COUPLING PARTICLE SYSTEMS AND GROWTH PROCESSES

THOMAS RAFFERTY
PAUL CHLEBOUN & STEFAN GROSSKINSKY
WWW.WARWICK.AC.UK\TRAFFERTY

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PROBLEM

We study attractive particle systems with stationary product measures. We utilize the property of attractivity and its link to coupling to build a growth process that samples from the stationary measure of the zero-range process, on fixed and finite lattices, with computation times scaling linearly with the number of particles N.

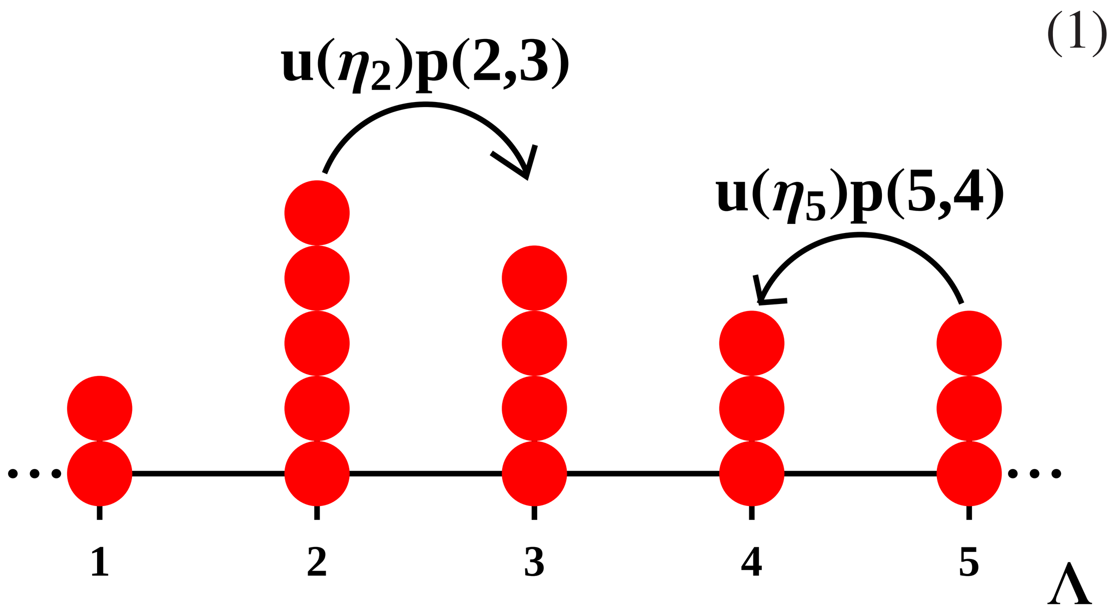
THE ZERO-RANGE PROCESS

The zero-range process, $(\eta(t) : t \geq 0)$, is defined as follows, [1];

- State-space $S = \mathbb{N}^{\Lambda_L}$ where $\Lambda_L = \{1, 2, \ldots, L\}$.
- $p(x, y)$; irreducible random-walk on Λ_L where $p(x, x) = 0$ for all $x \in \Lambda_L$.
- The jump-rate $u_x : \mathbb{N} \to \mathbb{R}_+$.

The generator is given by

$$\mathcal{L}f(\eta) = \sum_{x,z\in\Lambda} u_x(\eta_x)p(x,z)(f(\eta^{x\to z}) - f(\eta)). \quad (1)$$



STATIONARY MEASURE

Definition

- $\mu(\mathcal{L}(f)) = \sum_{\eta\in S} \mu(\eta)\mathcal{L}f(\eta) = 0$ for all observables f.

