Monotonicity and Condensation in Stochastic Particle Systems

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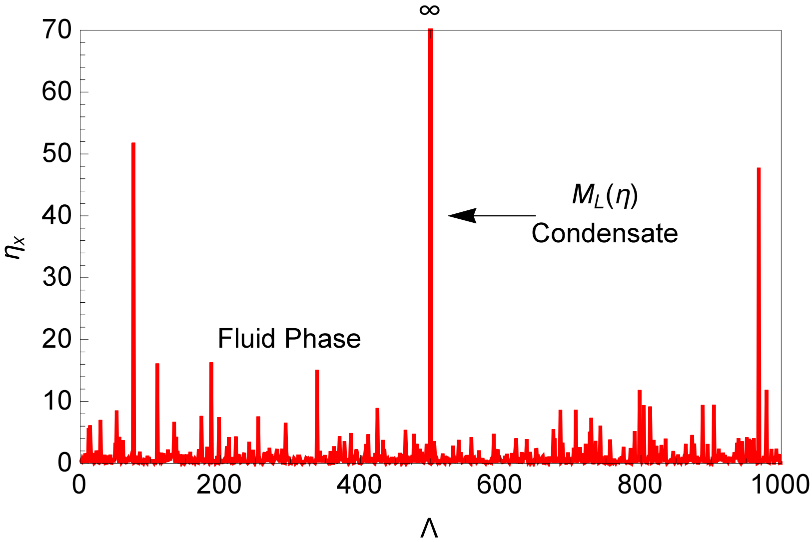
Introduction

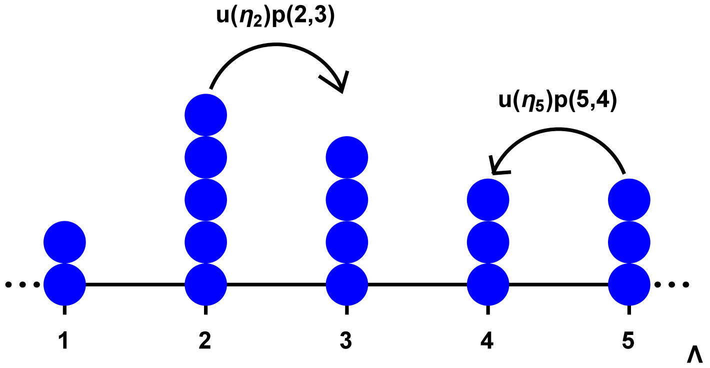
Condensation



Example: Zero-Range Processes (ZRP)

Generator [Spitzer, 1970]

$$\mathcal{L}f(\eta) = \sum_{x,y \in \Lambda} u_x(\eta_x) p(x,y) \left(f(\eta^{x,y}) - f(\eta) \right)$$



Invariant measures

Theorem [Andjel, 1982]

Consider the ZRP on the state-space $\mathbb{N}^L$ with generator

- $\mathcal{L}f(\eta) = \sum_{x,y\in\Lambda} u(\eta_x)p(x,y)\left(f(\eta^{x,y}) - f(\eta)\right)$.
- Assume $p(x,y) = q(y-x)$ translation invariant jumps.

Then the family

$$\left\{\nu_\phi^L[d\eta] = \prod_{i=1}^{L} \tilde{\nu}_\phi[\eta_x]d\eta : \nu_\phi[.] \text{ exists}\right\}$$

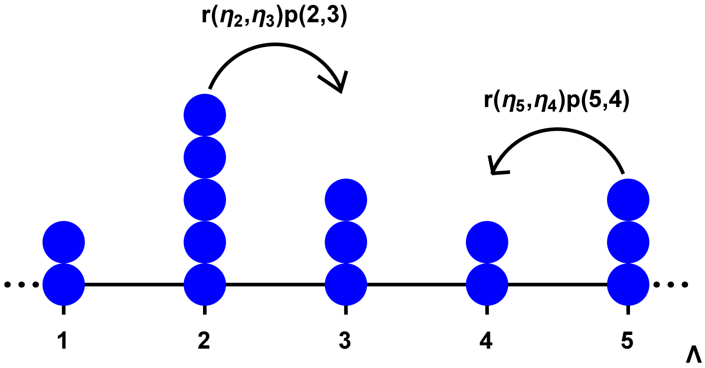
is invariant for the ZRP with marginals

$$\tilde{\nu}_\phi[\eta_x = n] = \frac{\phi^n w(n)}{z(\phi)} \quad \text{and} \quad \frac{w(n-1)}{w(n)} = u(n) \ .$$

