Driven diffusive systems and growing stationary configurations

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Overview

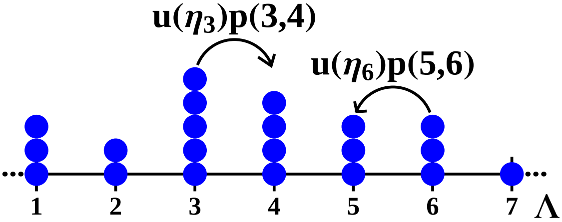
Introduction

- Interacting particle system
- Stationary product measures
- Condensation transition
- Relaxation times of order N^2
- Fixed and finite lattice

The Aim

- Construct a growth rule
- CPU times scales linearly with N
- Observe condensation

The zero-range process - Definition



Definition

$$\text{Lattice:} \quad \Lambda = \{1, 2, \ldots, L\}$$

$$\text{State space:} \quad S = \mathbb{N}^\Lambda$$

$$\text{Configurations:} \quad \eta = (\eta_x)_{x\in\Lambda} \text{ where } \eta_x \in \mathbb{N}$$

$$\text{Jump rates:} \quad u_x : \mathbb{N} \to [0, \infty)$$

$$u_x(n) = 0 \Leftrightarrow n = 0$$

The zero-range process - Stationary measures

The zero-range process - Stationary measures

Site marginals (Grand-canonical ensemble)

$$\nu_{\phi}^{x}[\eta_x = n] = \frac{w_x(n)(\phi)^n}{z_x(\phi)} \quad \text{and } w_x(n) = \prod_{k=1}^{n} \frac{1}{u_x(k)}$$

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- Partition function $z_x(\phi) = \sum_{n=0}^{\infty} w_x(n)(\phi)^n$
- Fugacity $\phi \in [0, \phi_c)$ where $\phi_c \in [0, \infty]$ called critical fugacity

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- Average density $R_x(\phi) = \mathbb{E}_\phi[\eta_x] = \frac{1}{z_x(\phi)} \sum_{k=1}^{\infty} k w_x(k)(\phi)^k$
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Unique stationary measure (Canonical ensemble)

$$\pi_{L,N}[\eta] = \nu_\phi^L\Big[\eta_x = n \Big| \sum_{x \in \Lambda} \eta_x = N\Big] = \frac{1}{Z(L,N)} \prod_{x \in \Lambda} w_x(\eta_x)$$

- Reference: [Spitzer, 1970] [Andjel, 1982]

The zero-range process - Stationary measures

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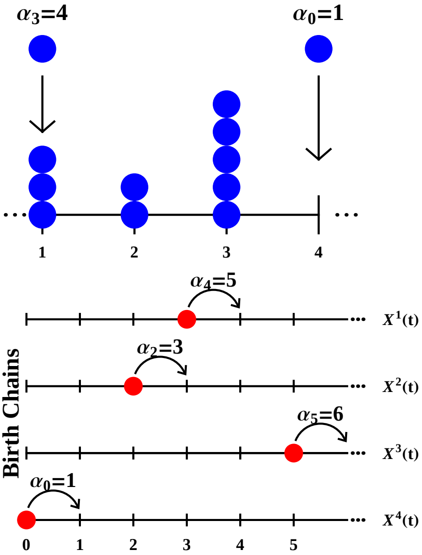
Growth - Properties

- Grow L chains independently
- Product measures
- Continuous time birth process
- Correct marginals
- Time vs Fugacity
- Condition on N particles to regain canonical stationary measure



The zero-range process - Constant rates

- $u_x(k) = \alpha > 0 \quad \forall k \geq 1 \implies w_x(n) = \frac{1}{\alpha^n}$

Stationary measures (Grand-canonical ensemble)

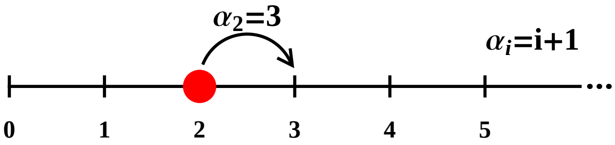
$$\nu_\phi^x[\eta_x = n] = \left(1 - \frac{\phi}{\alpha}\right)\left(\frac{\phi}{\alpha}\right)^n \quad \text{(Geometric RV)}$$

Stationary measures (Canonical ensemble)

$$\pi_{L,N}[\eta] = \frac{1}{Z(L,N)}$$

- Defined for all $\phi \in [0, \alpha)$
- Average density: $\rho_x(\phi) = \frac{\phi}{\alpha - \phi}$
- Critical density: ∞

