THERMOECONOMICS: APPLYING THERMODYNAMIC IDEAS TO THE ECONOMY

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Abstract. What can thermodynamics bring to the study of the economy? In this project we use an alternative concept of entropy to explore thermodynamic analogies to economics. We suggest that there may be mathematical similarity between entropy and the economic concept of utility, and simulate a model bartering economy to explore this idea. The simulation also indicates a possible link between the economic concept of marginal utility to that of a chemical potential.

1. Introduction

From its very inception economics has adapted ideas from the physical sciences and used them to shed light on market processes. It is thought that Adam Smith, the father of modern economics, was directly influenced by Newton's physical laws [Hetherington, 1983]. The idea of applying ideas thermodynamics and statistical physics to the economy is hardly new, but has caused much discussion. There are those who argue that the disciplines of physics and economics are too separate for anything other than coincidental similarities between the two [Mirowski, 1984], or that the similarities are not useful ones. Others consider economics from a directly physical point of view, looking at the economy as an out-of-equilibrium physical system and applying the concept of Maximum Entropy Production [Corning, 2002, Schwartzman]. There are also many those who argue that there are similarities that are more than superficial [Ausloos et al., 1999, Dragulescu and Yakovenko, 2000, Egorov, 2007] and that there is more to be gained by comparing physical and economic systems, particularly by using methods from statistical physics. It was the aim of this project to find meaningful economic analogies to thermodynamics, in the spirit of this last group.

1.1. Ideas from economics.

It is important to consider the ideas from economics which could be related to those in the physical sciences. An important quantity in economic theory is 'utility', which is used to rank all possible situations depending on how desirable they are to a particular person Snyder and Nicholson [2008]. The properties of this ranking relation, called 'is less preferable than' and which I shall denote $\prec$, are given by the Axioms of Rational Choice:

- Completeness: If A and B are any two situations, an individual can always specify one of the three possibilities

$$\begin{array}{ccc} A & \prec & B \\ B & \prec & A \\ A \prec B & \text{and} & B \prec A \end{array}$$

- Transitivity: If $A \prec B$ and $B \prec C$ then $A \prec C$.
- Continuity: A small change in the conditions of a decision should not change the ordering.

Utility is not a unique function - any function which represents the order of preference of different situations will be correct. A trade between two people is likely only to take place if it increases the utility of both parties. In general, the utility of owning a particular amount of a good is subject to 'diminishing marginal returns', in the sense that while a person who has none of the good may rank obtaining the good very highly, that person may well wish to swap some of that good for another if they have too much of it. In other words, the more of a good you have, the less utility one more unit of that good would give you.

A related concept is the idea of an 'indifference curve map', a figure which plots contours of constant utility on a plane of the quantities of two goods that a person could own. Along the contours a person would be able to swap a certain quantity q_0 of one good for a set amount q_1 of the other while maintaining the same total utility. This is described by the equation

$$dU = \frac{\partial U}{\partial q_0} dq_0 + \frac{\partial U}{\partial q_1} dq_1 = 0$$

Along an indifference curve we can define a 'marginal rate of substitution',

$$MRS = - \left. \frac{dq_1}{dq_0} \right|_U = \frac{\partial U / \partial q_0}{\partial U / \partial q_1}$$

Due to diminishing returns these curves are usually convex, and the the property of transitivity in the Axioms of Rational Choice means that the curves cannot cross.

1.2. **Ideas from thermodynamics.** Thermodynamics arose around the time of the Industrial Revolution, as a way to analyse heat engines and make them more efficient [Carnot, 1824]. It was used successfully even at times before the discovery of atoms and while the 'Phlogiston' theory of heat was still accepted by some [Conant, 1964], because it does not require an understanding of the microscopic behavior of a system.

