My PhD

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Overview

1. Introduction
2. Mini-project
3. More recent work

Introduction

Key points

- Some background from mini-project
- Coupling and Attractivity
- Particle systems with stationary product measures
- Calculate mixing and relaxation times

Coupling, Monotonicity and Attractivity

- A coupling of two probability distributions μ and ν is a pair of random variables (X, Y) defined on a single probability space such that the marginal distribution of X is μ and the marginal distribution of Y is ν. That is, a coupling (X, Y) satisfies $\mathbb{P}(X = x) = \nu(x)$ and $\mathbb{P}(Y = y) = \nu(y)$
- The partial ordering of two configurations η and ζ defined on a state-space of the form $S = X^{\Lambda}$ where Λ is a lattice or network and for interacting particle systems $X \subseteq \mathbb{N}$. $\eta \leq \zeta$ if we have $\eta_x \leq \zeta_x$ for all $x \in \Lambda$
- For probability measures μ_1, μ_2 on S: $\mu_1 \leq \mu_2$ provided that $\mu_1(f) \leq \mu_2(f)$ for all increasing f
- A process $(\eta(t) : t \geq 0)$ on S is attractive if the property of stochastic monotonicity, or the partial ordering of configurations, is preserved through time.

Mini-project

Key Points

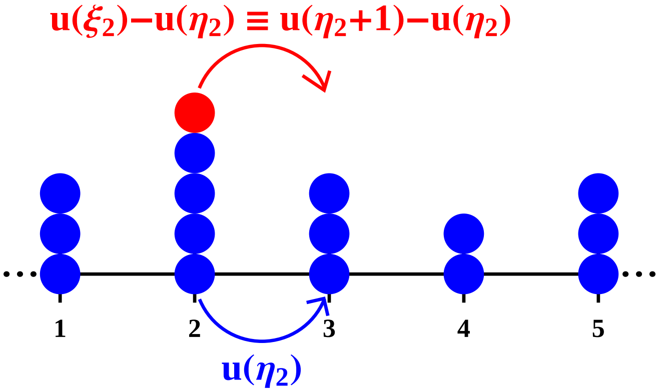
- Consider the zero-range process (ZRP). $S = X_{L,N}$
- $\mathcal{L}f(\eta) = \sum_{x,z\in\Lambda} u(\eta_x)p(x,z)(f(\eta^{x\to z}) - f(\eta))$
- Attractive $\Leftrightarrow$ $u(x) \leq u(x+1)$ for all $x \in \mathbb{N}$

Coupling Construction

Consider the joint state space $(X_{L,N}, X_{L,N+1})$ between process η and ξ such that, $\xi = \eta + \delta_y$ for some $y \in \Lambda_L$.

$$\left.\begin{array}{l}\xi_y = n+1\\ \eta_y = n\end{array}\right\} \xrightarrow{u(\xi_y)-u(\eta_y)} \left\{\begin{array}{l}\xi_y = n\\ \eta_y = n\end{array}\right.$$
$$\left.\begin{array}{l}\xi_y = n+1\\ \eta_y = n\end{array}\right\} \xrightarrow{u(\eta_y)} \left\{\begin{array}{l}\xi_y = n\\ \eta_y = n-1\end{array}\right. \tag{1}$$

Some pictures



Stationary measure (Canonical)

Some equations

$$\nu_\phi^x[\eta_x = n] = \frac{w_x(n)(\phi)^n}{z_x(\phi)}$$

$$w_x(n) = \prod_{k=1}^{n} \frac{1}{u_x(k)}$$

$$\pi_{L,N}[\eta] = \nu_\phi^\Lambda\left[\eta \middle| \sum_L (\eta) = N\right] = \frac{\mathbb{I}_{X_{L,N}}(\eta)}{Z_{L,N}} \prod_{x\in\Lambda_L} w_x(\eta_x), \qquad (2)$$

where $Z_{L,N} = \sum_{\eta\in X_{L,N}} \prod_{x\in\mathbb{N}^{\Lambda_L}} w_x(\eta_x)$

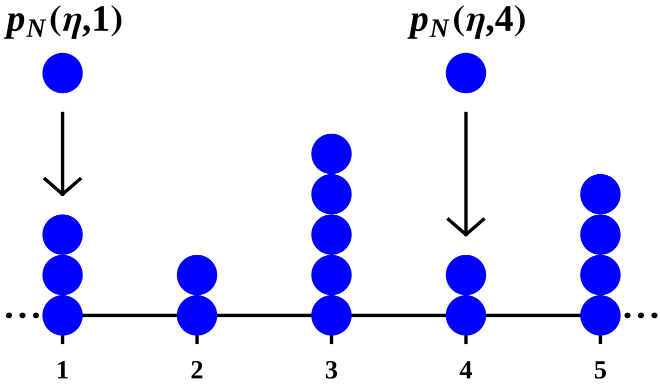
Coupled generator

$$\mathcal{L}f(\eta, y) = \sum_{x,z \in \Lambda_L} [u(\eta_x)p(x,z)(f(\eta^{x \to z}, y) - f(\eta, y))]$$
$$+ \sum_{z \in \Lambda_L} (u(\eta_y + 1) - u(\eta_y))p(y,z)(f(\eta, z) - f(\eta, y)). \quad (3)$$