Example: Misanthrope Processes (MP)

Generator [Cocozza-Thivent, 1985]

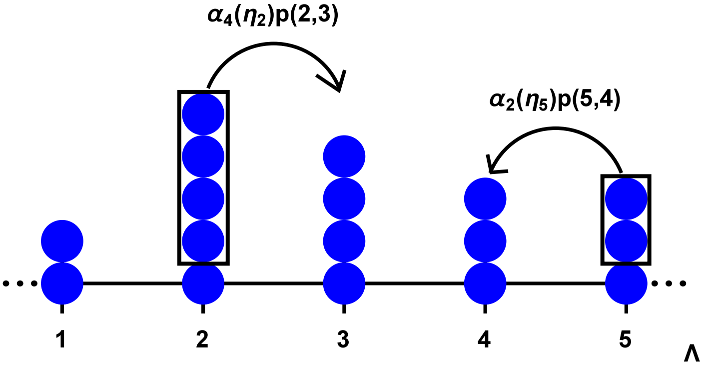
$$\mathcal{L}f(\eta) = \sum_{x,y\in\Lambda} r(\eta_x, \eta_y)p(x,y)\left(f(\eta^{x,y}) - f(\eta)\right)$$



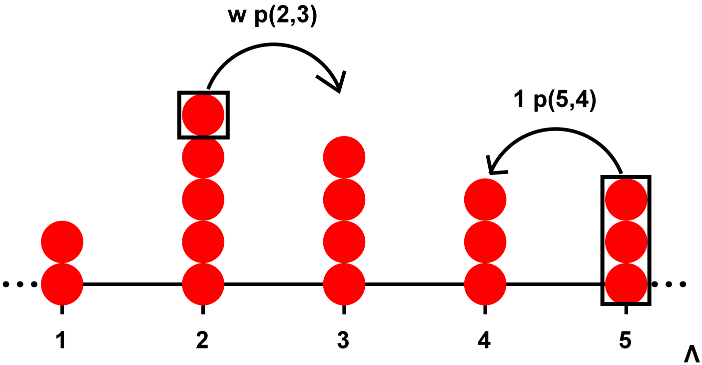
Example: Generalised Zero-Range Processes (gZRP)

Generator [Evans et al., 2004]

$$\mathcal{L}f(\eta) = \sum_{x,y\in\Lambda} \sum_{k=1}^{\eta_x} \alpha_k(\eta_x)p(x,y)\left(f(\eta^{x,(k)y}) - f(\eta)\right)$$



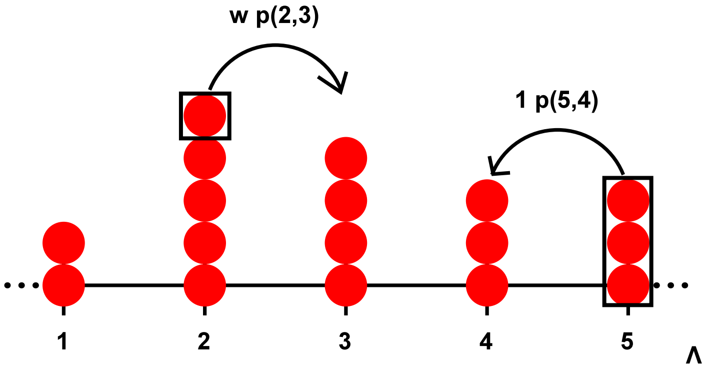
Example: Chipping Processes



Generator [Rajesh and Majumdar, 2001]

$$\mathcal{L}f(\eta) = \sum_{x,y\in\Lambda} w\mathbb{1}(\eta_x > 0)p(x,y)\left(f(\eta^{x,y}) - f(\eta)\right)$$
$$+ \sum_{x,y\in\Lambda} \mathbb{1}(\eta_x > 0)p(x,y)\left(f(\eta + \eta_x(\delta_y - \delta_x)) - f(\eta)\right)$$

Example: Chipping Processes



Link to gZRP

$$\alpha_k(n) = \begin{cases} w & \text{if } k = 1 \text{ and } n \geq 1 \ , \\ 1 & \text{if } k = n \text{ and } n \geq 1 \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Background and definitions

Stochastic particle system

- State space $\Omega_L = \mathbb{N}^L$.

$$\mathcal{L}f(\eta) = \sum_{\xi \neq \eta} c(\eta, \xi)(f(\xi) - f(\eta)) \ .$$

- Conserves particle number $F(\eta) = \sum_{x=1}^{L} \eta_x = N$, i.e. $\mathcal{L}F = 0$.
- Irreducible on the state space $\Omega_{L,N} = \{\eta \in \Omega_L : \sum_{x=1}^{L} \eta_x = N\}$.