The grand-canonical ensemble [2]

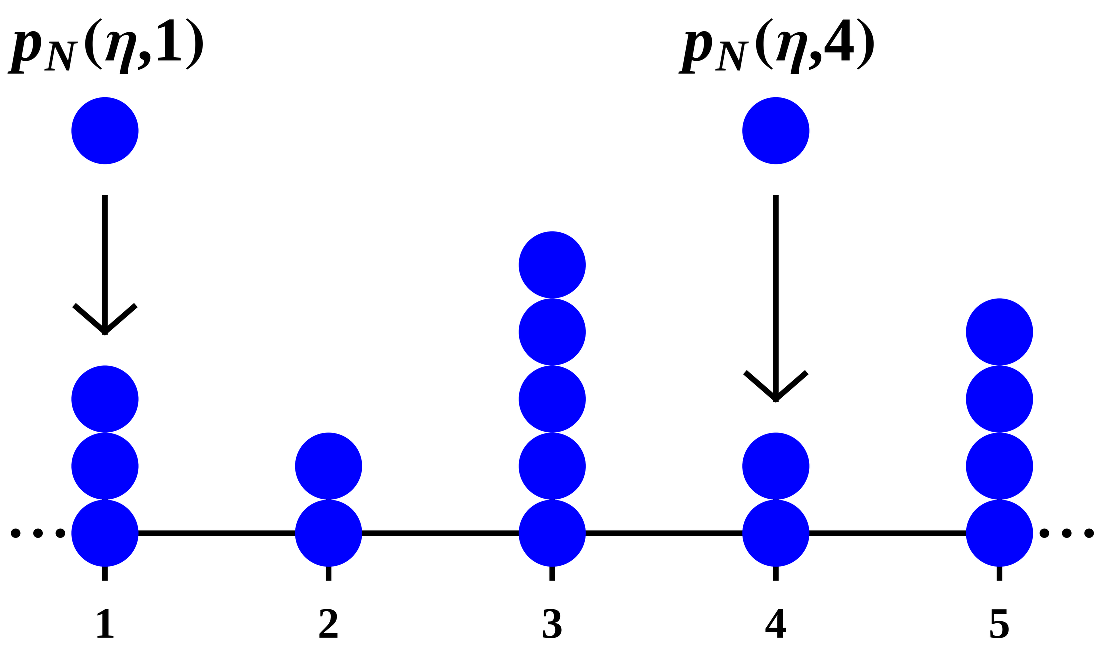
Fugacity	$\phi > 0$
Product measure	$\nu_\phi^{\Lambda_L}[\eta] = \prod_{x\in\Lambda_L} \nu_\phi^x[\eta_x]$
Site marginals	$\nu_\phi^x[n] = w_x(n)\phi^n(z_x(\phi))^{-1}$
Stationary weights	$w_x(n) = \prod_{k=1}^n (u_x(k))^{-1}$
Partition function	$z_x(\phi) = \sum_{n=0}^\infty w_x(n)\phi^n$
Density function	$\rho_x(\phi) = \sum_{n=1}^\infty n\nu_\phi^x[n]$
Critical fugacity	$\phi_x^c = \lim_{n\to\infty} \lvert\frac{w_x(n)}{w_x(n+1)}\rvert$
Critical density	$\rho_x^c(\phi_x^c) \in [0, \infty]$

The canonical ensemble [2]

State space	$X_{L,N} = \{\eta \in S : \sum \eta = N\}$
Stationary measure	$\pi_{L,N}[\eta] = \nu_\phi^{\Lambda_L}[\eta\lvert\sum\eta = N]$
Product measure	$\pi_{L,N}[\eta] = \frac{\prod_{x\in\Lambda_L} w_x(\eta_x)}{Z_{L,N}}$
Partition function	$Z_{L,N} = \sum\prod_{x\in\Lambda_L} w_x(\eta_x)$
Site marginals	$\pi_{L,N}[\eta_1 = n] = \frac{w(n)Z_{L-1,N-n}}{Z_{L,N}}$