Stationary coupled measure

- Let $\mu(\eta, y) = \mu(y|\eta)\pi_{L,N}(\eta)$ be the stationary measure of the coupled process.
- $\mu(\mathcal{L}f(\eta, y)) = 0$ for all f
- $\mu(y|\eta) = \alpha_\eta(y)$ is a possible growth rule

Another picture



A result

A result

If $\mu(\eta, y)$ is stationary then $\alpha_\eta(y)$ must satisfy

$$\sum_{x,z\in\Lambda_L} u_z(\eta_z)p(x,z)\alpha_{\eta^{z\to x}}(y) + \sum_{z\in\Lambda_L}\left[u_z(\eta_z+1)-u_z(\eta_z)\right]p(z,y)\alpha_\eta(z)$$
$$= \sum_{x,z\in\Lambda_L} u_x(\eta_x)p(x,z)\alpha_\eta(y) + \sum_{z\in\Lambda_L}\left[u_y(\eta_y+1)-u_y(\eta_y)\right]p(y,z)\alpha_\eta(y). \tag{4}$$

Results

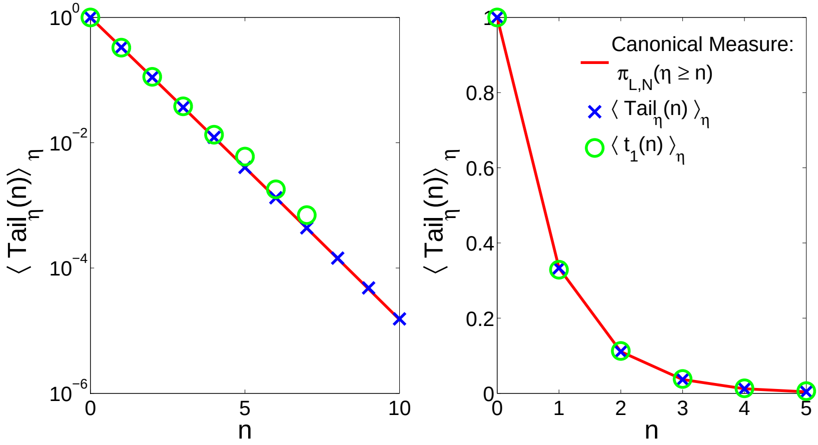


Figure: $N = 256$

Results

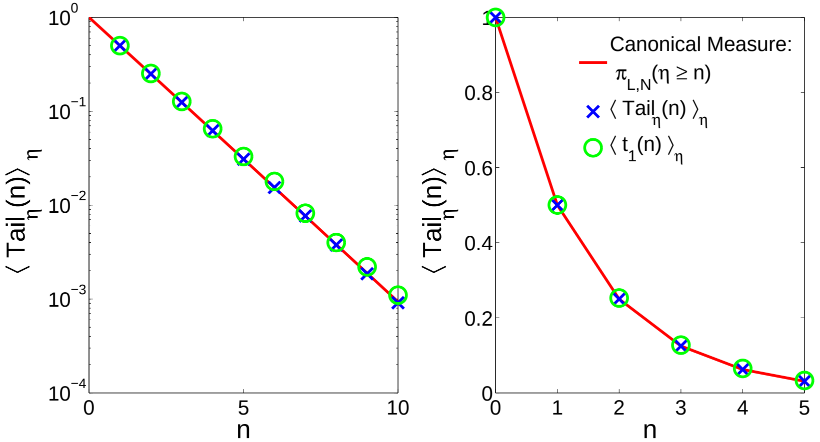


Figure: $N = 512$

Results

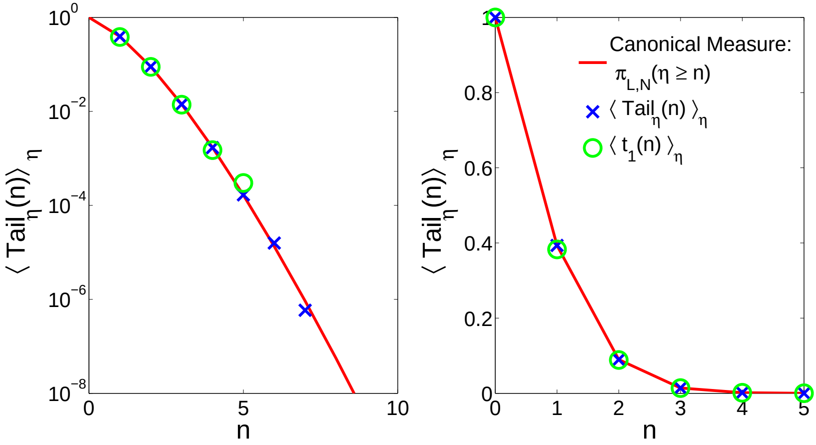


Figure: $N = 512$

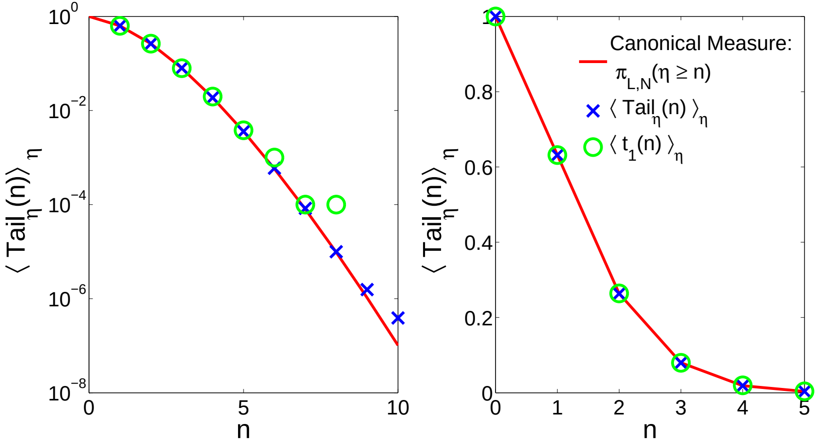

10^0
10^−2
10^−4
10^−6
10^−8
0
5
10
n
⟨ Tail_η(n)⟩_η
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
n
⟨ Tail_η(n)⟩_η
Canonical Measure:
π_L,N(η ≥ n)
⟨ Tail_η(n) ⟩_η
⟨ t_1(n) ⟩_η

Figure: $N = 512$

Mixing and relaxation times

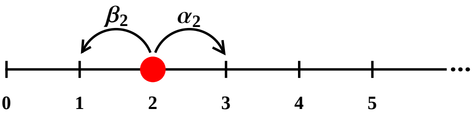
Mixing

- $d(t) = \max_{x \in S} ||P_t(x,.) - \pi||_{TV}$
- ϵ-Mixing: $t_{mix}(\epsilon) := \min\{t : d(t) \leq \epsilon\}$
- $t_{mix} := t_{mix}(1/4)$

Relaxation

