1+3-Dimensional Directed Random Polymers

Engineering and Physical Sciences Research Council (EPSRC)

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The Setup:

For $\omega_j \in \mathbb{Z}^d$, $j \in \mathbb{N}$, let the sequence $\omega = (\omega_1, \omega_2, \omega_3, ...)$ represent the path of a d-dimensional directed random polymer. We construct the n-th step measure on this path as follows,

$$\mu_n(\{\omega\}) = \frac{1}{Z_n} \exp\left(\beta \sum_{j=1}^{n} \eta(j, \omega_j)\right) P(\{\omega\})$$

where:

- P - the $\mathbb{Z}^d$ simple-symmetric random walk measure of a path ω.
- β - a parameter whose physical interpretation is inverse temperature.
- $(\eta)_{\mathbb{N}\times\mathbb{Z}^d}$ - a family of iid rvs we call the random environment.
- Z_n - a normalisation factor we call the the partition function.

Then μ_n reweights the measure P, giving more weight to those paths which often encounter a positive environment $(\eta(j, \omega_j) > 0)$ and less to those which often encounter a negative environment $(\eta(j, \omega_j) < 0)$. Hence the model attempts to capture the idea that a polymer will grow towards an environment which is favourable. *(fig. 1)*

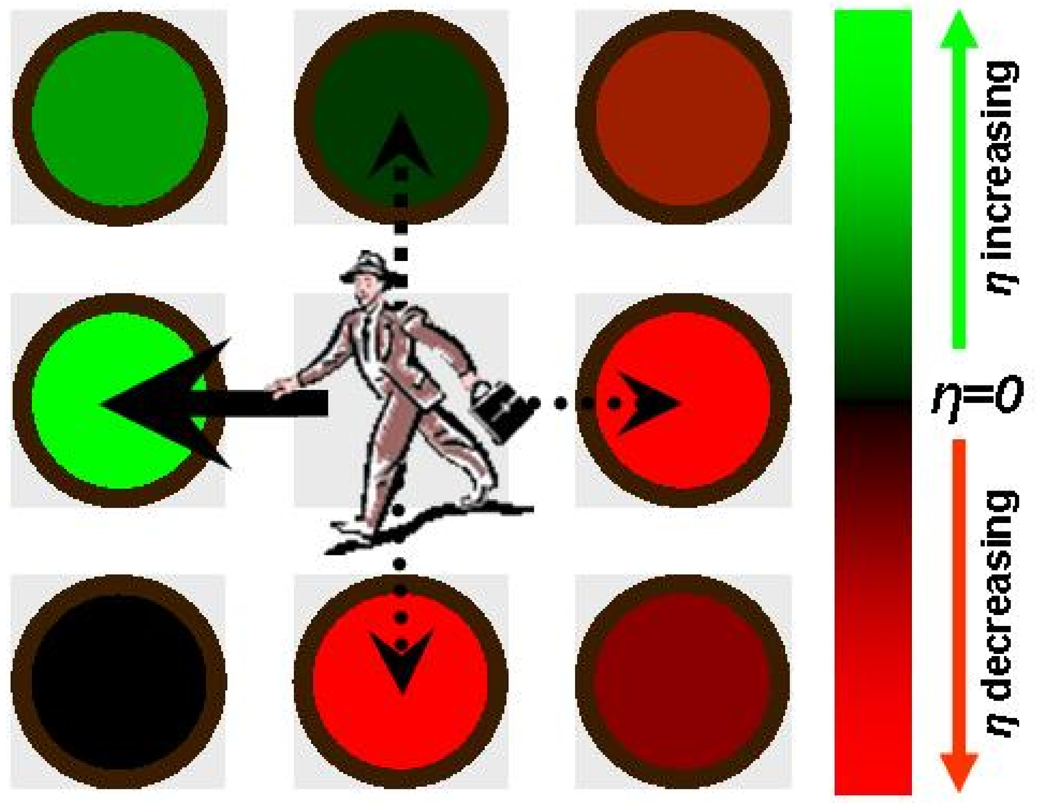

fig. 1

Under the measure μ_n the walker's next step is more likely to be left, where the environment is +ve and less likely to be right or down.

The Random Environment:

The random environment $(\eta)_{\mathbb{N}\times\mathbb{Z}^d}$ is a family of real valued, non constant iid rvs. We define Q to be the measure on members of this family. Furthermore, η has a finite moment generating function. Formally, $Q(e^{\beta\eta}) < \infty$ for all $\beta \in \mathbb{R}$. Additionally we will introduce notation for the log of the moment generating function as a function of the inverse temperature, which we will denote by $\lambda_\eta(\beta)$. Formally, $\lambda_\eta(\beta) = \log(Q(e^{\beta\eta}))$.

