The Relaxation Time of a Condensing Zero Range Process

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Model

- Finite lattice $\Lambda = \{1, \dots, L\}$
- State space $\Omega_L = \mathbb{N}^\Lambda$
- Configurations $\boldsymbol{\eta} = (\eta_1, \dots, \eta_L) \in \Omega_L$
- For $f : \mathbb{N}^\Lambda \to \mathbb{R}$ the generator is

$$\mathcal{L} f(\boldsymbol{\eta}) = \frac{1}{L-1} \sum_{x,y} g_x(\eta_x)\left(f(\boldsymbol{\eta}^{x,y}) - f(\boldsymbol{\eta})\right)$$

- (Conservative) Irreducible on $\Omega_{L,N} := \{\boldsymbol{\eta} \in \mathbb{N}^\Lambda : \sum_{x=1}^L \eta_x = N\}$
- Zero range processes with slow defect sites ($r < 1$) [Evans, M.,2000]

$$g_x(n) = \begin{cases} 1 & \text{if } x \notin \{d_1, d_2\} \text{ and } n \geq 1 \\ r & \text{if } x \in \{d_1, d_2\} \text{ and } n \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Relaxation

- Unique stationary distribution on $\Omega_{L,N}$ given by $\pi_{L,N}[\boldsymbol{\eta}] = Z_{L,N}^{-1} \prod_{x=1}^L w_x(\eta_x)$
- Stationary weights $w_x(n) = \prod_{k=1}^n (g_x(\eta_x))^{-1}$
- Heat kernel $H_t(\boldsymbol{\eta}, \boldsymbol{\xi}) = \mathbb{P}_{\boldsymbol{\eta}}(X_t = \boldsymbol{\xi})$ and $H_t f(\boldsymbol{\eta}) = \sum_{\boldsymbol{\xi}} H_t(\boldsymbol{\eta}, \boldsymbol{\xi}) f(\boldsymbol{\xi})$
- Relaxation time is the optimal constant satisfying

$$||H_t(f) - \mathbb{E}_\pi(f)||_2^2 \leq \exp\left\{-2\frac{t}{T_{rel}}\right\} Var_\pi(f)$$

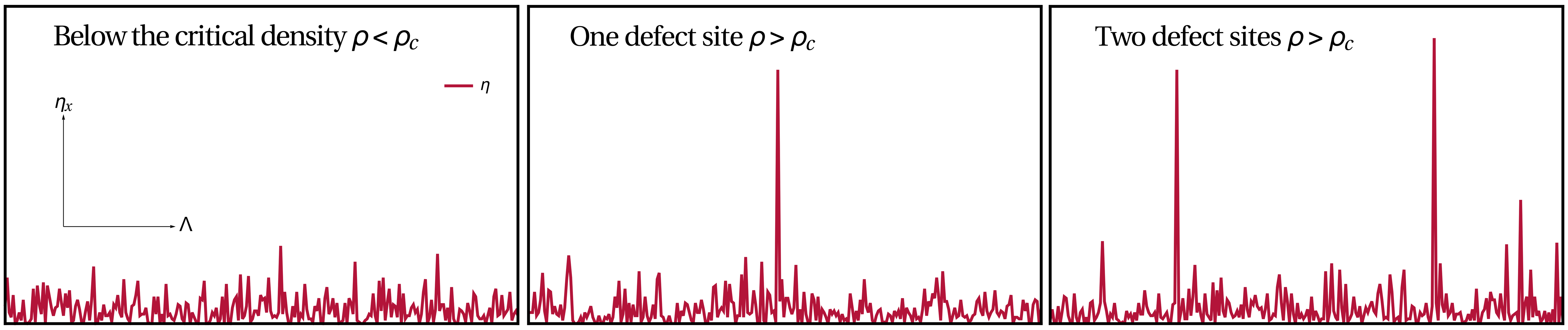
- Controls the asymptotic rate of decay of correlations
- Characterised by variational principle

$$\frac{1}{T_{rel}} = \inf\left\{\frac{D_\pi(f)}{Var_\pi(f)} : Var_\pi(f) \neq 0\right\} \quad \text{where} \quad D_\pi(f) := \mathbb{E}_\pi\left(f(-\mathcal{L}f)\right)$$