- Smallest non-zero eigenvalue (Reversible chains)
- $\lambda = \inf\{\frac{\mathcal{D}(f)}{var_\pi(f)} : var_\pi(f) \neq 0\}$
- $\mathcal{D}(f) = \sum_{x,y \in S} r_{x,y} |f(y) - f(x)|^2$
- $||P_t f - \pi(f)||_2^2 \leq e^{-2\lambda t} Var_\pi(f)$

A birth death process



Properties

- $S = \{0, 1 \ldots, N\}$
- Birth rates $\alpha_k = 1$
- Death rates $\beta_k = g(k) = 1 + \frac{b}{k^\gamma} \approx e^{b/k^\gamma}$
- Stationary distribution $\pi(n) \propto \frac{1}{g(n)!}$
- $e^{\frac{b}{1-\gamma} - b} e^{-b\frac{n^{1-\gamma}}{1-\gamma}} \geq \frac{1}{g(n)!} \geq e^{\frac{b}{1-\gamma}} e^{-b\frac{(n+1)^{1-\gamma}}{1-\gamma}}$

Mixing times are hitting times of large sets (Peres, Sousi (2013))

Hitting times

- $\tau_i = \inf\{t > 0 : X_t = i\}$
- $T_i(j) = \mathbb{E}(\tau_i | X_0 = j)$
- $\mathcal{L} T_i(j) = -1$
- $T_i(i) = 0$
- $T_s N = \sum_{n=0}^{N-s} \sum_{k=n}^{N-s} \frac{\pi[N-n]}{g(N-k)\pi[N-k]}$

Mixing and hitting large sets

- Define A_α such that $\pi(A_\alpha) \geq \alpha$
- $t_H(\alpha) = \max_{x, A_\alpha : \pi(A_\alpha \geq \alpha)} T_{A_\alpha}(x) \quad (= T_{A_\alpha}(N))$
- There exist $c_\alpha, c'_\alpha > 0$ such that $c'_\alpha t_H(\alpha) \leq t_{mix} \leq c_\alpha t_H(\alpha)$

Results

$$t_{mix} \sim N^{1+\gamma}$$
$$t_{rel} \sim N^{2\gamma}$$

The End

Thanks

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