Existence of a Phase Transition:

We study the normalised partition function, W_n, defined as follows,

$$W_n = Z_n e^{-n\lambda_\eta(\beta)}$$

Notice $Q(W_n) = 1$ for all n and W_n is a martingale wrt filtration $(\mathcal{G}_n)$, where $\mathcal{G}_n = \sigma\{\eta(j, x); j \leq n, x \in \mathbb{Z}^d\}$.

It can be shown analytically that $W_\infty := \lim_{n\to\infty} W_n$ converges Q-as and either $W_\infty > 0$, Q-as or $W_\infty = 0$, Q-as.

Further more this transition is monotonic in β and hence the model shows a phase transition namely,

$$W_\infty > 0 \ Q\text{-as for } \beta < \beta_c$$
$$W_\infty = 0 \ Q\text{-as for } \beta > \beta_c$$

where β_c is the critical point at which phase transition occurs.

$W_\infty > 0$ Q-as corresponds to a diffusive behaviour. Formally

$$\lim_{n\to\infty} \frac{\mu_n(||\omega_n||^2)}{n} = 1, \ Q\text{-as}$$

$W_\infty = 0$ Q-as corresponds to a regime of strong disorder. *(fig. 2)*

For $d = 1, 2$ it has been shown that $\beta_c = 0$.

For $d \geq 3$ the following analytical bounds exist for β_c[1],

- $\lambda_\eta(2\beta_c) - 2\lambda_\eta(\beta_c) \geq \log\frac{1}{\pi_d}$, (lower)
- $\beta_c\lambda'_\eta(\beta_c) - \lambda_\eta(\beta_c) \leq \log 2d$, (upper)

where π_d is the probability that a d-dimensional random walk returns to its starting point.

Recent work has shown that the lower bound is not tight for $d \geq 4$. A major objective of this project is to investigate numerically whether the lower bound is tight for $d = 3$.

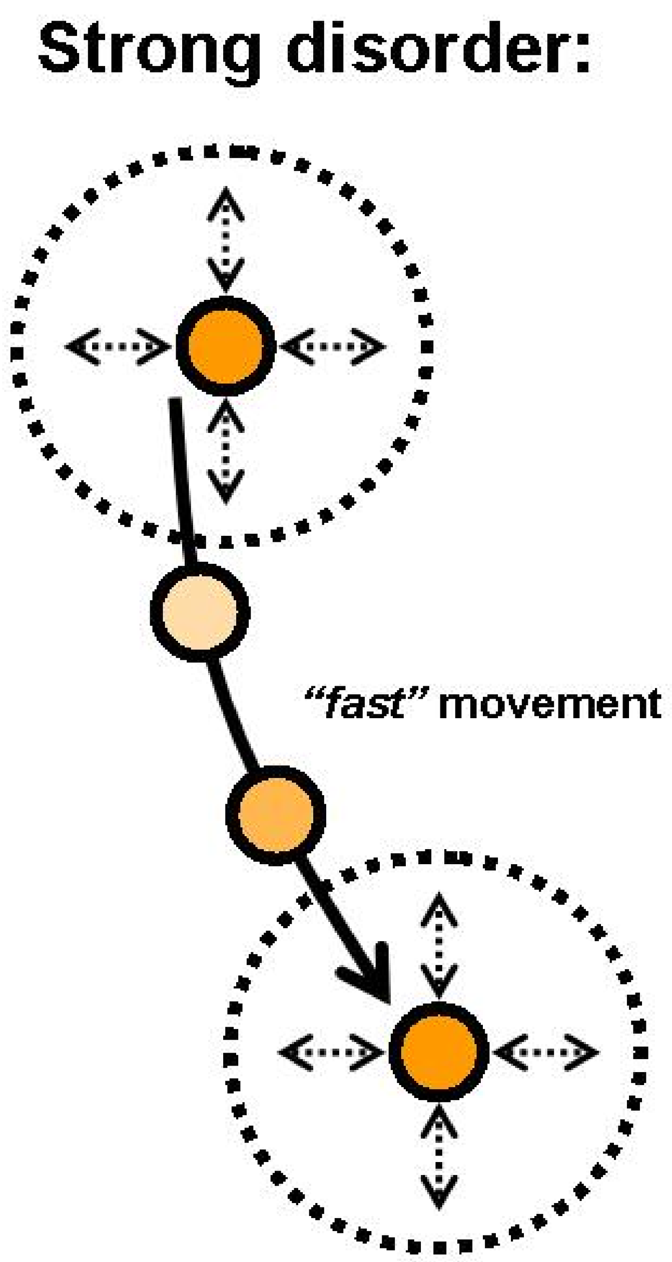

fig. 2

Strong disorder: Favourable paths stay localised at a favourite site and exhibit occasional fast movement through a +ve environment.

