the general qualitative outcomes for SCE solutions, in the figure on the left, for a given randomly drawn robust population, described by the density plot on the right. In this setup, robust populations are characterized by patient investors with a low intertemporal elasticity. The stability diagram is dominated by stable solutions, all eigenvalues have negative real parts, and unstable solutions[22], all eigenvalues have positive real parts. This diagram suggests that a stable SCE solution requires institutional scenarios ordered in the following fashion, $r_k \leq r$. There is also a transition region, described in yellow, that is consistent with saddle type solutions for this game. In this region we have eigenvalues with positive and negative real parts. Bifurcations of equilibria can also arise as a result of a degenerate *Jacobian*, $\det(J) = 0$[23]. In the vicinity of these singularities, higher-dimensional nonlinear phenomena may occur.

To better understand the qualitative results portrayed in Figure 1, it is convenient to illustrate the distributions for the endogenous variables, $\bar{Z}_2$ and d. In Figure 2, we observe that stability requires that the investment leaders are bond sellers and institutional risk premium is positive but small. On the other hand, when leaders diversify their portfolios and the regulator sets risk premium at a small but negative rate, SCE solutions are unstable. The critical transition region is characterized by leaders accumulating vast amounts of financial/foreign assets or vast amounts of debt. The regulator in this region sets risk premium close to zero.

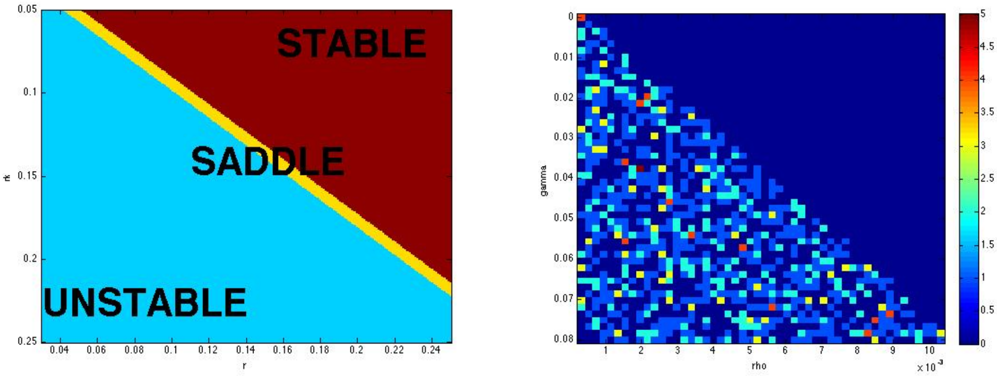
We then we compare the qualitative dynamics of this system with the analysis of a game where $Z_{1,i}(0) \sim \exp(1)$. The results portrayed in Figure 3 and Figure 4 do not show any major difference from the results discussed

[20]If we consider that the fixed point defined by $\bar{Z}_{1,i} = 0$ is always a repelling solution, following the discussion on $Z_{4,i}(t)$ dynamics, then we can assume that SCE is a globally stable solution if it fulfils the *Routh-Hurwitz Stability Criterion*. If we additionally assume that $Z_{4,i}(t)$ dynamics are *Lipschitz* continuous, then it is possible to give a full description of this game dynamics.

[21]The *Life Cycle* consumption hypothesis was forwarded by Brumberg and Modigliani [7] and Friedman [13]. This theory suggests that individuals make saving and consumption decisions, in order to maintain a stable consumption pattern throughout their lives. Evidence suggests that the *Life Cycle* hypothesis is not consistent with saving and consumption patterns observed in older members of the population. Older generations show patterns of precautionary saving that can be explained by: (i) intergenerational altruism; (ii) increased caution in spending; and (iii) poor retirement planning based on optimistic assumptions about life expectancy.

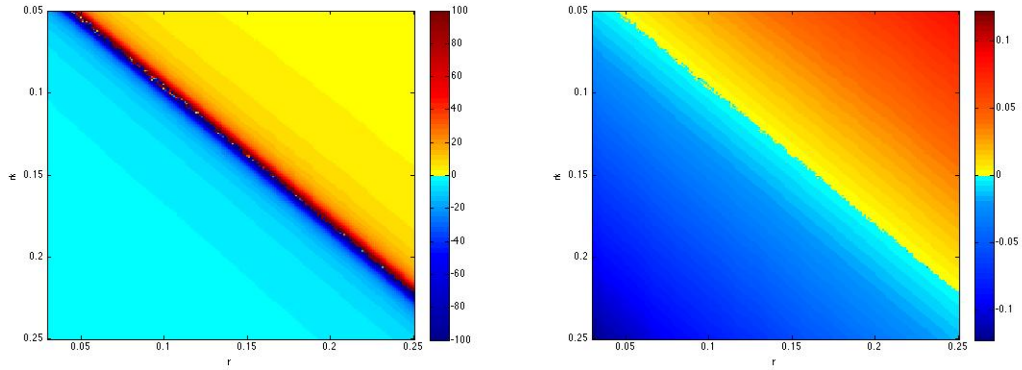
[22]By unstable solutions we mean that the SCE is a completely unstable solution, which implies that the SCE solution is time-reverse stable.

[23]In these simulations we only evaluate the real part of the two leading eigenvalues, $\min\{\mathrm{Re}(\Lambda)\}$ and $\max\{\mathrm{Re}(\Lambda)\}$, where Λ stands for the set of eigenvalues solving the characteristic polynomial of (35) in the vicinity of SCE.



