Condensation and Attractive Particle Systems

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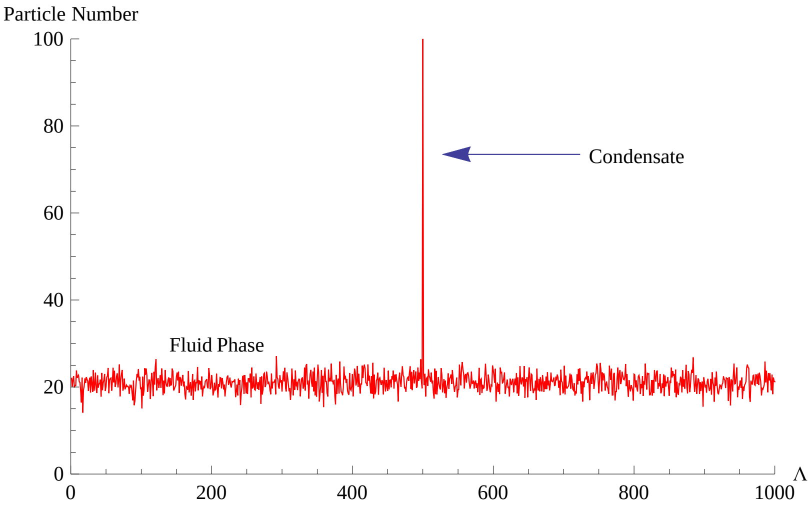
Overview

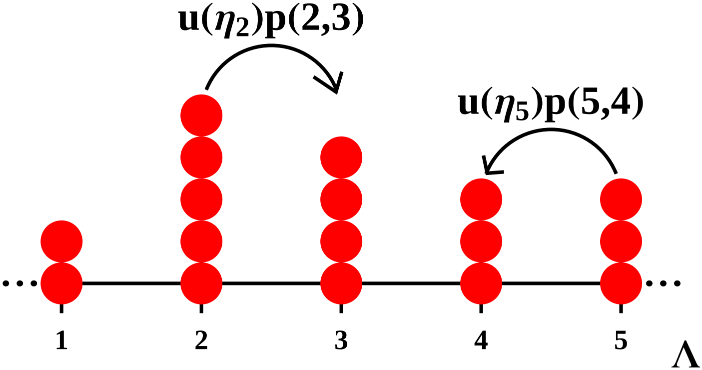

CONDENSATION

Condensation



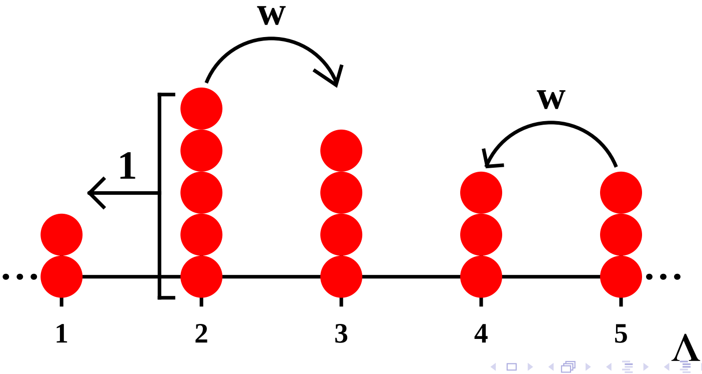
Definition [Spitzer, 1970]

- A particle leaves a site at rate $u(k)$.
- $p(x, y)$ corresponds to a random walk.
- Stationary distributions are conditional product measures. [Andjel, 1982]



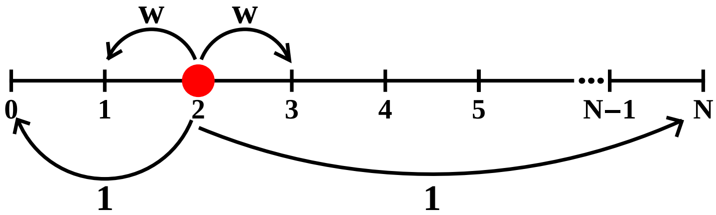
Definition [Majumdar et al., 2000]

- A particle leaves a site at rate w.
- Blocks jump at rate 1.
- Stationary distributions are **not** conditional product measures.
- Prediction for background density $\rho_{BG}(w) = \sqrt{1+w} - 1$.



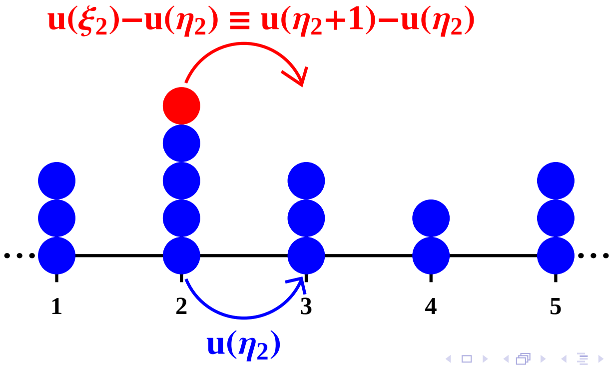
2 sites

- Random walk with resetting.
- Can find stationary distribution.
- Prediction for background density, $\rho_{BG}(w) = \frac{\sqrt{1+2w}-1}{2}$.