Thermodynamics is an axiomatic theory based on four laws, held to be true for all systems:

- Zeroth Law: Two systems in thermal equilibrium with a third are also in thermal equilibrium with each other.
- First Law: The total energy of a system is conserved, any any changes to it are due to energy transfer as work or as heat.
- Second Law: In any neighbourhood of any state, there are states that cannot be reached from it by an adiabatic process.
- Third Law: The entropy of a perfect crystal approaches zero as the absolute temperature approaches zero.

These laws, as stated, rely intimately on physical notions such as as thermal equilibrium, work, heat, crystal etc. We shall see later that it is possible to consider the second law without resorting to these terms.

Another statement of the first law of thermodynamics is

$$dE = dQ + dW$$

where dE is the change in the internal energy of the system, dQ is the change in heat and dW is the change in work. For a reversible change (one that happens slowly compared to the time taken to for the system to equilibrate), $dQ = TdS$, where T is the temperature and dS the change in entropy. An example of a work term is $dW = -PdV$, which applies to an ideal gas, but this will turn out to not be important in the context of this project. Finally, a term must be added if there is a transfer of particles as well as energy between sub-systems. This term is

$$\sum_i \mu_i dN_i$$

where μ_i is the 'chemical potential' of particle species i, and dN_i is the change in the number of particles of that species.

The change in internal energy of a thermodynamic system in full is therefore given by:

$$dE = TdS - PdV + \sum_i \mu_i dN_i$$

Here T is the temperature, dS is the change in entropy, P is the pressure, dV is the change in volume. Later we will consider a system in which the energy, temperature and volume are fixed, and therefore

$$dS = -\frac{1}{T}\sum_i \mu_i dN_i$$

This leads to a definition of the chemical potential of i in this situation as

$$\mu_i = -T\frac{\partial S}{\partial N_i}$$

1.3. **Previous approaches.** There have been countless previous applications of both thermodynamics and statistical physics to the economic and finance. Statistical physics approaches have been used to explain the power laws often found in economic data[Bouchaud, 2000], studing optimal allocation in markets [Blume, 1993], and wealth distribution [Milakovic, 2001]. In this project, however, we would like to take an approach in a more thermodynamic manner, considering only macroscopic (in the physical sense) quantities, without looking at low-level interactions.

Although this approach has also been tried many times before [Press, 2006, Lemoy et al., 2011, Saslow, 1999], and has led to many different (and sometimes conflicting) physical analogies for economic systems, what this project hopes to bring to the area is an economic interpretation of a paper which shows entropy to be a more abstract and general concept than has been previously considered. It is hoped that this new view of entropy will enable deeper analogies to be drawn, and this project investigated some of these ideas.

1.4. **A new look at entropy.** This project was motivated by a paper by Lieb and Yngvason [2002], in which the authors derive the existence and uniqueness of an entropy function simply from basic properties of a relation between equilibrium states in a system. They abstracted the concept of (thermodynamic) entropy and allowed it to be used for other, non-physics based, systems, so long as the system had a relation between states which satisfied some basic axioms. It is in this spirit that we approach economics: is it possible to find a relationship between economic states which satisfies the axioms given by Lieb and Yngvason (abbreviated to LY), and use this to deduce the effects of one economic quantity on another without reference to low-level behavior?

LY proceed by considering a state space Γ, with some relation between these states denoted by

$$\prec$$

It is a preorder, because $X \prec Y$ does not imply $X = Y$, and they denote states for which $X \prec Y$ and $Y \prec X$ by $X \overset{A}{\sim} Y$.

and LY ask under what conditions it can be encoded in a real-valued function on the set, called the entropy S, which satisfies:

- Monotonicity: When two states are comparable (i.e. $X \prec Y$ and/or $Y \prec X$),
$$X \prec Y \text{ if and only if } S(X) \leq S(Y)$$
- Additivity and Extensivity: If X and Y are states of some (possibly different) systems, and (X, Y) is the corresponding state in the compound system, then the entropy is additive for these states, i.e.
$$S(X, Y) = S(X) + S(Y)$$
S is extensive, so for each $\lambda > 0$, each state X and its scaled copy $\lambda X \in \Gamma^{(\lambda)}$ the entropy satisfies
$$S(\lambda X) = \lambda S(X)$$
which means that the entropy of a state in one system twice the size of another should have twice the entropy of the other.