(a) Stability diagram (b) Robust population distribution

Figure 1: Stability diagram and density distribution of preferences for $Z_{1,i}(0) \sim U(0,1)$



(a) $\bar{Z}_2$ outcomes (b) d outcomes

Figure 2: Game and institutional risk premium outcomes for $Z_{1,i}(0) \sim U(0,1)$

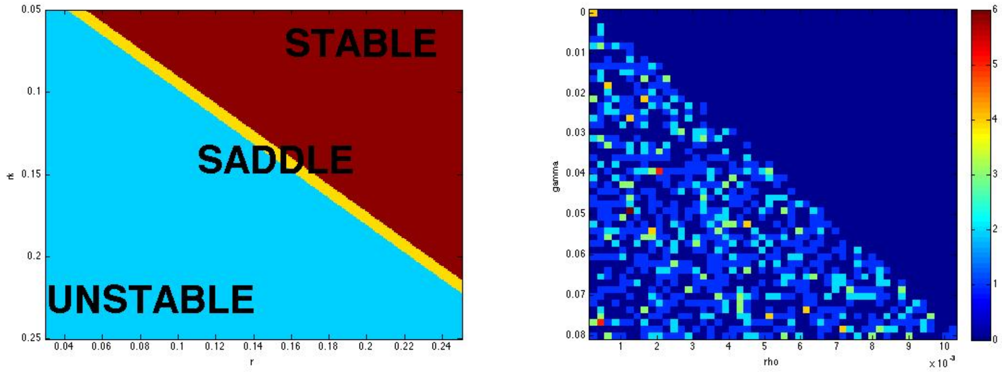
above in Figure 1 and Figure 2. This outcome suggests that scaled consumption values do not seem to interfere much with the qualitative dynamics of this game. To have a better insight in this matter, we have to consider sampling results for larger populations sets, assuming different distribution hypotheses for initial consumption[24].

We finish this presentation with a discussion on the qualitative dynamics of the state-separable problem in the vicinity of SCE solutions. The solution and qualitative analysis of the state-separable solution, assuming $Z_{4,i}(t) = \bar{Z}_{4,i}, \forall t \in T$, is given in Appendix B.1. The condition describing qualitative dynamics in the vicinity of equilibrium is given generally by (B.6). Rearranging this expression, we obtain a condition for stability in terms of the marginal revenue on domestic capital for the state-separable game in the vicinity of the SCE equilibrium,

$$r_k > \frac{2r(\gamma_i - 1)}{\gamma_i - 2} + \delta - \frac{\rho_i}{\gamma_i - 2}. \tag{41}$$

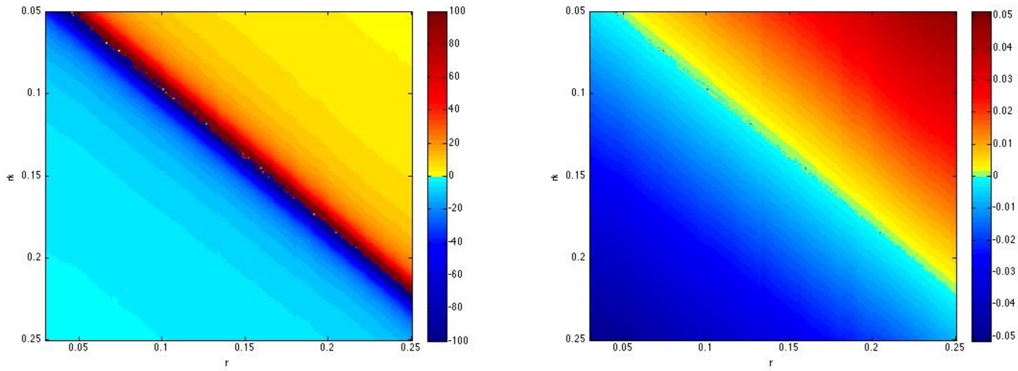
Following (41), we can evaluate numerically the qualitative dynamics of the state-separable solutions for the institutional parameter scenarios described. In this analysis, we consider the percentage of players that have a stable solution in the vicinity of a SCE. These results are given in Figure 5, below. Given that consumption does not play a role in the state-separable solution, the differences observed are related to player asymmetries. A quick glance at the figures below, shows that the game stable and unstable regions, portrayed in Figure 1 and Figure 3, coincide with regions, where players have stable or unstable solutions. The saddle region is now characterized by region where only a share of players have a stable solution. We can draw a simple conclusion from this outcome. If we consider the game solution has a sum of its individual parts, we might be tempted to consider that this region

[24] Our numerical simulations suggest that there are no significant differences between these two hypotheses for games with a share population of leaders ranging from 10% to 90%. Given the constraints imposed for the existence of robust populations, it is likely that this result holds for a broad range of scaled consumption values in large populations.



(a) Stability diagram (b) Robust population distribution

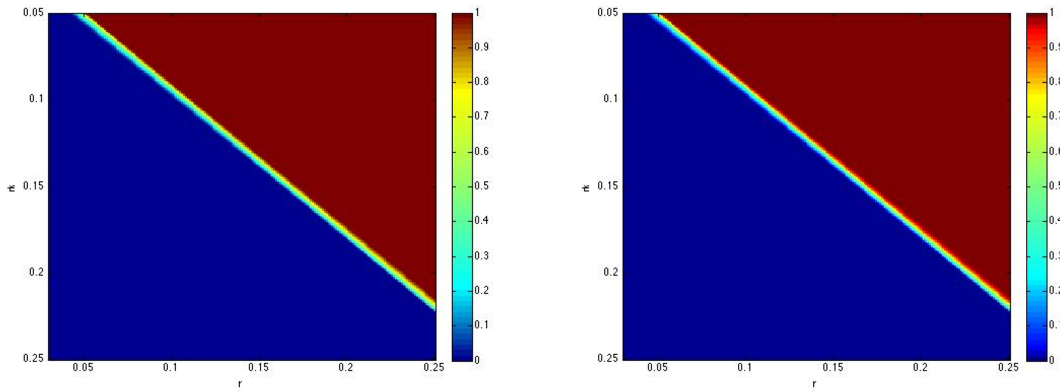
Figure 3: Stability diagram and density distribution of preferences for $Z_{1,i}(0) \sim exp(1)$



(a) $\bar{Z}_2$ outcomes (b) d outcomes

Figure 4: Game and institutional risk premium outcomes for $Z_{1,i}(0) \sim exp(1)$

is not consistent with stable SCE solutions, as a percentage of players does not agree with this SCE. However, the game solution suggests the existence of a saddle equilibrium and bifurcation phenomena. Thus we conclude that there are institutional regimes, where game dynamics in the vicinity of a SCE cannot be described by just the sum of its individual parts, as defined by the state-separable outcomes, and weak emergence phenomena might occur in this economy, as a result of aggregate interactions consistent with higher-dimensional nonlinear phenomena.



