Condensation and Monotone Particle Systems

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Theorem

Consider a spatially homogeneous stochastic particle system as defined by Eq. (2), which exhibits condensation as defined by Eq. (3), and has stationary product measures. If we have finite critical density,

$$\rho_c := \frac{1}{z(1)} \sum_{n=0}^{\infty} n\, w(n) < \infty \quad \text{where} \quad z(1) := \sum_{n=0}^{\infty} w(n), \qquad (1)$$

then the canonical measures, $\pi_{L,N}$, are not stochastically ordered and the process is necessarily not monotone.
The same is true if the finite mean assumption is replaced by the assumption that $w(n) = n^{-b} f(n)$, $w(0) = 1$ and $f(n) \to c \in (0,\infty)$ as $n \to \infty$ with $b \in (3/2, 2]$, *i.e.* $\rho_c = \infty$.

Definitions and Background

- Finite lattices $\Lambda = \{1, \ldots, L\}$.
- State space $X_L = \mathbb{N}^L$.
- Configurations $\eta \in X_L$, $\eta_x \in \mathbb{N}\ \forall\, x \in \Lambda$.
- Jump rates $c(\eta,\xi) \geq 0$ from configurations $\eta \to \xi$.
- $f \in C(X_L)$, generator given by

$$\mathcal{L}f(\eta) = \sum_{\{\xi \in X_L : \xi \neq \eta\}} c(\eta,\xi)\big(f(\xi) - f(\eta)\big)\,. \qquad (2)$$

- Conserved quantity $S_L(\eta) := \sum_{x\in\Lambda} \eta_x$.
- Conditioned on $S_L = N$ processes is irreducible and therefore ergodic with unique stationary measure $\pi_{L,N}$.
- Spatially homogeneous systems such that the marginal distributions $\pi_{L,N}[\eta_x \in .]$ are identical for all $x \in \Lambda$.

Product Structure

Definition

- $\mu(\mathcal{L}(f)) = \sum_{\eta\in S} \mu(\eta)\mathcal{L}f(\eta) = 0$ for all $f \in C(X_L)$.

The canonical ensemble

State space	$X_{L,N} = \{\eta \in S : \sum \eta_x = N\}$
Stationary measure	$\pi_{L,N}[\eta] = \frac{\prod_{x\in\Lambda_L} w(\eta_x)}{Z_{L,N}}$
Partition function	$Z_{L,N} = \sum \prod_{x\in\Lambda_L} w(\eta_x)$
Site marginals	$\pi_{L,N}[\eta_1 = n] = \frac{w(n) Z_{L-1,N-n}}{Z_{L,N}}$

Condensation

- Maximum occupation $M_L(\eta) := \max_{x\in\Lambda} \eta_x$.
- A stochastic particle system with canonical measures $\pi_{L,N}$ on X_L with $L \geq 2$ exhibits condensation if

$$\lim_{K\to\infty} \lim_{N\to\infty} \pi_{L,N}[M_L \geq N - K] \to 1\,. \qquad (3)$$

- Condensation is equivalent to the convergence of

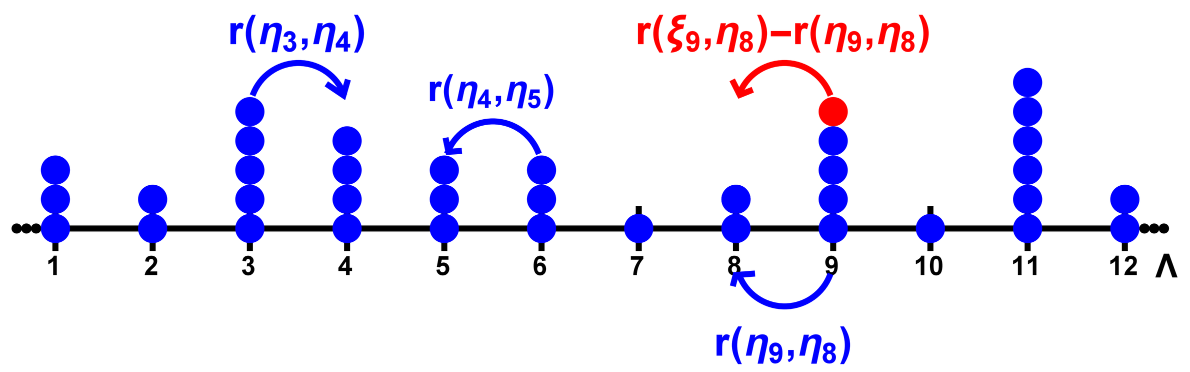
$$\lim_{N\to\infty} \frac{Z_{L,N}}{w(N)} = L\left(\sum_{n=0}^{\infty} w(n)\right)^{L-1} \quad \forall L \geq 2\,. \qquad (4)$$