Grand canonical distribution and condensation

Factorisable $\mathbb{P}_\phi[\boldsymbol{\eta} = \mathbf{n}] = \prod_{x=1}^L \mathbb{P}_\phi[\eta_x = n_x]$ where $\mathbb{P}_\phi[\eta_x = n] = \begin{cases} \phi^n(1-\phi) & \text{for } x \notin \{d_1, d_2\} \\ \left(\frac{\phi}{r}\right)^n\left(1 - \frac{\phi}{r}\right) & \text{for } x \in \{d_1, d_2\} \end{cases}$

Density $\rho_x(\phi) := \begin{cases} \frac{\phi}{1-\phi} & \text{if } x \notin \{d_1, d_2\} \\ \frac{\phi}{r-\phi} & \text{if } x \in \{d_1, d_2\} \end{cases}$ which implies $\lim_{\phi \to r} \rho_x(\phi) = \begin{cases} \frac{r}{1-r} & \text{if } x \notin \{d_1, d_2\} \\ +\infty & \text{if } x \in \{d_1, d_2\} \end{cases}$



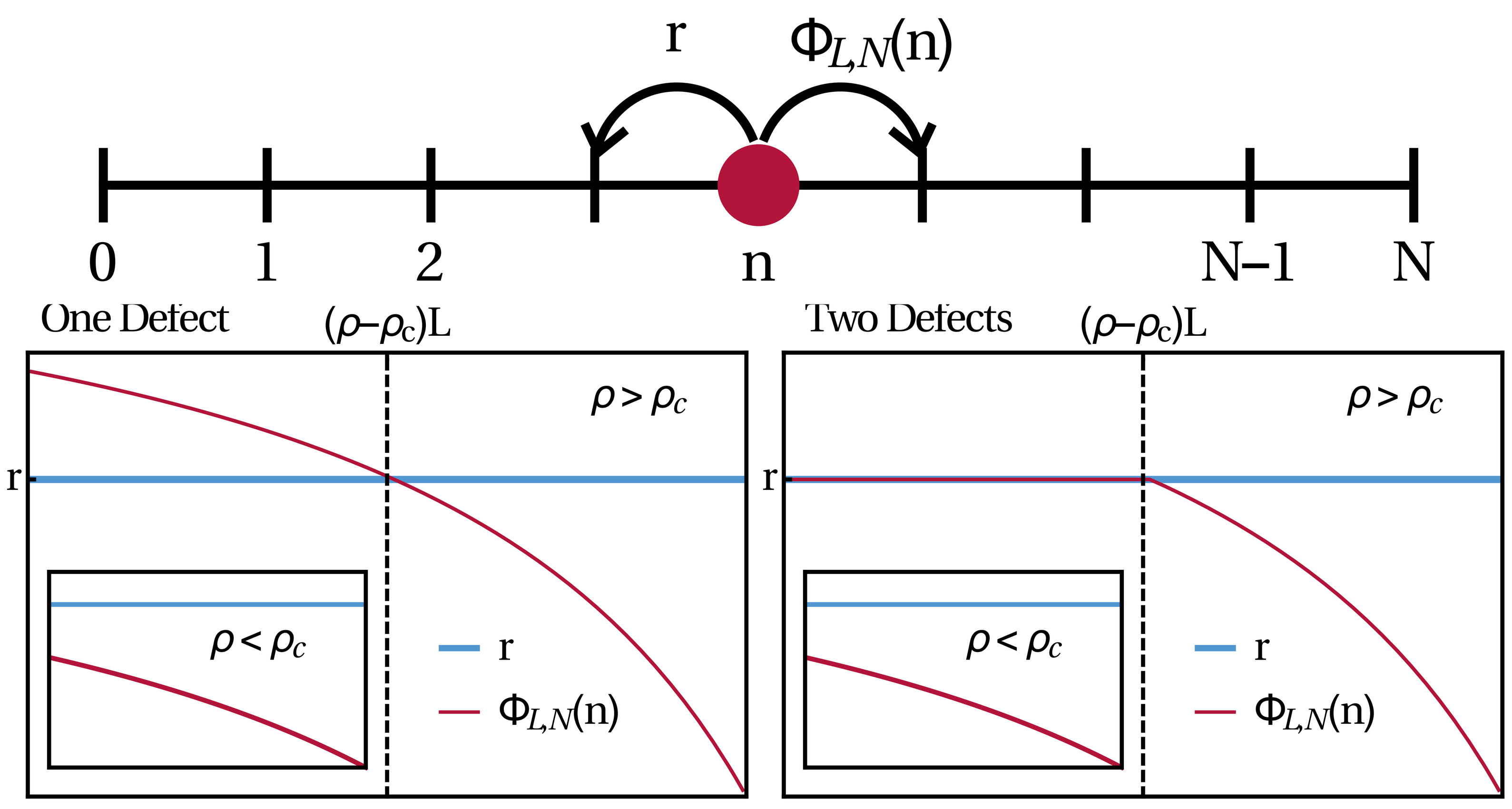
State space decomposition

- Decompose state space: $\Omega_{L,N} = \bigcup_{i=0}^N \Omega_{L,N}^i = \bigcup_{i=0}^N \{\boldsymbol{\eta} : \eta_{d_1} = i\}$
- Restriction chain: A ZRP restricted to the partition $\Omega_{L,N}^i$
- Projection chain: A process between partitions with transition rates

$$\hat{q}(\Omega_{L,N}^i, \Omega_{L,N}^j) = \begin{cases} r & \text{if } j = i-1 \\ \Phi_{L,N}(n) & \text{if } j = i+1 \end{cases}$$