(a) $Z_{1,i}(0) \sim U(0,1)$ (b) $Z_{1,i}(0) \sim exp(1)$

Figure 5: Percentage of players with stable dynamics

In this section, we showed that when beliefs are unique, strategies are *Lipshitz* continuous and the game solution is bounded asymptotically, it is possible to perform a qualitative analysis of SCE solutions. This setup allowed us to show that weak emergence phenomena may arise for some parameter regions. In the next section, we show that the introduction of further nonlinearities leads to solutions where a SCE is only achievable when learning dynamics are considered. We show that feasible SCE solutions can be identified geometrically and some conjectures about possible game outcomes can be put forward, based on the qualitative analysis of the state-separable problem.

4. A consumption/investment game with coupled endogenous risk premium

The second game we propose is largely based on the setup discussed in the previous section with a simple exception. We now drop the institutional risk premium hypothesis and consider that individual risk premium depends on the ratio of net financial assets to productive capital. Interest payments/revenues are now given by $rB_i(t)\left[1+\left(B_i(t)/K_i(t)\right)\left(B(t)/K(t)\right)\right]$. In this setup, institutional conditions driving risk premium no longer depend on market driven beliefs, but on information regarding the player financial balances. The inclusion of an additional nonlinearity in the risk premium mechanism allows for the introduction of several novel features in this economy. Bond buyers are now rewarded when the aggregate economy is a net buyer of bond contracts, and penalized when the economy is a net issuer of bonds. Bond issuers benefit from smaller interest premiums, when the aggregate economy is net buyer of bonds and are penalized if the aggregate economy is a net issuer of bonds. Given this very brief introduction, the non-cooperative differential game faced by player i is given by the following dynamic optimization problem:

$$\begin{aligned}
&\underset{C_i(t),I_i(t)}{MAX}\int_0^\infty e^{-\rho_i t}C_i(t)^{\gamma_i}\,dt\\
&\text{subject to the solution of:}\\
&\dot{B}_i(t)=C_i(t)+I_i(t)+rB_i(t)\left(1+\tfrac{B_i(t)}{K_i(t)}\tfrac{B(t)}{K(t)}\right)-r_kK_i(t);\\
&\dot{K}_i(t)=I_i(t)-\delta K_i(t);
\end{aligned}\tag{42}$$

satisfying the transversality conditions, (A.18) and (A.19), guaranteeing that solutions to (42) do not grow too fast. The optimal *Keynes-Ramsey* consumption strategies for (42), following the procedure described in 3, and given the maximum conditions in Appendix A.2, are now defined by:

$$\dot{C}_i(t)=\frac{C_i(t)}{\gamma_i-1}\left(\rho_i-r-2r\frac{B(t)\,B_i(t)}{K(t)\,K_i(t)}\right);\tag{43}$$

$$\dot{C}_i(t)=\frac{C_i(t)}{\gamma_i-1}\left(\rho_i+\delta-r\frac{B(t)\,B_i(t)^2}{K(t)\,K_i(t)^2}-r_k\right).\tag{44}$$

The optimality condition for indifference in capital accumulation is obtained by setting (43) equal to (44). Following the discussion in 3, we solve this equality in terms of the aggregate coupled risk premium mechanism, $B(t)/K(t)$. The optimal belief for this game is,

$$\frac{B(t)}{K(t)}=(\delta+r-r_k)\left[r\frac{B_i(t)}{K_i(t)}\left(\frac{B_i(t)}{K_i(t)}-2\right)\right]^{-1}.\tag{45}$$

Substituting the optimal belief, (45), in the optimal consumption strategy, (43), and then scaling(43) and (A.16), following the rule given in (18), we obtain the stationary dynamical system defining the general open loop dynamic solution for the non-cooperative game given in (42), in terms of player i dynamics:

$$\dot{Z}_{1,i}(t)=Z_{1,i}(t)\left[\frac{\rho_i-r-2r\,(r+\delta-r_k)\left(Z_{2,i}(t)-2\right)^{-1}-(\gamma_i-1)\left(Z_{4,i}(t)-\delta\right)}{\gamma_i-1}\right];\tag{46}$$

$$\dot{Z}_{2,i}(t)=Z_{1,i}(t)+Z_{4,i}(t)+Z_{2,i}(t)\left[r+rZ_{2,i}(t)\,Z_2(t)-Z_{4,i}(t)+\delta\right]-r_k;\tag{47}$$

where capital dynamics, $K_i(t)$, and state dynamics, $Z_2(t)$, are given by (21) and (22). Again, there is no information regarding investment strategies. We start by defining the equilibrium condition for state dynamics, $\lim_{t\to\infty}Z_{2,i}(t)=\bar{Z}_{2,i}$, assuming the existence of an asymptotic equilibrium solution for investment strategies, $\lim_{t\to\infty}Z_{4,i}(t)=\bar{Z}_{4,i}$, consistent with $\dot{Z}_{1,i}(t)=0$, and the growth and transversality condition defined in (34). Assuming solutions are consistent with a feasible economic outcome, $Z_1(t)>0,\forall t\in T$, the individual state equilibrium solution is given by:

$$\lim_{t\to\infty} Z_{2,i}(t) = \bar{Z}_{2,i} = \frac{2r(r+\delta-r_k)}{\rho_i - r - \left(\bar{Z}_{4,i}-\delta\right)(\gamma_i-1)} + 2. \tag{48}$$

Following these assumptions on the long run dynamics of $Z_{2,i}(t)$ and $Z_{4,i}(t)$ and the result in (48), aggregate state dynamics is again defined asymptotically by (26), following the discussion in 3. This set of assumptions is again sufficient to define the conditions for the existence of SCE solutions consistent with the player's optimal beliefs. We start by redefining player beliefs asymptotically as a function of the scaled variables. Optimal beliefs, $Z_2^{b,i}(t)$, in the long run are now given by:

$$\lim_{t\to\infty} Z_2^{b,i}(t) = \bar{Z}_2^{b,i} = (r+\delta-r_k)\left[r\bar{Z}_{2,i}\left(\bar{Z}_{2,i}-2\right)\right]^{-1}. \tag{49}$$