They show that the relation can be encoded by an entropy as long as $\prec$ satisfies certain axioms:

- Reflexivity: $X \overset{A}{\sim} X$
- Transitivity: If $X \prec Y$ and $Y \prec Z$, then $X \prec Z$
- Consistency: If $X \prec X'$ and $Y \prec Y'$, then the joint system $(X, Y) \prec (Y, Y')$
- Scaling Invariance: If $\lambda > 0$ and $X \prec Y$, then $\lambda X \prec \lambda Y$
- Splitting and recombination: $X \overset{A}{\sim} ((1 - \lambda)X, \lambda X) \, \forall \, 0 < \lambda < 1$
- Stability: If $(X, \epsilon Z_0) \prec (Y, \epsilon Z_1)$ for some Z_0, Z_1 and a sequence of ϵ's tending to zero, then $X \prec Y$
- Comparison Hypothesis: Although they prove this from other axioms later in the paper, LY initially hypothesise that for all states $X, Y \in \Gamma$, either $X \prec Y$, $Y \prec X$, or both.

From these axioms and the Comparison Hypothesis, LY prove that there is an entropy function S on Γ which is unique, up to a scaling of
$$S(X) \to aS(X) + B$$
where $a > 0$.

The importance of this work is that it shows entropy to be an abstract ordering on a set of states, one application of which is to physical systems, but which is in principle applicable to many others. This suggests a framework in which to examine the links between thermodynamics and economics: find a relation between states which satisfies the axioms they provide, and use this to define an entropy and hence other thermodynamic relations. If this is not possible directly, then it may still be possible to find a function similar to entropy - their work shows that an entropy function can exist which has nothing to do with information or any other physical concepts.

2. A possible economic Entropy

After thinking of several possible entropy-like quantities for the economy, none of which seemed to have the required properties, utility was chosen as a possible candidate. The Axioms of Rational Choice (ARC), mentioned previously, seem to exactly match some of those that Lieb and Yngvason required for a relation between thermodynamic states to be represented by an entropy. Both require reflexivity, transitivity and consistency of the relation. Scale invariance is required for both, since the ARC state that it does not matter much 'more' utility one good gives over another. The continuity axiom in the ARC is replaced by 'stability' in LY, because LY do not initially assume an underlying topology - they do introduce one later in the paper, however. Completeness in ARC corresponds to the 'Comparison Hypothesis' in LY.

There was not enough time to check the correspondence of all the axioms, but those already mentioned do seem to force both entropy and utility to have some similar properties. Indifference curves for systems with diminishing marginal returns are convex, which is also true for the curves separating accessible states in LY. Transitivity in both cases prohibits the intersection of curves. This could point to deeper mathematical links between entropy and utility, which could be used to relate thermodynamics and economics.

3. Simulation

A model economy was simulated to explore the idea that utility could be similar to entropy. The model consisted of N agents and M goods. Each agent had a 'preference' for each good, $p_i \in [0, 10)$, chosen uniformly at random. The economy was initialised to have a certain quantity of each good, q_j, and these goods were randomly distributed among the population. The utility, U_i, that a stock s_i of a particular good gave to an agent was given by the utility function

$$U_i(s_i, p_i) = p_i \left(1 - e^{-s_i/\theta}\right) \tag{3.1}$$

where θ is a parameter used to set the rate of diminishing returns and, for this report, is the same for all goods.

An example of this utility function is plotted in figure 3.1 for an agent with different preferences for two goods, where the utility is plotted against the stock of that good. This particular function was chosen because it displayed the property of diminishing returns and is analytically tractable, but other concave-down functions could also be used.