Background and definitions

A family of stationary product measures (SPM)

- Single site marginal $\nu_\phi[n] = \frac{\phi^n w(n)}{z(\phi)}$.
- Fugacity $\phi \in [0, \phi_c]$ where $\phi_c = \lim_{n\to\infty} \frac{w(n-1)}{w(n)}$.
- $\nu_\phi^L[\eta] = \prod_{i=1}^L \nu_\phi[\eta_x]$ satisfies

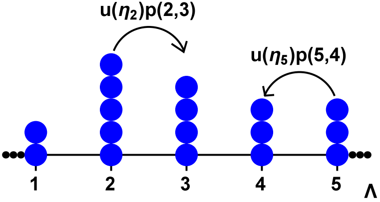
$$\nu_\phi^L(\mathcal{L}f) = 0 \quad \text{for all} \quad f \in C(\Omega_L) \ .$$

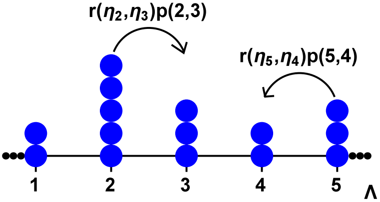
- Density $\rho(\phi) = \sum_{n=1}^\infty n\,\nu_\phi[n]$.
- Canonical measure
$\pi_{L,N}[\eta] = \nu_\phi^L[\eta | \sum_{x=1}^L \eta_x = N] = \prod_{x=1}^L w(\eta_x) Z_{L,N}^{-1}$.

Review: [Chleboun and Grosskinsky, 2013]

Examples

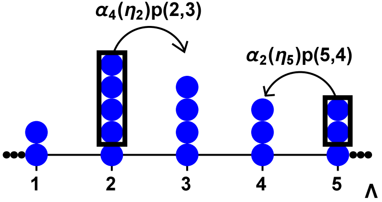
Weights $w(n) = \prod_{k=1}^{n} \frac{1}{u(k)}$

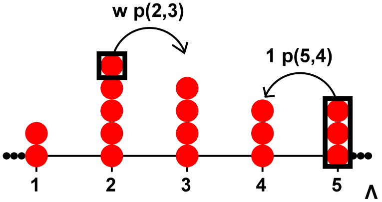




Weights $w(n) = \prod_{k=1}^{n} \frac{r(1, n-1)}{r(n, 0)}$

If $\alpha_k(n) = g(k) \frac{h(n-k)}{h(n)}$ then $w(n) = h(n)$



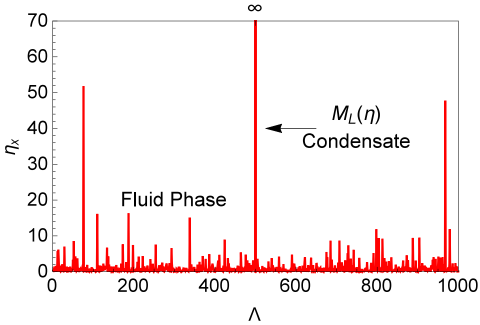


Never has SPM

Background and definitions

Thermodynamic condensation

Condensation if $\frac{N}{L} \to \rho > \rho_c$ when $N, L \to \infty$ with 'fluid phase' distributed according to ν_{ϕ_c} and excess mass accumulated on a single site.



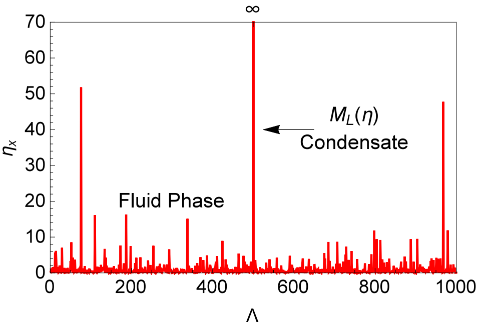
Review: [Chleboun and Grosskinsky, 2013].