Email: T.Rafferty@warwick.ac.uk

THE GROWTH PROCESS



Definition

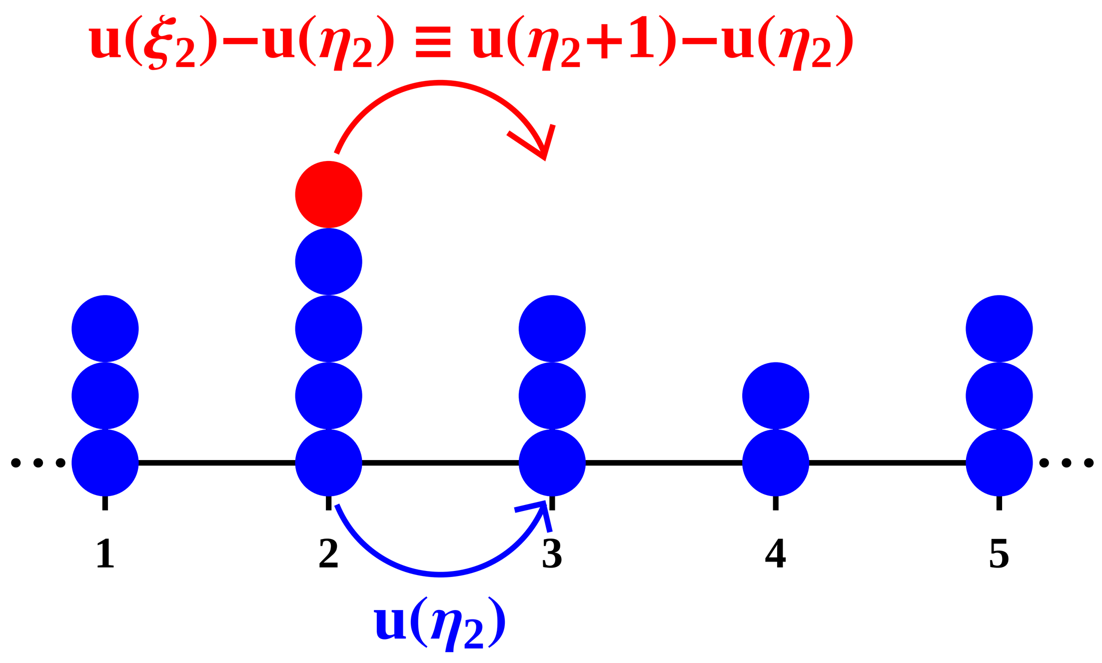
1. Zero-Range process on $X_{L,N}$.
2. $\eta \in X_{L,N}$ generated by $\pi_{L,N}$.
3. Sample from $\pi_{L,N+1}$ by adding a particle according to $p_N(\eta, x)$.

Therefore, for all $\xi \in X_{L,N+1}$ we need

$$\pi_{L,N+1}(\xi) = \sum_x \pi_{L,N}(\xi - \delta_x)p_N(\xi - \delta_x, x).$$

COUPLING: ZRP

A coupling of two probability distributions μ and ν is a pair of random variables (X, Y) defined on a single probability space such that the marginal of X is μ and the marginal of Y is ν. This definition may be extended to a coupling of a stochastic process.



$$\left.\begin{array}{l}\xi_y = n+1\\ \eta_y = n\end{array}\right\} \xrightarrow{u(\xi_y)-u(\eta_y)} \left\{\begin{array}{l}\xi_y = n\\ \eta_y = n\end{array}\right.$$

$$\left.\begin{array}{l}\xi_y = n+1\\ \eta_y = n\end{array}\right\} \xrightarrow{u(\eta_y)} \left\{\begin{array}{l}\xi_y = n\\ \eta_y = n-1\end{array}\right.$$

The stationary measure of coupled dynamics can be written as $\mu(\eta, y) = \alpha_\eta(y)\pi_{L,N}[\eta]$. $\alpha_\eta(y)$ is a valid growth rule according to $p_N(\eta, y)$.