Misanthrope & Monotonicity

- Transition rates $c(\eta, \eta^{x,y}) = r(\eta_x, \eta_y) p(x,y)$ [1].
- Stationary measures if translation invariant dynamics $p(x,y) = q(x-y)$ and for all $n \geq 1$ and $m \geq 0$ the rates satisfy

$$\frac{r(n,m)}{r(m+1,n-1)} = \frac{r(n,0)r(1,m)}{r(m+1,0)r(1,n-1)}\,.$$

- Monotone if and only if $r(n,m) \leq r(n+1,m)$ and $r(n,m) \geq r(n,m+1)$ [2], see coupling example below



- $(\eta_t)_{t\geq 0}$ monotone, if $\eta_0 \leq \xi_0$ implies $\eta_t \leq \xi_t$ for all $t \geq 0$.
- The existence of a coupling implies monotonicity.
- Monotonicity implies the stationary measures are stochastically ordered, *i.e.* $\pi_{L,N} \leq \pi_{L,N+1}$.

Sufficient Conditions

Condensation occurs if either of the following hold

- $\sup_{1\leq k\leq n/2} \frac{w(n-k)}{w(n)} \leq K \in (0,\infty)$, or
- $w(n) = e^{-n\psi(n)}$,

where $\psi(x)$ is a smooth function on $\mathbb{R}$ with $\psi(x) \searrow 0$ and $x^2|\psi'(x)| \nearrow \infty$ as $x \to \infty$, and $\int_0^\infty dx\, e^{-\frac{1}{2}x^2|\psi'(x)|} < \infty$.

Overshoot: Test Function and Background Density

- $H_L(N) := \frac{Z_{L,N}}{w(N)} \frac{1}{Lz(1)^{L-1}}$, monotonicity implies $H_L(N) \leq H_L(N+1)$ for all $N \in \mathbb{N}$.
- Background density $R_L^{bg}(N) := \frac{1}{L-1}\pi_{L,N}(N - M_L)$, monotonicity implies $R_L^{bg}(N) \leq R_L^{bg}(N+1)$ for all $N \in \mathbb{N}$.