Background and definitions

Condensation [Ferrari et al., 2007]

For $\eta \in \mathbb{N}^L$ let $M_L(\eta) = \max_{1\leq x\leq L}\{\eta_x\}$ then we have **condensation** if

$$\lim_{K\to\infty}\lim_{N\to\infty}\pi_{L,N}[M_L \geq N-K] = 1 \ .$$



Sub-exponential

What is sub-exponential [Goldie and Klüppelberg, 1998]

- $\mathbb{P}(X = n) = \frac{\phi^n w(n)}{z(\phi)}$
- Ratio-test $\lim_{n\to\infty} \frac{w(n-1)}{w(n)} = \phi_c < \infty$.
- $\lim_{N\to\infty} \frac{\mathbb{P}\sum_{i=1}^{L} X_i = N]}{\mathbb{P}[\max_{1\leq i\leq L} X_i = N]} = 1$.
- Existence of critical measure $z(\phi_c) < \infty$.

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Examples

- Power law weights $w(n) \sim n^{-b}$ where $b > 1$.
- Stretched exponential weights $w(n) \sim \exp\{-n^{\gamma}\}$ where $\gamma \in (0, 1)$.
- Log-normal weights $w(n) \sim \exp\{-\frac{1}{2\sigma^2}(\log(n) - \mu)^2\}$ where $\mu, \sigma \in \mathbb{R}$.
- Almost exponential weights $w(n) \sim \exp\left\{ - \frac{n}{\log(n)^{\beta}} \right\}$ where $\beta > 0$.

Condensation and sub-exponential tails

Proposition: T-R, P. Chleboun and S. Grosskinsky

Consider a stochastic particle system with stationary product measures with the regularity assumption

$$\lim_{n\to\infty} \frac{w(n-1)}{w(n)} = \phi_c \in (0,\infty] \ .$$

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Then the process exhibits condensation if and only if $\phi_c < \infty$, the grand-canonical partition function satisfies $z(\phi_c) < \infty$, and

$$\lim_{N\to\infty} \frac{Z_{L,N}}{w(N)} \in (0,\infty) \quad \text{exists}\ .$$

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i.e. for fixed $n_1, \ldots n_{L-1}$

$$\pi_{L,N}\left[\eta_1 = n_1, \ldots, \eta_{L-1} = n_{L-1} | M_L = \eta_L\right] \to \prod_{k=1}^{L-1} \nu_{\phi_c}[\eta_k = n_k] \text{ as } N \to \infty \ .$$

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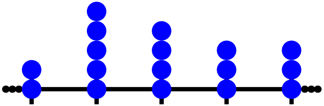
Stochastic monotonicity

Configurations $\eta, \xi \in \mathbb{N}^L$ then $\boldsymbol{\eta} \leq \boldsymbol{\xi}$ if $\eta_x \leq \xi_x$ for all $x \in \{1, \dots L\}$.



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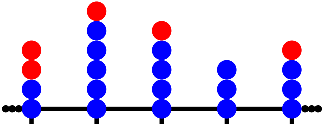

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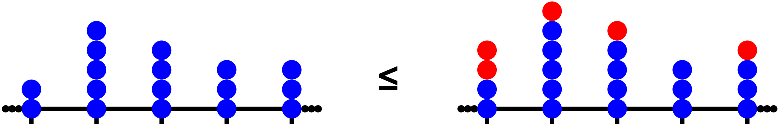
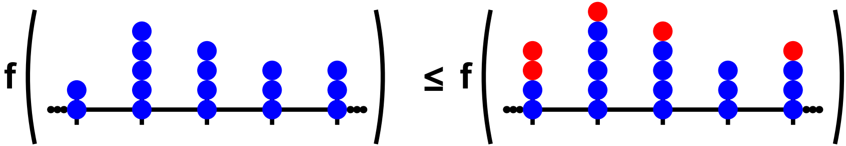


$f : \mathbb{N}^L \to \mathbb{R}$ is **increasing** if $\eta \leq \xi$ implies that $f(\eta) \leq f(\xi)$.