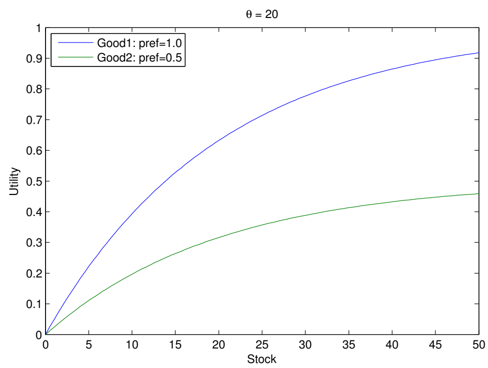

Figure 3.1. Utility curves for two goods, with preferences 1.0 and 0.5, against an agent's quantity (stock) of that good.

When the simulation was run two different agents were selected at random and allowed to barter. Two bartering algorithms were tried, a simple one (1) where agents traded one unit of the goods which provide the maximum benefit to both of them, and a more probabilistic one (2) in which one agent offered the other a random quantity of one good in exchange for enough of another good to equal its lost utility, with some profit. The possiblity of an agent accepting a trade which decreased its utility was included, controlled by a parameter β. For most of these simulations this parameter was set so high that negative-utility swaps almost never occured. This second algorithm is based on the idea of a shop: the shop offers a customer some good in exchange for enough money to cover wholesale cost and some profit. In the case of either algorithm, if bartering was successful the two agents would swap the agreed quantity of two of their goods. These agents were then returned to the pool of possible agents, and the selection process started again.

Algorithm 1 Simple bartering

Cycle through all pairs of goods i, j
For each:
Calculate utility change $\Delta U1$ for Agent 1 swapping 1 unit of i for 1 of j
Calculate utility change $\Delta U2$ for Agent 2 swapping 1 unit of j for 1 of i
Choose the pair (i, j) such that:
$\Delta U1 > 0$ and $\Delta U2 > 0$ and $(\Delta U1 + \Delta U2)$ is maximized
Perform this swap of one unit of i for one unit of j.

Algorithm 2 Random bartering

Start with a random quantity $q \in [0, 1)$ of good i to trade for a good j.
Agent 1 calculates how much j it would need to receive in order to make up for the loss of q units of i
Agent 1 adds some fixed profit (arbitrarily set to 10^{-6}) to this quantity to give r: the amount of j it is asking for.
Agent 1 offers this deal to Agent 2, who can decide to accept or reject the deal:
Always accepts if the deal increases the utilty of Agent 2
Sometimes accepts even if the deal decreases the utility of Agent 2, with probability $P_{accept} = e^{\beta(\Delta U)}$, where ΔU is change in utility of Agent 2 and β is a parameter.
If accepted, perform this swap of q units of i for r units of j.

Some of the tests performed on the simulation required there to be two or more sub-economies, which I have called 'islands', which could be stopped from trading with each other. Each island had a population of agents separate from the others. When two islands were allowed to trade an agent was picked at random from one of the islands and it proposed a trade to a randomly-picked agent from the other island, using the random-bartering algorithm.

Most of the results were from the situation where the random barter algorithm was used, since it gave richer behavior than the simple bartering algorithm, and also for which $\frac{\beta}{\Delta U} \gg 1$ - that is, with almost no negative-utility decisions.

4. Results

Figure 4.1 shows some examples of the utility timeseries for the simulated model. For both bartering systems, and for different values of β, the total utility of all the agents increased with trading until reaching some equilibrium value. Simulations with the same parameters were run $N_{ens} = 20$ times, and the mean utility after $t_{eq} = 15000$ timesteps and its standard error for each realisation were plotted, as shown in figure 4.1. These plots, for different values of β, show that, even when the trading is noisy and agents often make poor decisions, the mean utility still converges to some fixed value (within error). The simple bartering model is smooth, but converges slower and to a lower value than the random bartering models with high β. For lower β values, i.e. where agents make bad decisions more often, the converged total utility is lower and more noisy. The random models all converge faser than the simple bartering model. The model which leads to the best final utility is the random bartering model with no bad decisions. This is due to the