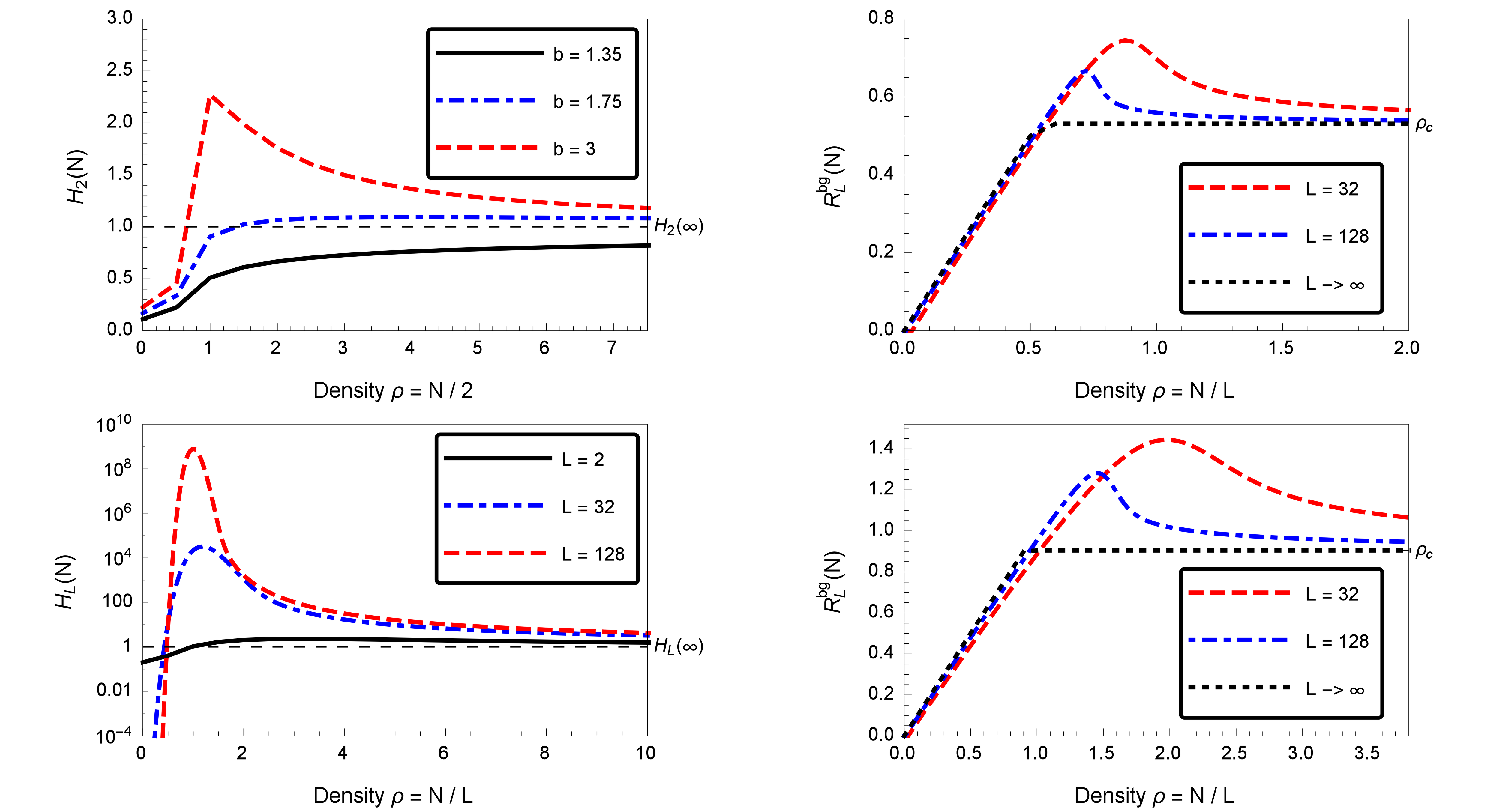


Figure: Overshoot and non-monotonicity of the functions $H_L(N)$ and $R_L^{bg}(N)$ in condensing systems. (Top Left and Right) Power law tails $w(n) \sim n^{-b}$. (Bottom Left and Right) Log-normal tails $w(n) \sim \frac{1}{n}\exp(-(\log(n))^2)$.

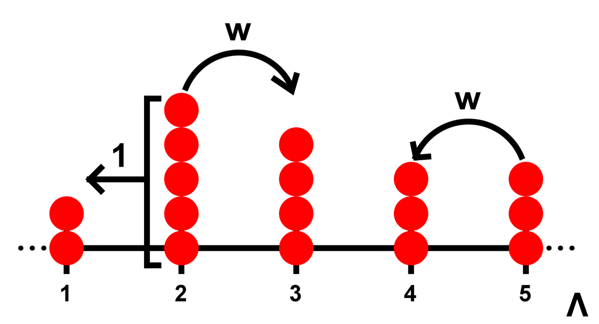
Examples of Condensation

In addition to power law and log-normal tails, condensation also occurs if the weights are of the following form [3]:

- Stretched exponential $w(n) \sim \exp(-n^\gamma)$ where $\gamma \in (0,1)$.
- Almost exponential $w(n) \sim \exp(-n/(\log(n))^\beta)$ where $\beta > 0$.
- Log-gamma $w(n) \sim (log(n))^{\alpha-1} n^{\beta-1}$ where $\alpha > 0$ and $\beta > 1$.

The Chipping Model

- Particles 'chip' at rate $w > 0$ and blocks jump at rate 1.
- Condensation with critical density $\rho_c(w) \sim \sqrt{w}$ [4].



- Process does not exhibit stationary product measures.
- Constructing of a basic coupling implies monotonicity of the process.

Previous Work

- Condensation on finite lattices with power law tails [5].
- Non-monotonicity of condensing zero-range processes with stationary product measures [6].
- Non-monotonicity of condensing Misanthrope process [7].

References

[1] C Cocozza-Thivent, Z. Wahrscheinlichkeitstheorie 70 (1985), no. 4, 509-528.
[2] T Gobron and E Saada, Ann. I. H. Poincare-PR **46** (2010), no. 4, 1132-1177.
[3] C M Goldie and C Klüppelberg, 1998, pp. 453-459 Birkhäuser.
[4] S N Majumdar, S Krishnamurthy, and M Barma, Journal of Statistical Physics 99, no. 1-2, 1-29 (en).
[5] P A Ferrari, C Landim, and V Sisko, J. Stat. Phys. 128 (2007), no. 5, 1153-1158.
[6] P Chleboun and S Grosskinsky, J.Stat. Phys. 140 (2010), no. 5, 846-872.
[7] B Waclaw and M R Evans, Phys. Rev. Lett., 108(7):070601, 2012.