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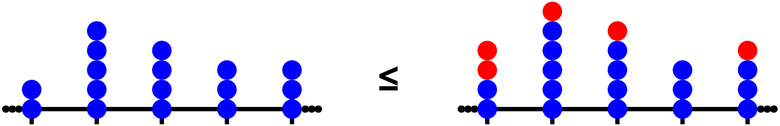

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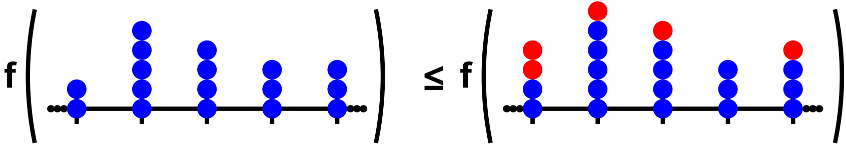

Then $\mu_L \leq \mu'_L$ if for all increasing function $f : \mathbb{N}^L \to \mathbb{R}$ we have $\mu_L(f) \leq \mu'_L(f)$.

Monotone processes

A process is called **monotone** if for all ordered initial conditions $\eta \leq \xi$ and all increasing test function $f : \mathbb{N}^L \to \mathbb{R}$ we have

$$\mathbb{E}_\eta \left[f(\eta(t)) \right] \leq \mathbb{E}_\xi \left[f(\eta(t)) \right] \quad \text{for all} \quad t \geq 0 \ .$$

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This implies canonical measures satisfy

$$\pi_{L,N} \leq \pi_{L,N+1} \quad \text{for all} \quad N \in \mathbb{N} \ .$$

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Misanthrope processes are monotone if and only if

$$\begin{aligned} r(n,m) &\leq r(n+1,m) \quad \textit{i.e.} \text{ increasing in } n, \\ r(n,m) &\geq r(n,m+1) \quad \textit{i.e.} \text{ decreasing in } m. \end{aligned}$$

[Cocozza-Thivent, 1985, Gobron and Saada, 2010].

Misanthrope coupling

Monotonicity and convexity of the entropy

- The **canonical entropy** $s(\rho) := \lim_{\substack{N,L\to\infty \\ N/L\to\rho}} \frac{1}{L} \log Z_{L,N}$.
- Equivalence of ensembles implies $s(\rho)$ is the (logarithmic) Legendre transform of the pressure $p(\phi) := \log z(\phi)$ [Grosskinsky et al., 2003].

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- Discrete derivative of $\log Z_{L,N}$ we have

$$\Delta\left(\log Z_{L,N}\right) = \log Z_{L,N+1} - \log Z_{L,N} = \log\left(\frac{Z_{L,N+1}}{Z_{L,N}}\right) \geq 0\ .$$

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- Implies convexity of $N \mapsto \frac{1}{L} \log Z_{L,N}$.

Condensing processes with SPM are not monotone

Theorem: T-R, P. Chleboun and S. Grosskinsky

Consider a spatially homogeneous stochastic particle system which exhibits **condensation** and has **stationary product measures**, and has finite critical density

$$\rho_c = \rho(\phi_c) = \sum_{n=1}^{\infty} n\, \nu_{\phi_c}[n] < \infty \ .$$

Then the canonical measures $(\pi_{L,N})$ are not ordered in N and the process is necessarily non-monotone.

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Then the canonical measures $(\pi_{L,N})$ are not ordered in N and the process is necessarily non-monotone.

The same is true if the weights are of the form $w(n) \sim n^{-b}$ with $b \in (3/2, 2]$.

Outline of proof

- Pick a monotone (decreasing) test function $f : \mathbb{N}^L \to \mathbb{R}$,

$$f(\eta) = \mathbb{1}\left(\eta_1 = \ldots = \eta_{L-1} = 0\right) \ .$$

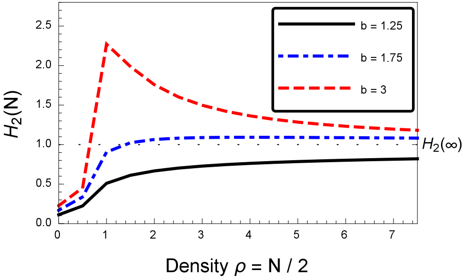
- Take expectations of f with respect to $\pi_{L,N}$,