Calculation:

We will attempt to calculate W_n for $d = 3$ and a Gaussian random environment, $\eta \sim N(0, 1)$. For large n we hope to observe the phase transition and estimate β_c.

We will exploit the following recursion relation,

$$W_n(x) = \frac{e^{\beta\eta(n,x)-\lambda_\eta(\beta)}}{2d} \sum_{\delta\in\Delta} W_{n-1}(x+\delta)$$

with $W_0(x) = 1$ if $x = 0$ and $W_0(x) = 0$ if $x \neq 0$, where

$$W_n(x) = P\left(e^{\sum_{j=1}^{n}(\beta\eta(j,\omega_j)-\lambda_\eta(\beta))}\mathbb{I}_{\{\omega_n = x\}}\right)$$

and

$$\Delta = \{\delta \in \mathbb{Z}^d : ||\delta||_1 = 1\}^\dagger$$

We calculate W_n by summing over lattice points,

$$W_n = \sum_{x\in\mathbb{Z}^d} W_n(x)$$

remembering that $W_n(x) = 0$ when $||x||_1 > n$ because a random walker could not have traveled more that n steps away from the origin.

Another aim of the project is to investigate a possible scaling law of of the form $W_n \sim e^{-nf(\beta-\beta_c)}$ for $\beta > \beta_c$, where the function f will be some scaling exponent.

Preliminary Findings:

We have written c code to compute W_n up to $n = 290$ for several realisations of the Gaussian random environment. Figure 3 shows W_{290} plotted against β for one such environment. The phase transition can be clearly observed and the preliminary findings indicate it is plausible that the analytical lower bound for $d = 3$ and $\eta \sim N(0, 1)$ of $\beta_c \geq 1.03$ may not be tight. It will be necessary to compute W_n to larger n to confirm this.

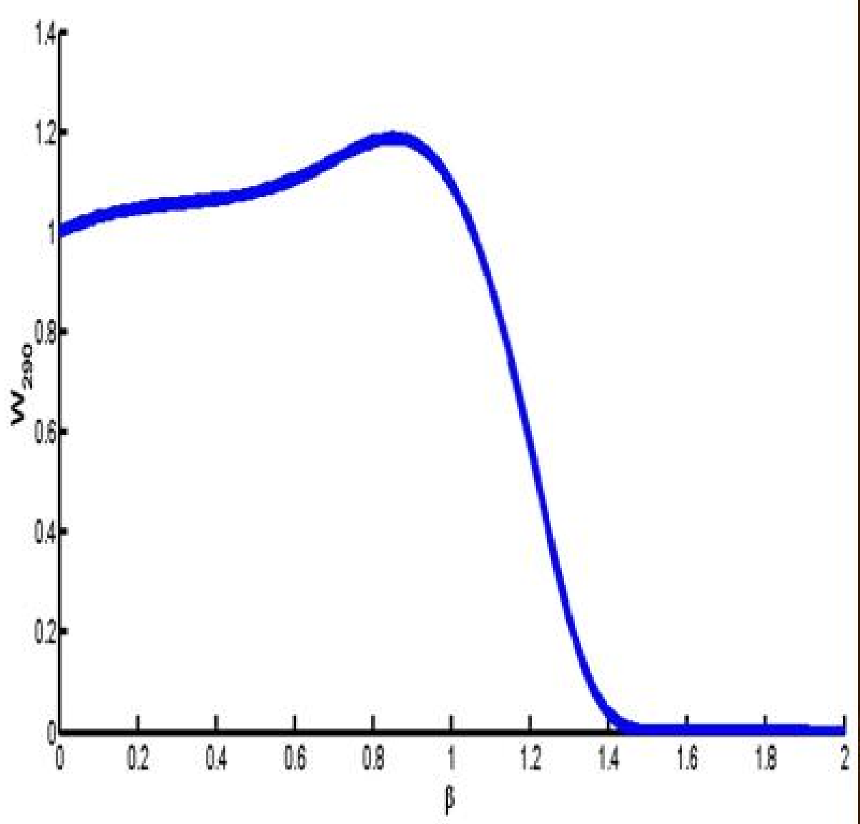

fig. 3

[1] Comets, F.; Shiga, T.; Yoshida, N.: Probabilistic Analysis of Directed Polymers in a Random Environment: a Review, Adv. Studies Pure Math. **39** (2004) 115-142.

†For $x \in \mathbb{R}^d$ st $x = (x_1, ..., x_d)$, $||x|| = \left(\sum_{i=1}^d x_i^2\right)^{\frac{1}{2}}$, where as $||x||_1 = \sum_{i=1}^d |x_i|$