Rearranging (49) and solving in terms of $\bar{Z}_{2,i}$, we obtain:

$$\lim_{t\to\infty} Z_{2,i}(t) = \bar{Z}_{2,i} = 1 \pm \sqrt{1+\frac{r+\delta-r_k}{r}\left(\bar{Z}_2^{b,i}\right)^{-1}}. \tag{50}$$

Recall now that the existence of a SCE requires that $\bar{Z}_2 = \bar{Z}_2^{b,i}$ is fulfilled. From the result in (50), and following the equilibrium condition for individual state dynamics, (48), it becomes clear that agents have to pursue scaled investment strategies that involve learning dynamics and are consistent with the existence of a self confirming equilibrium asymptotically, $\lim_{t\to\infty} Z_{4,i}\{E[Z_2(t)]\} \to \bar{Z}_{4,i}\left(\bar{Z}_2\right)$[25]. This feature of the game becomes clear when we substitute (48) in (50), and obtain the asymptotic condition for scaled investment strategies that fulfils the optimal belief condition and the existence of a SCE. Investment strategies have to be consistent with

$$\lim_{t\to\infty} Z_{4,i}(t) = \bar{Z}_{4,i} = \delta + \frac{\rho_i - r}{\gamma_i - 1} - \frac{2r(r+\delta-r_k)}{(\gamma_i-1)\left(-1 \pm \sqrt{1+\frac{r+\delta-r_k}{r}\bar{Z}_2^{-1}}\right)}, \tag{51}$$

where we already assumed $\bar{Z}_2 = \bar{Z}_2^{b,i}$. This solution is an asymptotic optimal solution to (42) if tranversality conditions, given by (A.18) and (A.19), are fulfilled. Following the discussion in the previous section and the results described in (30) to (33), transversality and growth conditions impose that long run scaled investment strategies are consistent with (34). We can finish the description of this game equilibrium with the definition of scaled consumption strategies, $\bar{Z}_{1,i}$. We define long run scaled consumption strategies by setting $\dot{Z}_{2,i} = 0$ and assuming that a SCE solution, as defined in (49) to (51), is achieved:

$$\bar{Z}_{1,i} = r_k - \bar{Z}_{4,i} - \bar{Z}_{2,i}\left(r + r\bar{Z}_{2,i}\bar{Z}_2 - \bar{Z}_{4,i} + \delta\right). \tag{52}$$

From (52) it is obvious that that the best strategy for player i consists in the maximization of the expected value of scaled consumption strategies, $E\left(\bar{Z}_{1,i}\right)$, given some learning process driving scaled investment strategies to a feasible SCE. A static version of this game, describing the individual asymptotic outcomes, given the equilibrium solution defined in (48) to (52), is given by the following multi-objective maximization problem under uncertainty:

$$\begin{aligned} &\underset{\bar{Z}_{4,i}}{MAX}\, E\left(\bar{Z}_{1,i}\right) \\ &\text{Subject to the solution of:} \\ &\bar{Z}_{2,i} = 2r(r+\delta-r_k)\left[\rho_i - r - \left(\bar{Z}_{4,i}\left[E\left(\bar{Z}_2^i\right)\right]-\delta\right)(\gamma_i-1)\right]^{-1} + 2, \end{aligned} \tag{53}$$

such that $\bar{Z}_{4,i}\left[E\left(\bar{Z}_2^i\right)\right]$ fulfils (34), where $E\left(\bar{Z}_2^i\right)$ is the individual expectation about the evolution of the asymptotic state of the game, $\bar{Z}_2$. The players objective, $E\left(\bar{Z}_{1,i}\right)$, is defined by:

$$E\left(\bar{Z}_{1,i}\right) = r_k - \bar{Z}_{4,i}\left[E\left(\bar{Z}_2^i\right)\right] - \bar{Z}_{2,i}\left(r + r\bar{Z}_{2,i}E\left(\bar{Z}_2^i\right) - \bar{Z}_{4,i}\left[E\left(\bar{Z}_2^i\right)\right] + \delta\right), \tag{54}$$

and the *N-tuple* of strategies, Π_i, for player i are given by the set of feasible investment strategic actions, following the result in (51):

[25] For reasons of simplicity, we considered that the learning process takes only into account the expected value of $Z_2(t)$. Given the specific nature of this learning process it might be reasonable to consider other decision criteria that takes into account the uncertainty faced by players.

$$\Pi_i = \left\{ \delta + \frac{\rho_i - r}{\gamma_i - 1} - \frac{2r(r+\delta-r_k)}{(\gamma_i - 1)\left(-1 - \sqrt{1 + \frac{r+\delta-r_k}{r} E\left(\bar{Z}_2^i\right)^{-1}}\right)}, \delta + \frac{\rho_i - r}{\gamma_i - 1} - \frac{2r(r+\delta-r_k)}{(\gamma_i - 1)\left(-1 + \sqrt{1 + \frac{r+\delta-r_k}{r} E\left(\bar{Z}_2^i\right)^{-1}}\right)} \right\}. \quad (55)$$

The asymptotic dynamics of this game can thus be examined as a HMM, more concretely by a *Markov* switching regime model that mimics the co-evolutionary learning process faced by players. Recall that in this setup players have to learn a SCE, otherwise solutions are not optimal. Since in an open loop setup, agents have no information about the evolution of the state of the game, it is not clear how players can learn a SCE if their decisions can only rely on their individual beliefs. We suggest that players are able to learn the past moments of the evolution of the state of the game by simple extrapolation. Starting with an initial guess, $E\left[\bar{Z}_2^i\right]$, players are able to determine the last outcome of $\bar{Z}_2$, by measuring their individual forecasting errors. At any given moment of the game, the agent observed forecast error, ϵ_i, is given by:

$$\bar{Z}_{1,i} - E\left[\bar{Z}_{1,i}\right] = \bar{Z}_{1,i}\left[\bar{Z}_{4,i}, \bar{Z}_{2,i}, \bar{Z}_2\right] - \bar{Z}_{1,i}\left[E\left(\bar{Z}_{4,i}\right), E\left(\bar{Z}_{2,i}\right), E\left(\bar{Z}_2^i\right)\right] = \epsilon_i. \quad (56)$$