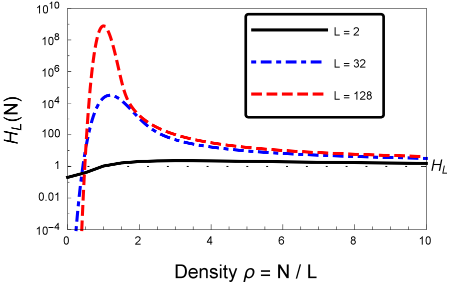
$$\pi_{L,N}(f) = \sum_{\eta \in \Omega_{L,N}} \pi_{L,N}[\eta] f(\eta) = \frac{w(0)^{L-1} w(N)}{Z_{L,N}} \ .$$

- If the process is monotone then $\frac{Z_{L,N+1}}{w(N+1)} \geq \frac{Z_{L,N}}{w(N)}$.
- Condensation implies $\frac{Z_{L,N}}{w(N)} \to Lz(\phi_c)^{L-1}$ as $N \to \infty$ for all $L \geq 2$.
- Show convergence of $\frac{Z_{L,N}}{w(N)}$ is from above.

Numerics: Expected value of test function

$$H_L(N) := \frac{1}{Lz(\phi_c)^{L-1}} \frac{Z_{L,N}}{w(N)} \ .$$

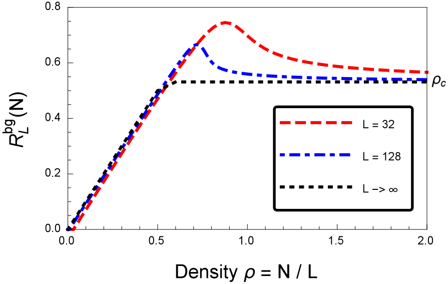


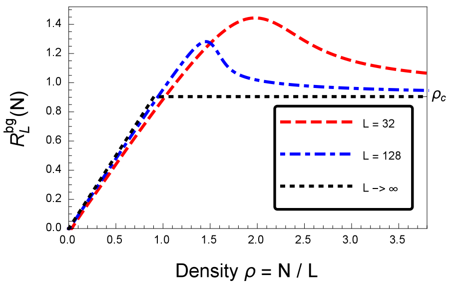


- (Left) Power law weights $w(n) = n^{-b}$ on two sites $L = 2$.
- (Right) Log-normal weights $w(n) = \exp\{-(\log(n))^2\}$.

Numerics: Background density

$$R_L^{bg}(N) := \frac{1}{L-1}\pi_{L,N}(N - M_L) \ .$$

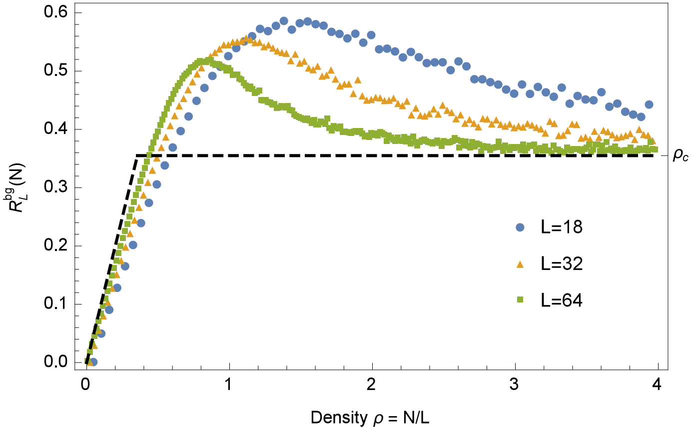




- (Left) Power law weights $w(n) = n^{-b}$ with $b = 5$.
- (Right) Log-normal weights $w(n) = \exp\{-(\log(n))^2\}$.

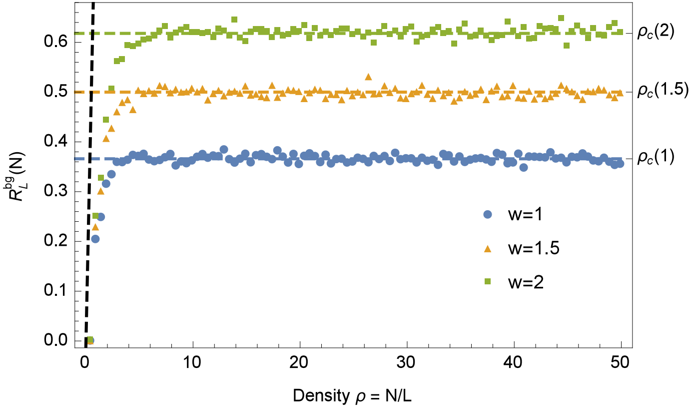
Examples: Non-monotone ZRP and condensation

- ZRP with jump rates $u(k) = 1 + \frac{b}{k}$ and $b = 5$ [Evans, 2000].
- $R_L^{bg}(N) := \frac{1}{L-1} \pi_{L,N}\,(N - M_L)$.
- $\rho_c = \rho(\phi_c) = \sum_{n=1}^{\infty} n\, \nu_\phi[n]$.