Results - Growth



Master Equation

$$\frac{d}{dt}\mathbb{P}(X_t = n) = n\mathbb{P}(X_t = n-1) - (n+1)\mathbb{P}(X_t = n)$$

$$\frac{d}{dt}\mathbb{P}(X_t = 0) = -\mathbb{P}(X_t = 0)$$

Results - Pure-birth process - Chain distribution

- Use generating function: $F(s,t) = \sum_{k=0}^{\infty} s^k \mathbb{P}(X_t = k)$
- Solve the PDE: $\frac{\partial}{\partial t} F(s,t)$
- Boundary conditions $\begin{cases} F(0,t) = 0 \ \forall t \geq 0 \\ F(s,0) = 1 \ \forall s \in [0,1] \end{cases}$
- Use the identity: $\frac{1}{k!} \frac{\partial^k}{\partial s^k} F(s,t) = \mathbb{P}(X_t = k)$

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Chain distribution

$$\mathbb{P}(X_t = n) = e^{-t}\left(1 - e^{-t}\right)^n \quad \text{(Geometric RV)}$$

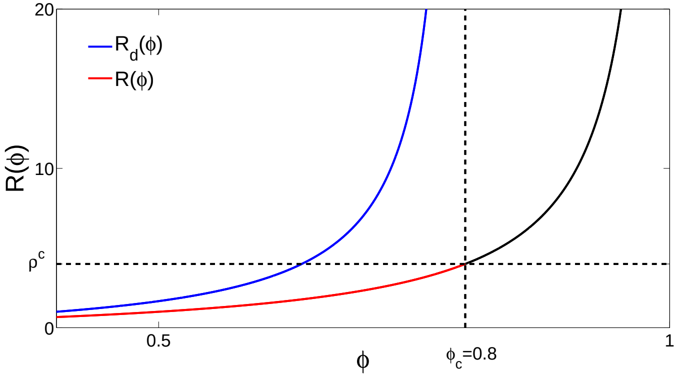
$$\nu_\phi^x[\eta_x = n] = \left(1 - \frac{\phi}{\alpha}\right)\left(\frac{\phi}{\alpha}\right)^n \quad \text{(Geometric RV)}$$

- $\frac{\phi}{\alpha} = 1 - e^{-t}$

Section 5

Condensation

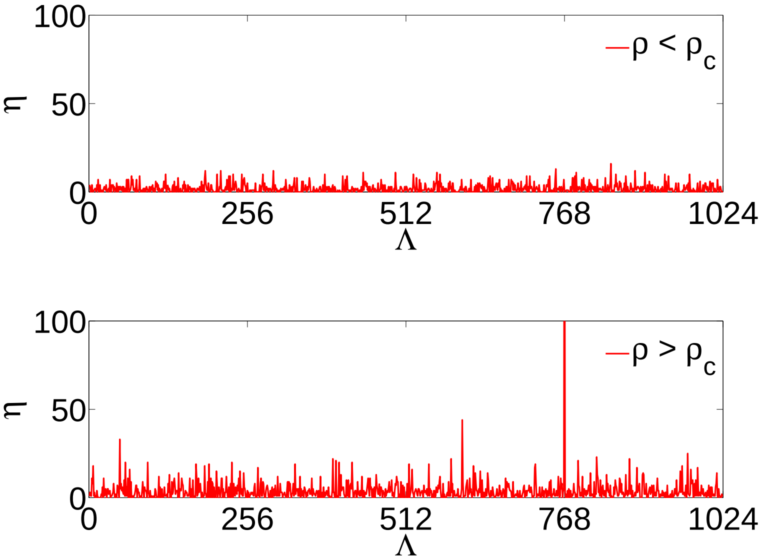
The zero-range process - Condensation



Conditions for condensation [Angel et al., 2004] [Ferrari et al., 2007]

- Constant jump rates $u_x(k) = 1 \quad \forall x \in \Lambda \setminus \{d\} \quad \forall k \geq 1$
- Single-site defect $u_d(k) = r < 1 \quad \forall k \geq 1$

Condensation



Results - Pure-birth process - Time inhomogeneous

- $\alpha_i = i + 1 \longrightarrow \alpha_i(t) = h(t)(i+1)$
- Geometric random variable
- $\mathbb{P}(X_t = n) = e^{-H(t)} \left(1 - e^{-H(t)}\right)^n$
- $H(t) = \int_0^t h(s) ds$
- $\nu_\phi^d[\eta_d = n] = \left(1 - \frac{\phi}{r}\right) \left(\frac{\phi}{r}\right)^n$