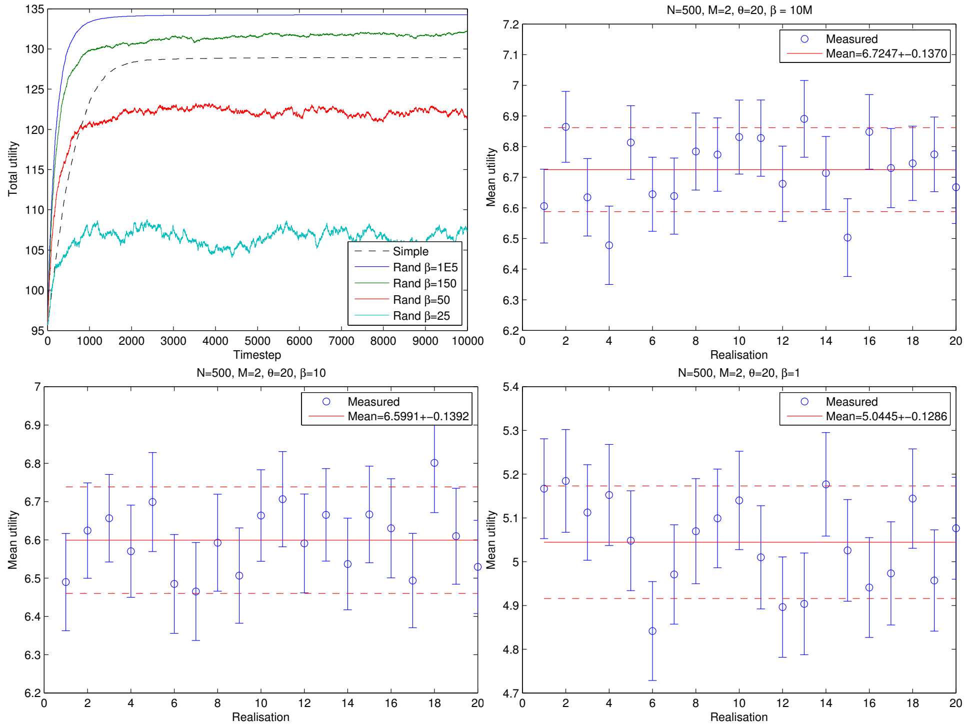

FIGURE 4.1. (Top left) Example utility timeseries for different bartering algorithms and values of β. (Others) Mean utilities after $t_{eq} = 15000$ timesteps, for $N_{ens} = 20$ realisations. The economies were given an initial store of $q_0 = q_1 = 10000$, randomly assigned to the population. Three different values, $\beta = 10^6$, $\beta = 10$ and $\beta = 1$, are shown. All converge, and $\beta = 1$ is significantly different from the other two, implying that when there is a lot of imperfect trading the mean utility is decreased.

ability of the random model to swap units of less than 1.0, which allows more possible trades to occur.

For a given total initial store of each good, spread randomly over the agents, the economy always converged to a particular value of the mean utility, within error. This meant that the model could be run several times with randomly chosen preferences and the results averaged to give good thermodynamic quantities. This process of increasing utility until equilibrium is reached looks very similar to the maximization of entropy which occurs when thermodynamic systems equilibrate. This hints that utility could be seen as an analogue of entropy for this particular economic model.