- Birth-rate: $\Phi_{L,N}(n) := \mathbb{E}_{L-1,N-n}(g_x(\cdot)) = \dfrac{N-n}{N+L-n-2}$

The projection chain



Results

One defect:

- Restriction Chain: $T_{rel}^{RC}(L, \rho) \asymp (1+\rho)^2$ [Morris, B.,2006]
- Projection Chain: $T_{rel}^{PC}(L, \rho) \asymp \begin{cases} \left(\sqrt{r} - \sqrt{\frac{\rho}{1+\rho}}\right)^{-2} & \text{for } \rho < \rho_c \\ (1+\rho_c)^2 L & \text{for } \rho \geq \rho_c \end{cases}$

$$T_{rel}^1(L, \rho) \asymp \begin{cases} \left(\sqrt{r} - \sqrt{\frac{\rho}{1+\rho}}\right)^{-2} & \text{for } \rho < \rho_c \\ (1+\rho_c)^2 L & \text{for } \rho \geq \rho_c \end{cases}$$

Two defects:

- Restriction Chain: $T_{rel}^{RC}(L, \rho) \asymp T_{rel}^1(L, \rho)$ One defect ZRP
- Projection Chain: $T_{rel}^{PC}(L, \rho) \asymp \begin{cases} \left(\sqrt{r} - \sqrt{\frac{\rho}{1+\rho}}\right)^{-2} & \text{for } \rho < \rho_c \\ (1+\rho_c)^2 L & \text{for } \rho = \rho_c \\ (\rho-\rho_c)^2 L^2 & \text{for } \rho > \rho_c \end{cases}$

$$T_{rel}^2(L, \rho) \asymp \begin{cases} \left(\sqrt{r} - \sqrt{\frac{\rho}{1+\rho}}\right)^{-2} & \text{for } \rho < \rho_c \\ (1+\rho_c)^2 L & \text{for } \rho = \rho_c \\ (\rho-\rho_c)^2 L^2 & \text{for } \rho > \rho_c \end{cases}$$