To extrapolate the past moments of $\bar{Z}_2$, we just have to substitute expressions $\bar{Z}_{1,i}$ and $E\left[\bar{Z}_{1,i}\right]$ by the equivalent steady-state expressions, following the result in (52), and after rearranging we obtain:

$$\bar{Z}_2 = -\frac{\bar{Z}_{4,i} - E\left(\bar{Z}_{4,i}\right) + \bar{Z}_{2,i}\left(r - \bar{Z}_{4,i} - \delta\right) - E\left(\bar{Z}_{2,i}\right)\left[r + rE\left(\bar{Z}_{2,i}\right)E\left(\bar{Z}_2^i\right) - E\left(\bar{Z}_{4,i}\right) + \delta\right] + \epsilon_i}{r\bar{Z}_{2,i}^2}. \quad (57)$$

The result in (57) guarantees that players have access to a distribution of past moments of the evolution of the state of the game. Given the *Bayesian* nature of this decision process, portrayed by the existence of individual optimal beliefs, we suggest that learning has to take into account these beliefs as a form of individual bias. We argue that any inference process considered should be a *Bayesian* inference process that takes into account the existence of a posterior distribution of the past moments of the game, the evolution of players forecast errors and the players optimal beliefs, as a prior assumption on future outcomes. This hypothesis has its roots in modern economic reasoning. Morris [29], for example, puts forward a strong argument regarding the importance of considering individual priors as opposed to the common prior assumption usually found in orthodox economics literature.

> *"Perhaps the most compelling argument against the common prior assumption is the following reductio argument. If individuals had common prior beliefs then it would be impossible for them to publicly disagree with each other about anything, even if they started out with asymmetric information. Since such public 'agreeing to disagree' among apparently rational individuals seems to be common, in economic environments as elsewhere, an assumption which rules it out is surely going to fail to explain important features of the world."*.

We do not put forward any specific proposal regarding learning dynamics in this game and discard the numerical evaluation of SCE solutions has a HMM solution to (53). The reasons for this decision are the following. First, the evaluation of learning outcomes as to rely on the numerical simulation and sampling of outcomes for the proposed HMM, following the *Markov Chain Monte Carlo* method. It is well known that this method is computationally costly[26]. Second, given the uncertain nature of this decision process, it is not clear how a robust population can be defined in such a way that player strategies are in accordance with (34), and more important that player consumption outcomes are economically feasible, $\bar{Z}_{1,i} > 0$. Recall that in a co-evolutionary learning framework, such as this one, as players learn and choose their best strategies, they change the environment faced by other players in a dramatic fashion. Later in this section, we show that this phenomena is likely to occur in this setup, even when a robust population is considered. Finally, following Grandmont [20], the local stability in complex multi-agent environments with adaptive learning dynamics depends on the degree of confidence that an agent has regarding the local stability of the system. Grandmont [20] defined this property as the *uncertainty principle* faced by learning agents in complex environments. In other words, equilibrium may not be stable when complex learning strategies are considered. Since the evaluation of SCE solutions has to rely on complex learning

[26]The main difficulty faced when simulating and sampling *Markov* process under uncertainty is related to the computational cost of performing inference in a large scale.

strategies, and these are decided *a priori* by the modeller, the qualitative dynamics of this system will be always a function of the specific co-evolutionary process considered. We acknowledge that the best approach to this game asymptotic solution, should be based on the evaluation of (53) as a HMM. By discarding this approach, we cannot guarantee that a SCE solution is achievable and we cannot put forward any results regarding the stability of solutions. To overcome this issue and still provide some intuition on possible solutions to the problem defined in (53), we propose a method to evaluate the existence and economic feasibility of SCE solutions, based on geometrical evaluation of conjectural solutions when players commit to an initial investment strategy. We discuss the implications of feasible solutions based on the analysis of the state-separable system described in Appendix B.2.

The method we propose to evaluate the existence of SCE solutions to the problem defined in (53), relies in a straightforward geometrical approach that can be easily applied. To determine the existence of consistent SCE solutions, we start by evaluating the actual outcomes obtained, when players share a common conjecture about the state of the game. Finally a SCE solution, consistent with the general definition in (10), is defined by the intersection of the curve describing the actual observable outcomes, given a common conjectural expectation on $\bar{Z}_2$, and the 45 degree line that crosses the origin. We exemplify our approach schematically in Figure 6 below.

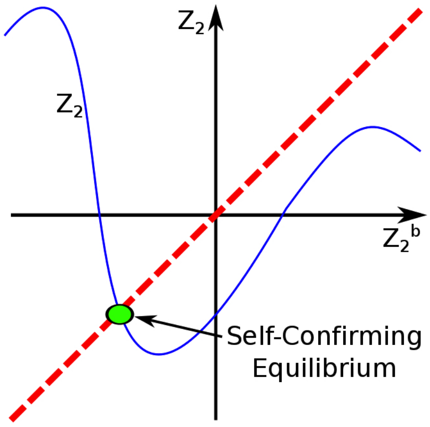


Figure 6: Conjectural and actual SCE solutions.

To exemplify our method, we start by testing institutional scenarios that are consistent with the existence of a robust population of $n = 1000$ players[27] in a reasonable interval of common conjectural outcomes of the state of the game, $\bar{Z}_2 \subset [-2.501 \;\; 2.501]$, where at least one of the strategies defined in (55) is consistent with $\bar{Z}_{1,i} > 0$[28]. Numerical simulations suggest that institutional scenarios consistent with the existence of a robust population require that, $r_k \approx r+\delta$ and $r_k \neq r+\delta$[29]. We evaluate two scenarios for r_k, assuming $r = 0.05$ and $\delta = 0.03$ fixed. The results for the first scenario, $r_k = 0.07999$, are described below in Figure 7. Robust populations are again described by a set of patient agents, but now consumption elasticities are evenly distributed. The figure in the center shows that there is a unique SCE solution in this institutional scenario. However, the figure on the right, describing the worst player actual scaled consumption outcome, suggests that strategic interactions in this framework might not be consistent with feasible economic solutions for the entire set of agents in this economy[30]. The reason behind this dramatic outcome can be explained by wrong conjectures regarding the actual outcome of the game. The figure in the center shows that when player's conjectures are consistent with $\bar{Z}_2^b < 0$, these conjectures are systematically flawed, as actual outcomes are consistent with $\bar{Z}_2 > 0$. This result suggests that the numerical simulation of this institutional scenario as a HMM, might not be consistent with the economic constraints imposed by our model. Finally, the qualitative analysis of the state-separable solution, given in Appendix B.2, suggests that individual solutions, when $r_k < r + \delta$, may be either stable or unstable. The description of qualitative equilibrium solutions for the state-separable problem is provided in Table B.1. These results suggest that the stability of solutions in this institutional scenario depends on the relation between individual strategic investment outcomes and the individual state outcomes. Given that this outcomes depend endogenously on the the state of the game, it is unlikely that the state-separable stability condition is fulfilled for the overall set of players. Further, as we showed in the previous section, the game qualitative dynamics in the vicinity of a SCE, cannot be fully explained by the individual