To explore this idea, the convergence property was used to study how the final mean utility depended on the total quantity of goods in the economy. In figure 4.2 the simulated curves are compared to the mean utility of one agent who holds who holds q/N goods, calculated from the utility function equation (3.1). This curve represents the mean utility if no trading was allowed and the same quantity of goods were randomly assigned among the population. The simulated and calculated curves have a similar form, but the simulated economies have greater mean utility for all values of the total quantity of goods. This shows that the trading economy is performing better than random allocation, as one would expect. A lower value of θ, the parameter controlling diminishing returns, causes the economy to saturate

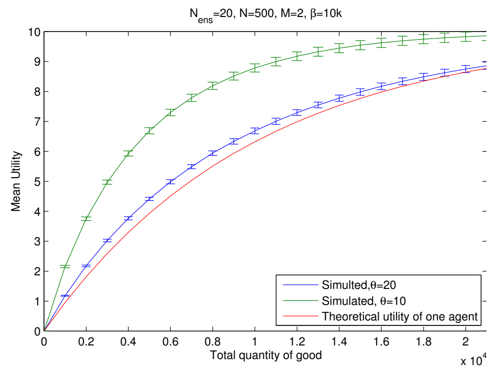

FIGURE 4.2. Final mean utility against total quantity of a goods in the economy, comparing two simulated economies (with $\theta = 10$ and 20) with the mean utility of one agent randomly assigned goods. The mean and uncertainty for each measured point was calculated by averaging $N_{ens} = 20$ ensembles.

faster. If the converged mean utility were indeed like an entropy then the gradient of this curve would be the equivalent of a chemical potential. This was a particularly interesting result: in thermodynamics the chemical potential drives the diffusion of particles until it equilibrates, so this economic potential could fulfill the same function.

An analogue of the thermodynamic diffusion situation would be island economies suddenly allowed to trade with each other, with goods diffusing between the islands. An important property of thermodynamic equilibrium is that it is transitive (the Zeroth Law), so to sharpen the analogy it was necessary to check that the economic equilibrium reached in a system with three island economies satisfied: Island1 $\leftrightarrow$ Island2 $\leftrightarrow$ Island3 implies Island1 $\leftrightarrow$Island3 (where $\leftrightarrow$ means 'can trade with'). Although not shown rigorously, the simulations seemed to satisfy this condition. Initially the islands were all disconnected. Then Island1 and Island2 were allowed to trade, and Island2 and Island3. Finally Island1 and Island3 were brought into direct contact, and no trades took place, indicating that the islands have reached equilibrium by being indirectly connected through island 2. This process was simulated several times, one example of which is shown in figure 4.3, and no trades were seen during the last stage.

With this evidence that the islands were equilibrating in a thermodynamic sense, the idea of a 'economic potential' was tested further. Two islands were simulated, each with $N = 200$ agents, and there were $M = 2$ goods (for simplicity). Each island was given a set initial stock, q_0 and q_1, of the two goods, and the preferences were chosen from uniform distributions the limits of which could be changed for each good. For the first simulations the preferences for each were drawn from the same uniform distribution $p = [0, 10)$. The simulation was run and the islands allowed to trade, both between islands and within them. The start and end position of the two islands on the (q_0, q_1) plane were plotted, overlaid onto contour map of the mean converged utility of that configuration. The contour map was determined by running other simulations with only one island of $N = 200$ agents and with an initial stock given by the (q_0, q_1) value of a particular grid square and letting the system equilibrate. figure 4.4a) and b) show the overlay and contour maps for islands with two different initial quantities of goods. figure 4.4c) shows a situation for which the preferences for the two goods are not drawn from the same uniform distribution: here $p_0 = [0, 20)$ and $p_1 = [0, 5)$. This changes how valuable the