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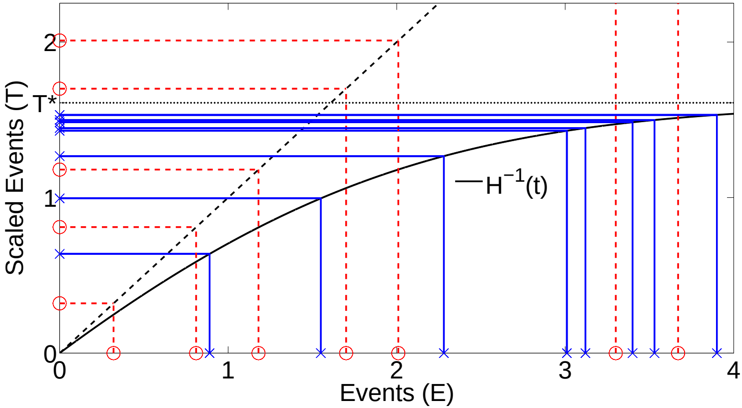
Intensity function

$$\text{Solve } e^{-H(t)} = 1 - \phi/r$$

$$H(t) = -\log\left(1 - \frac{1-e^{-t}}{r}\right)$$

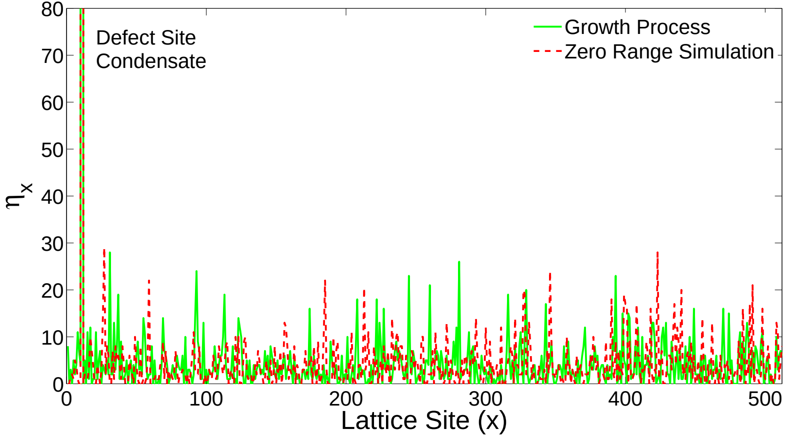
$$T^\star = -\log(1-r)$$

Results - Pure-birth process - Simulation

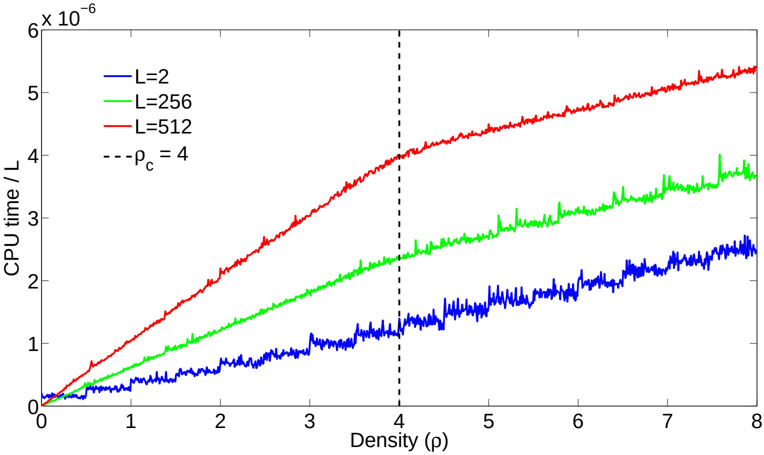


- Blue - Time inhomogeneous process
- Red - Time homogeneous process

Results - Pure-birth process - Simulation



Results - Pure-birth process - CPU time



Conclusion

Discussion

- Sampled from the stationary product measure with CPU times scaling linearly with N
- Used a time inhomogeneous pure-birth process
- The intensity function exhibited finite time blow up

Future work

- Use coupling techniques to sample form the stationary measure of the attractive ZRP with general rates
- Construct a coupling of general product measure to sample from general driven diffusive systems

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- Any questions???

References

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