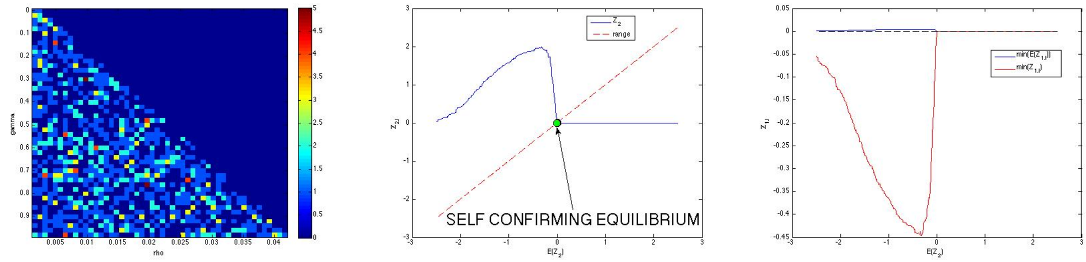
[27] Again we consider the same conditions described in the previous section.

[28] When both strategies are consistent with $\bar{Z}_{1,i} > 0$, we consider that players choose the strategy that yields the best consumption outcome.

[29] Recall that $r_k - \delta$ can be interpreted as the net marginal revenue of domestic capital. This result suggests that feasible solutions can only be considered for economies where the net marginal revenue of domestic assets is close to the international interest rate.

[30] Our numerical results suggest that this is a common outcome among the population.

dynamics of the state-separable problem. Given that co-evolutionary strategic interactions, arising from learning dynamics, play a role on the possible outcomes of this game. It is reasonable to assume the existence of emergence phenomena for this institutional setup.



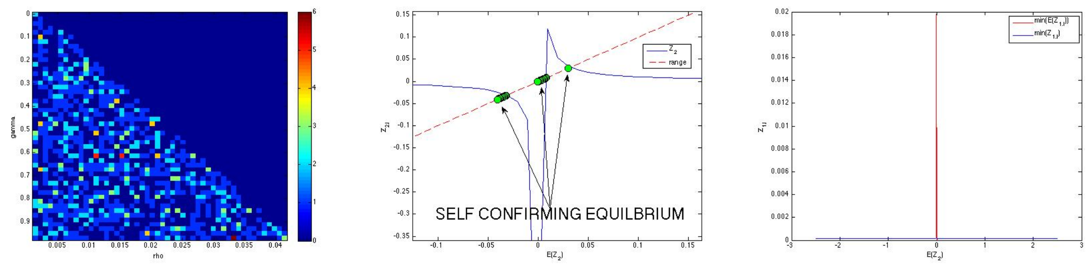
(a) Robust population distribution

(b) SCE

(c) Worst player

Figure 7: Robust population distributions and SCE outcomes for $r_k = 0.07999$.

We finish this presentation with a discussion of an institutional scenario consistent with $r_k > r + \delta$. We now consider $r_k = 0.0801$. The results for this scenario, depicted below in Figure 8, suggest that the feasibility problems arising from wrong conjectures about aggregate risk premium outcomes are no longer an issue. The computed population shares the same broad characteristics described in the previous example. However, we no longer have a unique SCE solution. Our numerical routine detects three distinct SCE solutions[31]. The first consequence of this result is that if players are able to learn and concur in a specific SCE solution, the final outcome is always conjectural. For example, Hu and Wellman [23] shows that multi-agent learning dynamics are highly sensitive to initial conditions in competitive models of conjectural equilibrium. Another hypothesis is that players do not concur on an unique solution and wonder between different equilibrium. This conjecture is suggested by the qualitative analysis of the state-separable solution. As portrayed in Table B.1, institutional scenarios where $r_k > r + \delta$, are always consistent with saddle solutions for the individual problem. Again, we stress that we can only speculate about the possible outcomes that may arise when complex learning strategies are considered. However, this final example suggests that the analysis of this game should be constrained to institutional scenarios consistent with $r_k > r + \delta$, where SCE solutions are of a conjectural nature and players' conjectures robust to strategic interactions under uncertainty. An alternative approach would be the analyses of the open loop feedback *Nash* solution to the games of leaders and followers. An example of this approach for a class of multi-player general sum differential games with leaders and followers is given in Bacchiega et al. [2]. If there are feedback *Nash* equilibrium solutions consistent with a SCE, then we can evaluate under what conditions is the Dynamic Programming problem consistent with an optimal solution to (42). However, given the coupled nature of the state of this game, it is unlikely that an analytical solution to the uncoupled leader/follower open loop feedback game can be derived.



(a) Robust population distribution

(b) SCE

(c) Worst player

Figure 8: Robust population distributions and SCE outcomes for $r_k = 0.0801$.

[31] The repetition of intersection dots is a consequence of our computational procedure. This routine controls for intersections to the left and to the right of the 45 degree line. Although we considered very small error tolerances, we were not able to eliminate this problem fully.