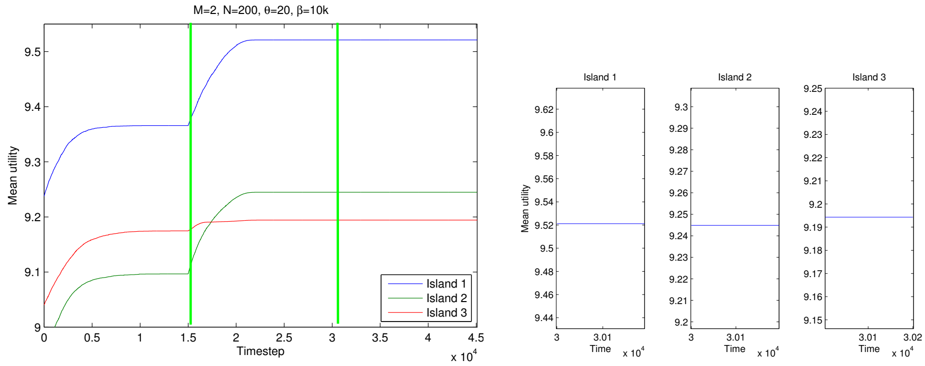

FIGURE 4.3. Left: One realisation of the zeroth-law test. The light green lines separate the plot into regions with different trading conditions: In the first region none of the islands can trade with each other. In the second, Island1 $\leftrightarrow$ Island2 $\leftrightarrow$ Island3 (at $t = 1.5 \times 10^4$), and in the third Island1 $\leftrightarrow$ Island3 (at $t = 3.01 \times 10^4$). Right: a zoomed version of the plot on the left, around time $t = 3.01 \times 10^4$.

two goods are compared to each other, rescaling the quantity axes. In all cases, including that of asymetric preferences, the islands end up on a line parallel to $q_0 = q_1$ (guide lines are shown in white on the plots). Since these simulations all used $\beta = 10^7$ the islands almost only ever increase their utilities, and hence can only stay on the same contour or climb upwards.

figure 4.4d) shows cross-sections of the surface along lines parallel to $q_0 = q_1$, overlaid with the mean utilities and standard errors of the scattered points in c). It demonstrates that the mean utilities of the two islands do stay on the utility surface found for a single island, and that the end points for both islands are higher in utility than the start, as expected.

The surfaces in figure 4.4 can also be analysed from the economic perspective, since they are the indifference curves mentioned earlier. Along the curves

$$dU = \frac{\partial U}{\partial q_0} dq_0 + \frac{\partial U}{\partial q_1} dq_1 = 0$$

By comparison with the physical interpretation, for which along curves separating accessible and inaccessible states,

$$dS = -\frac{\mu_0}{T} dq_0 - \frac{\mu_1}{T} dq_1 = 0$$

it can be seen that, if the temperature is constant and units are rescaled such that $\frac{\mu}{T} \to \mu$, the chemical potential μ_{ij} for good j on island i is very similar to what economists would call the marginal utility of good j:

$$\mu_{ij} = -\frac{\partial U_i}{\partial q_j}$$

Equilibrium is reached within one island when $dU = 0$, in which case

$$\mu_0 dq_0 = -\mu_1 dq_1 \tag{4.1}$$

Thus

$$\frac{dq_0}{dq_1} = -\frac{\mu_1}{\mu_0} = -\text{MRS}$$

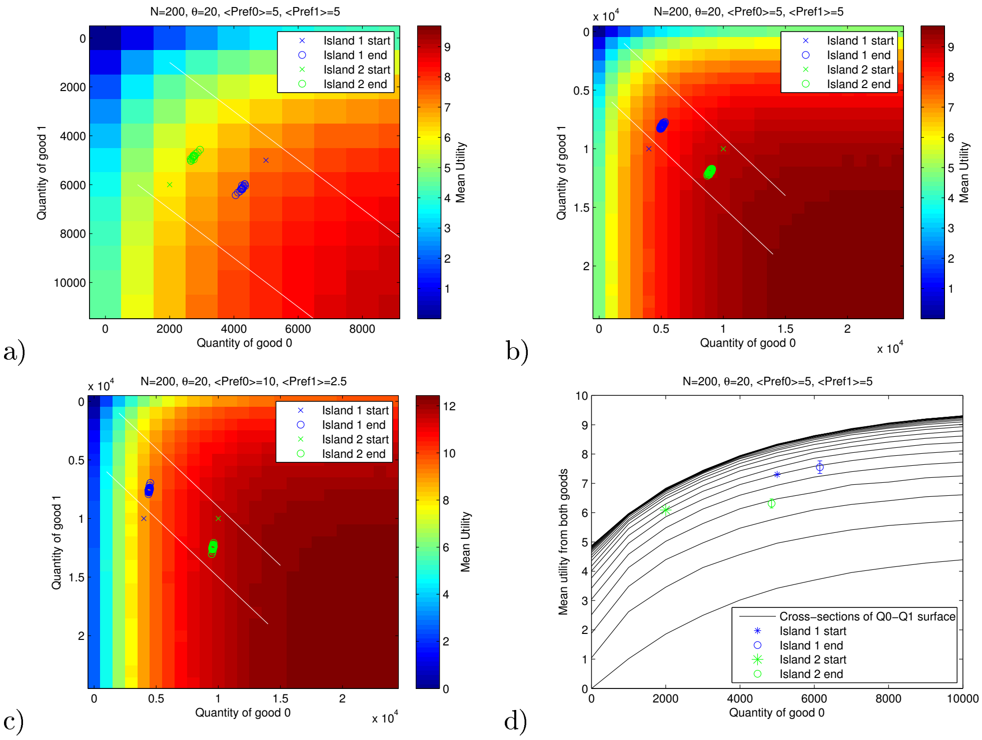

FIGURE 4.4. a) Utility surface (mean utility for a given quantity of good 0 and good 1) for a situation where preferences for the two goods are equal. Overlaid are the starting and ending positions on of the two islands, for $N_{ens} = 20$ realisations. The white lines are simply guides used to illustrate that the line between the endpoints of the two islands always end up parallel to the line $q_0 = q_1$. b) Same, but with a different initial distribution of goods. c) Same initial distribution of goods as b), but with different preferences for each good. d) Cross-sections of the surface along lines parallel to $q_0 = q_1$. The end points of both islands are all on the surface, within error.

This is the marginal rate of substution and describes how much of one quantity can be converted the other while maintaining the same total utility. On the surfaces plotted above this describes the slope of the contour lines on the (q_0, q_1) plane.

When considering two islands goods are conserved, so $dq_{0,0} = -dq_{0,1}$. Trading will occur when both $\mu_{0,0}dq_{0,0} > \mu_{1,0}dq_{1,1}$ and $\mu_{0,1}dq_{0,1} > \mu_{1,1}dq_{1,0}$. This explains the trajectories seen on the (q_0, q_1) plane in figure 4.4: the islands take diagonal trajectories from their starting locations that are parallel to each other because an increase in the quantity of a good on one island is accompanied by the decrease of that good on the other. The slope of the start to end lines is determined by the direction that allows both islands' utilities to increase. The line between the endpoints of the two islands is parallel to $q_0 = q_1$ because both islands had, on average, the same demand for the goods as each other. There was not enough time to examine a situation in which the demand on the islands was different.

5. Conclusion and further work

The similarities between the properties of entropy and utility encouraged further investigation. This was done by simulating a simple model economy which included diminishing marginal returns. The model demonstrated features that one would hope to find in a model economy: under all the schemes the mean utility increased until converging to a fixed value as trading took place between agents, and trading did a better job of resource allocation than randomly distributing. When the mean utility converged to some value, that equilibrium seemed to satisfy transitivity - a necessary property if and economy is to be treated as having thermodynamic states. The gradient of utility with respect to different quantities of goods in the economy looked like a chemical potential, and corresponded to the economic idea of the marginal utility for a particular good. Although this could be coincidental, it could also be due to the similarity between the definitions of utility and entropy. There is no clear analogue for energy or temperature in this model, and so they had to be arbitrarily set to be constant and thus removed. Further work would be required to check if either these are present, or if their absence leads to big differences between this system and a physical one.

An investigation is still required of exactly where the differences are between the axioms underlying utility and entropy, and whether they are similar enough that useful results from physical thermodynamics can be brought to economics. The evidence found in this project provides some additional support to the idea that they are similar enough that analogies being found between economics and physics are more than just coincidence, but further work is required to prove these claims rigorously and to apply the ideas more broadly.

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