General Idea

- For increasing $f : S \to \mathbb{R}$ we have $\frac{d}{dt}\mathbb{E}(f(\eta(t))) > 0$.
- To prove the processes is attractive we construct a coupling.
- Construct new process which simultaneously simulates a process with N and $N+1$ particles.



Monotonicity: Example Chipping Model

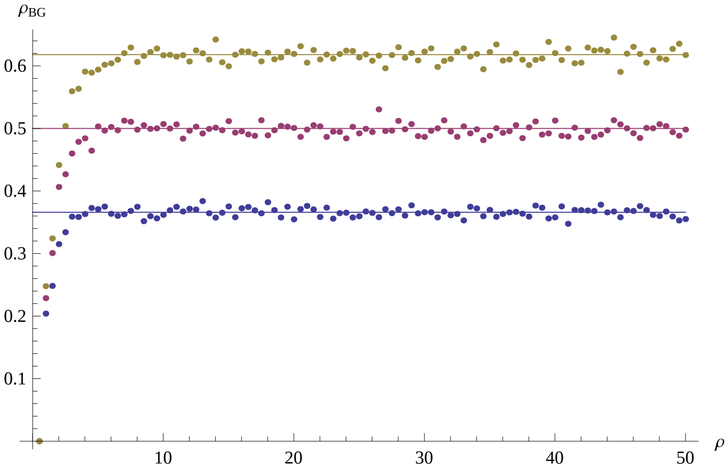


Figure: Measuring the average background density $\rho_{BG} = \langle \frac{N-\eta_{max}}{L-1} \rangle$ as a function of density for a two site chipping process. We compare simulation results against the predicted background density for $w \in \{1, 1.5, 2\}$.

Monotonicity: Example Zero-Range Process

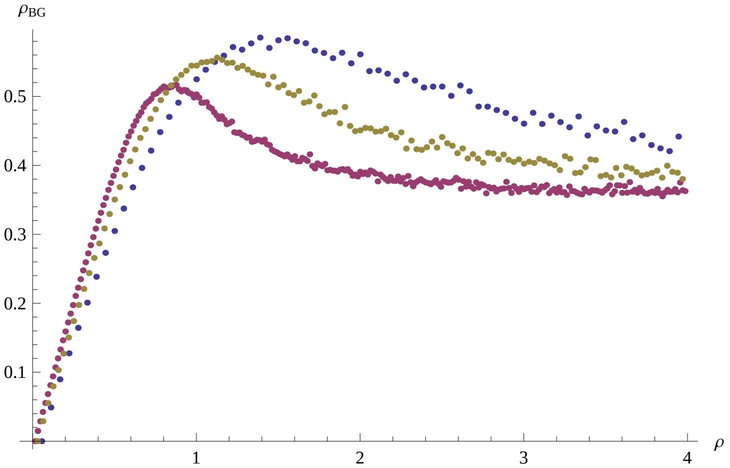


Figure: Measuring the average background density $\rho_{BG} = \langle \frac{N - \eta_{max}}{L-1} \rangle$ as a function of density for a condensing zero-range process. The jump-rate is given by $u(k) = 1 + \frac{b}{k}$.

Condensation

- Condensation does not occur in the attractive Zero-Range Process. $g(k) \leq g(k+1)$.

Why?

- We need $z(\phi) = \sum_{n=0}^{\infty} w(n)\phi^n$ to converge at $\phi_c < \infty$.
- $w(n) = \prod_{k=1}^{n} \frac{1}{u(k)}$.
- $\phi_c = \lim_{n\to\infty} \frac{1}{g(n)} = \begin{cases} \infty & \text{unbounded rates} \\ C & \text{bounded rates} \end{cases}$.
- $\big(g(1)\big)^{-n} \geq w(n) \geq \big(C\big)^{-n}$
- $\implies z(\phi_c) \geq \sum_{n=0}^{\infty} C^{-n}\phi_c^n = \sum_{n=0}^{\infty} 1 = \infty$.

Assumptions

- Given an ergodic Markov process which conserves the number of particles.
- The process converges to a stationary conditional product measure of the form $\pi_{L,N}(\eta) = \prod_{x=1}^{L} w(\eta_x) \; Z_{L,N}^{-1}$.

Statement

- For any conditional product measure we can construct a ZRP $g(n) := \frac{w(n-1)}{w(n)}$.
- If $\pi_{L,N} \leq \pi_{L,N+1} \iff$ Corresponding ZRP is attractive.

What this implies

- Processes that converge to ordered conditional product measures don't exhibit condensation.

ZRP attractive $\implies \pi_{L,N} \leq \pi_{L,N+1}$

- Construct a coupling.

$\pi_{L,N} \leq \pi_{L,N+1} \implies$ ZRP attractive

- $\pi_{L,N} \leq \pi_{L,N+1}$ means f increasing $\mathbb{E}_{\pi_{L,N}}(f) \leq \mathbb{E}_{\pi_{L,N+1}}(f)$.
- Assume ZRP is not attractive $\implies \exists K \in \mathbb{N}$ such that $g(K) > g(K+1)$.
- Find an increasing function f such that $\pi_{L,N}(f) > \pi_{L,N+1}(f)$.

Conclusion

- There exists an attractive particle system that condenses.
- Difficult to analyse since stationary measure is unknown.
- Restricting to two sites the process is a random walk with resetting.
- Potentially have a general statement concerning conditional product measures and condensation.

And finally..

- Thanks to my supervisors, Paul and Stefan.
- Any questions??

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