5. Conclusions and further research

In this paper, we proposed the existence of optimal beliefs for a class of differential games consistent with a specific state-separability property and related the existence of optimal solutions with the concept of SCE in a non-cooperative incomplete information game setup. In the first example discussed, we showed that a SCE outcome is feasible, when asymmetries between players are either limited or else further asymmetries are considered. Grandmont [20] had already suggested this relation as crucial for the existence and stability of self-fulfilling outcomes in large socio economic systems. We also showed that a qualitative analysis of equilibrium is feasible using standard dynamical system techniques and portrayed the existence of weak emergence phenomena, by comparing the game outcomes with the individual solution obtained from the analysis of the state-separable system. In the second example discussed, we showed that the introduction of further nonlinearities has dramatic implications. In this setup, we have to consider that agents are able to coordinate asymptotically their belief based decisions and learn a SCE. We suggested two possible paths to evaluate SCE solutions, based on a static version of the game equilibrium solution. First, we proposed that the asymptotic outcomes of this game can be evaluated as a multi-objective maximization problem. We then proposed a simple geometric approach to determine the existence of solutions consistent with a SCE and test the robustness of individual decisions under uncertainty. This approach allows for a description of feasible optimal solutions and a discussion of possible outcomes based on the analysis of the state-separable solution. Despite its simplicity, this approach allowed us to determine that only institutional scenarios where the net marginal revenue of domestic capital is slightly greater than the international interest rate, are robust for a co-evolving environment with strategic interactions. This result paves the way for a future evaluation of this game as a HMM, where different hypotheses regarding complex learning dynamics can be tested, with the objective of determining under what conditions a SCE solution can be achieved. An alternative approach would be the evaluation of open loop feedback *Nash* solutions to (42) consistent with a SCE for the uncoupled games of leaders and followers. This is an interesting option to explore in the future that may provide some insight on the qualitative dynamics in the vicinity of SCE equilibria.

Appendix

Appendix A. Optimal control conditions

Appendix A.1. Optimal control conditions for the non-cooperative game with coupled institutional risk premium

The current value *Hamiltonian* for the non-cooperative game in (12) is:

$$H\left[B_i(t), K_i(t), B(t), K(t), \lambda_i(t), q_i(t), C_i(t), I_i(t)\right]^* = C_i(t)^{\gamma_i} + \lambda_i(t)\dot{B}_i(t) + q_i(t)\dot{K}_i(t), \tag{A.1}$$

where $\dot{B}_i(t)$ and $\dot{K}_i(t)$ are given in (A.6) and (A.7). The general *Pontryagin* maximum conditions for the existence of optimal open loop solution are given by:

Optimality conditions

$$\gamma_i C_i(t)^{\gamma_i - 1} = -\lambda_i(t); \tag{A.2}$$

$$q_i(t) = -\lambda_i(t); \tag{A.3}$$

Multiplier conditions

$$\dot{\lambda}_i(t) = \lambda_i(t)\left(\rho_i - r - rd_i\frac{B(t)}{K(t)}\right); \tag{A.4}$$

$$\dot{q}_i(t) = q_i(t)(\rho_i + \delta) + \lambda_i(t)(r_k); \tag{A.5}$$

State conditions

$$\dot{B}_i(t) = C_i(t) + I_i(t) + rB_i(t)\left(1 + d_i\frac{B(t)}{K(t)}\right) - r_k K_i(t); \tag{A.6}$$

$$\dot{K}_i(t) = I_i(t) - \delta K_i(t); \tag{A.7}$$

Transversality conditions

$$\lim_{t\to\infty} \lambda_i(t) B_i(t) e^{-\rho_i t} = 0; \tag{A.8}$$

$$\lim_{t\to\infty} q_i(t) K_i(t) e^{-\rho_i t} = 0; \tag{A.9}$$

Admissibility conditions

$$B_{i,0}(t) = B_i(0), K_{i,0}(t) = K_i(0). \tag{A.10}$$

Appendix A.2. Optimal control conditions for the non-cooperative game with coupled endogenous risk premium

The current value *Hamiltonian* for the non-cooperative game in (42) is:

$$H[B_i(t), K_i(t), B(t), K(t), \lambda_i(t), q_i(t), C_i(t), I_i(t)]^* = C_i(t)^{\gamma_i} + \lambda_i(t)\dot{B}_i(t) + q_i(t)\dot{K}_i(t), \tag{A.11}$$

where $\dot{B}_i(t)$ and $\dot{K}_i(t)$ are given in (A.16) and (A.17). The general *Pontryagin* maximum conditions for the existence of optimal open loop solution are given by:

Optimality conditions

$$\gamma_i C_i(t)^{\gamma_i - 1} = -\lambda_i(t); \tag{A.12}$$

$$q_i(t) = -\lambda_i(t); \tag{A.13}$$

Multiplier conditions

$$\dot{\lambda}_i(t) = \lambda_i(t)\left(\rho_i - r - 2r\frac{B_i(t)}{K_i(t)}\frac{B(t)}{K(t)}\right); \tag{A.14}$$

$$\dot{q}_i(t) = q_i(t)(\rho_i + \delta) + \lambda_i(t)\left(r\frac{B_i(t)^2}{K_i(t)^2}\frac{B(t)}{K(t)} + r_k\right); \tag{A.15}$$

State conditions

$$\dot{B}_i(t) = C_i(t) + I_i(t) + rB_i(t)\left(1 + \frac{B_i(t)}{K_i(t)}\frac{B(t)}{K(t)}\right) - r_k K_i(t); \tag{A.16}$$

$$\dot{K}_i(t) = I_i(t) - \delta K_i(t); \tag{A.17}$$

Transversality conditions

$$\lim_{t\to\infty} \lambda_i(t) B_i(t) e^{-\rho_i t} = 0; \tag{A.18}$$

$$\lim_{t\to\infty} q_i(t) K_i(t) e^{-\rho_i t} = 0; \tag{A.19}$$

Admissibility conditions

$$B_{i,0}(t) = B_i(0), K_{i,0}(t) = K_i(0). \tag{A.20}$$

Appendix B. Qualitative analysis of solutions assuming state-separability

Appendix B.1. Qualitative analysis for the non-cooperative game with coupled institutional risk premium