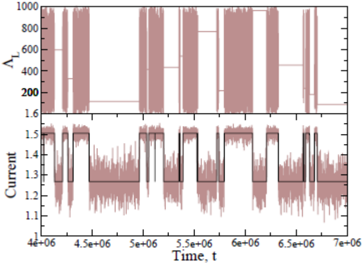
Examples: Monotone chipping processes and condensation

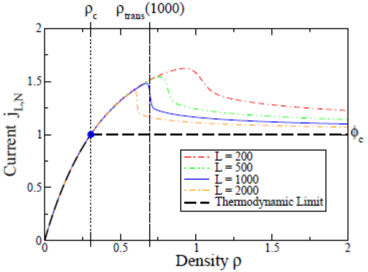
- Chipping process with $L = 2$.
- $R_L^{bg}(N) := \frac{1}{L-1}\pi_{L,N}(N - M_L)$.
- $\rho_c(w) \sim \sqrt{w}$ [Rajesh and Majumdar, 2001].



Implications of non-monotonicity

- Non-monotonicity of the canonical current and metastability in a condensing ZRP [Chleboun and Grosskinsky, 2010].
- The canonical current defined as $\pi_{L,N}\left(u(\eta_x)\right) = \frac{Z_{L,N-1}}{Z_{L,N}}$.
- ZRP with jump rates $u(k) = 1 + \frac{b}{k^\gamma}$.





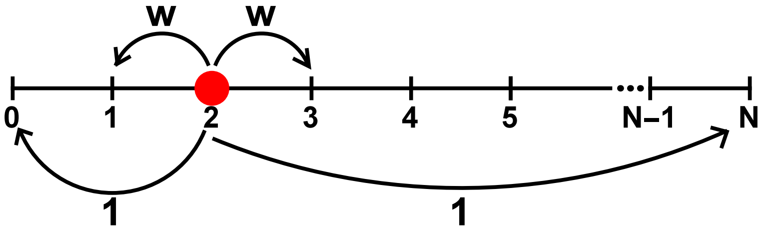
- (Left top) Position of the maximum. (Left bottom) Metastability of the canonical current.
- (Right) Numerics of the canonical current exhibiting non-monotone behaviour.

Conclusions

- Non-monotonicity linked with metastability of processes.
- Strong hydrodynamic limits for monotone Misanthrope processes [Gobron and Saada, 2010].
- Couplings are a powerful tool for studying relaxation times of processes [Nagahata, 2010].
- Condensation is equivalent to the stationary weights being sub-exponential.
- Extended known results on condensation in finite systems [Ferrari et al., 2007].
- Condensing stochastic particle systems with SPM and finite critical density are always **non-monotone**.
- For infinite critical density processes are non-monotone if stationary weights are power laws $w(n) \sim n^{-b}$ with $b \in (3/2, 2]$.
- Possible monotone example for $b \in (1, 3/2]$.

Critical density in the Chipping Process

- Consider the Chipping Process on two sites ($L = 2$) with N particles.
- $\rho_c(w) \sim \sqrt{w}$.



- Process is a random walk with resetting.
- After resetting processes diffuses.
- Processes reaches a typical distance of $\sqrt{w}$ from either boundary.

References

Andjel, E. D. (1982).
Invariant Measures for the Zero Range Process.
Ann. Probab., 10(3):525–547.

Chleboun, P. and Grosskinsky, S. (2010).
Finite Size Effects and Metastability in Zero-Range Condensation.
J. Stat. Phys., 140(5):846–872.

Chleboun, P. and Grosskinsky, S. (2013).
Condensation in Stochastic Particle Systems with Stationary Product Measures.
J. Stat. Phys., 154(1-2):432–465.

Cocozza-Thivent, C. (1985).
Processus des misanthropes.
Z. Wahrscheinlichkeitstheorie, 70(4):509–523.

Evans, M. R. (2000).
Phase transitions in one-dimensional nonequilibrium systems.
Braz. J. Phys., 30(1):42–57.