The state-separable solution to the game defined in (12), assuming $Z_{4,i}(t) = \bar{Z}_{4,i}, \forall t \in T$, is given by the system defined by scaled consumption, (19), scaled net financial assets, (20), and productive capital dynamics, (A.7), after substituting (17) in (19) and (20). The dynamics assuming state-separability are given by:

$$\dot{Z}_{1,i}(t) = Z_{1,i}(t)\left[\frac{\rho_i + \delta - r_k - \left(\bar{Z}_{4,i} - \delta\right)(\gamma_i - 1)}{\gamma_i - 1}\right]; \tag{B.1}$$

$$\dot{Z}_{2,i}(t) = Z_{1,i}(t) + \bar{Z}_{4,i} + Z_{2,i}(t)\left[2r + 2\delta - r_k - \bar{Z}_{4,i}\right] - r_k; \tag{B.2}$$

$$K_i(t) = K_i(0)\, e^{\left(\bar{Z}_{4,i}-\delta\right)t}. \tag{B.3}$$

The steady states consistent with a feasible solution, $\bar{Z}_1 > 0, \forall t \in T$, are defined by the following expressions:

$$\bar{Z}_{4,i} = \frac{\rho_i + \delta - r_k}{\gamma_i - 1} + \delta; \tag{B.4}$$

$$\bar{Z}_{2,i} = \frac{r_k - \bar{Z}_{4,i} - Z_{1,i}(0)}{2r + 2\delta - r_k - \bar{Z}_{4,i}}. \tag{B.5}$$

Given that agents commit to an initial investment strategy consistent with equilibrium for (B.2), the dynamics in the vicinity of (B.4) and (B.5), are described by the dynamics of net financial assets. Qualitative dynamics are thus defined by,

$$\frac{\partial Z_{2,i}(t)}{\partial Z2,i(t)} = 2r + 2\delta - r_k - \bar{Z}_{4,i}. \tag{B.6}$$

Appendix B.2. Qualitative analysis for the non-cooperative game with coupled endogenous risk premium

The solution to the state-separable optimal control problem defined in (42), assuming $Z_{4,i}(t) = \bar{Z}_{4,i}, \forall t \in T$, is given by the system defined by scaled consumption, (46), scaled net financial assets, (47), and productive capital dynamics, (A.17), after substituting (45) in (46) and (47). The dynamics assuming state-separability are given by:

$$\dot{Z}_{1,i}(t) = Z_{1,i}(t)\left[\frac{\rho_i - r - 2r(r + \delta - r_k)(Z_{2,i}(t) - 2)^{-1} - (\gamma_i - 1)\left(\bar{Z}_{4,i} - \delta\right)}{\gamma_i - 1}\right]; \tag{B.7}$$

$$\dot{Z}_{2,i}(t) = Z_{1,i}(t) + \bar{Z}_{4,i} + Z_{2,i}(t)\left[r + \frac{(r + \delta - r_k)}{Z_{2,i}(t) - 2} - \bar{Z}_{4,i} + \delta\right] - r_k; \tag{B.8}$$

$$K_i(t) = K_i(0)\, e^{\left(\bar{Z}_{4,i}-\delta\right)t}. \tag{B.9}$$

The steady states consistent with a feasible solution, $Z_1(t) > 0, \forall t \in T$, are defined by the following expressions:

$$\bar{Z}_{1,i} = r_k - \bar{Z}_{4,i} - \left(\frac{2(r + \delta - r_k)}{\rho_i - r - \left(\bar{Z}_{4,i} - \delta\right)(\gamma_i - 1)} + 2\right)\left[r + \frac{\rho_i - r - \left(\bar{Z}_{4,i} - \delta\right)(\gamma_i - 1)}{2} - \bar{Z}_{4,i} + \delta\right]; \tag{B.10}$$

$$\bar{Z}_{2,i} = \frac{2(r + \delta - r_k)}{\rho_i - r - \left(\bar{Z}_{4,i} - \delta\right)(\gamma_i - 1)} + 2. \tag{B.11}$$

Qualitative dynamics in the vicinity of (B.10) and (B.11), are defined by the following *Jacobian* matrix[32],

$$J = \begin{bmatrix} 0 & \frac{2\bar{Z}_{1,i} r(r+\delta-r_k)}{\left(\bar{Z}_{2,i}-2\right)^2(\gamma_i-1)} \\ 1 & r - \bar{Z}_{4,i} + \delta - 2\frac{(r+\delta-r_k)}{\left(\bar{Z}_{2,i}-2\right)^2} \end{bmatrix}. \tag{B.12}$$

[32] The *Jacobian* is non-degenerate when $J_{1,2}\left(\bar{Z}_{m,i}\right) \neq 0$.

Following the usual conditions for qualitative dynamics in hyperbolic autonomous planar systems[33]. We define the main qualitative features of this solution based on the general eigenvalue solution, Λ, to the characteristic equation, $\det(J - \Lambda I) = 0$[34], of the *Jacobian* defined in (B.12):

$$\Lambda = \frac{-tr(J) \pm \sqrt{tr(J)^2 - 4\det(J)}}{2}, \tag{B.13}$$

where $\det(J) = -J_{1,2}$ and $tr(J) = J_{2,2}$. Substituting the steady state expressions, (B.10) and (B.11), the main qualitative dynamic features of the state-separability in the vicinity of equilibrium are given in Table B.1.

Saddle point	Attractor	Repellor	Hopf Bifurcation
$r_k > r + \delta$	$r_k < r + \delta \ \wedge \ \bar{Z}_{4,i} > \Omega$	$r_k < r + \delta \ \wedge \ \bar{Z}_{4,i} < \Omega$	$r_k < r + \delta \ \wedge \ \bar{Z}_{4,i} = \Omega$

Table B.1: Local qualitative dynamics for open loop equilibrium solutions assuming full state separability

Where

$$\Omega = r + \delta - 2\frac{(r + \delta - r_k)}{\left(\bar{Z}_{2,i} - 2\right)^2}. \tag{B.14}$$

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[33] Recall that in planar systems we have: (i) an attractor when $\det(J) > 0$ and $Tr(J) < 0$; (ii) a saddle point when $\det(J) < 0$; and (iii) a repelling solution when $\det(J) > 0$ and $Tr(J) > 0$.

[34] Where I is the identity matrix.

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