RESULTS

Solving for the stationary measure of the coupled dynamics, we find $\alpha_\eta(y)$ satisfies the following equation;

$$\begin{aligned}&+\alpha_\eta(y)\left[u(\eta_y + 1) - u(\eta_y)\right]\\ &-\alpha_\eta(y-1)\left[u(\eta_{y-1} + 1) - u(\eta_{y-1})\right]\\ &= \sum_x u(\eta_x)\left[\alpha_{\eta^{x\to x-1}}(y) - \alpha_\eta(y)\right].\end{aligned}$$

Jump Rate	$\alpha_\eta(y) \propto$
$u(k) = 1$	$\eta_y + 1$
$u(k) = k$	1
$u(k) = k^2$ & $L = 2$	$3N + 1 - 2\eta_y$

References

[1] F Spitzer. Interaction of Markov processes. *Adv. Math.*, 5:246-290, 1970.

[2] E D Andjel. Invariant Measures for the Zero Range Process. *Ann. Probab.*, 10(3):525-547, August 1982.

[3] A G Angel, M R Evans, and D Mukamel. Condensation transitions in a one-dimensional zero-range process with a single defect site. *Journal of Statistical Mechanics: Theory and Experiment*, 2004(04):P04001, April 2004.

RESULTS

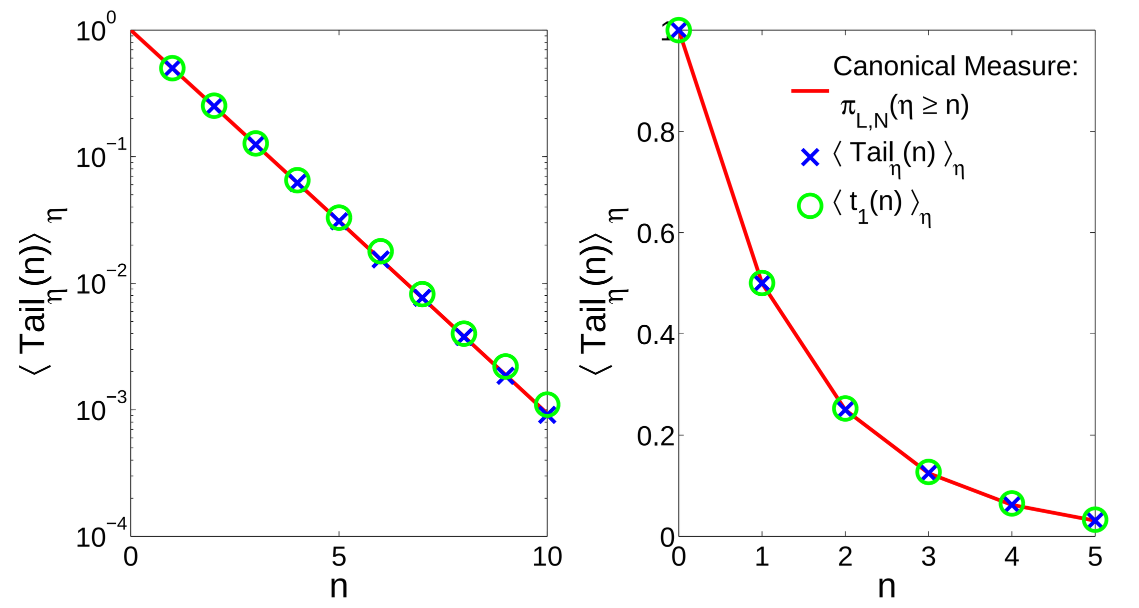


Figure. Comparing the growth process with the single-site marginal of the canonical stationary measure (red line) of the zero-range process for system size L=513 N=512 for a jump rate of the form $u_x(k)$=1 for all $x\in\Lambda_L$. $Tail_\eta(n)=\frac{\#\{i\in\Lambda_L|\eta_i\geq n\}}{L}$ and $t_x(n)=\mathbb{I}_{\eta_x\geq n}$

CONDENSATION & GROWTH

Condensation can occurs in the ZRP when the particle density exceeds a critical value and the system phase separates into a condensed and a fluid phase. For example constant rates with a single site defect, [3].

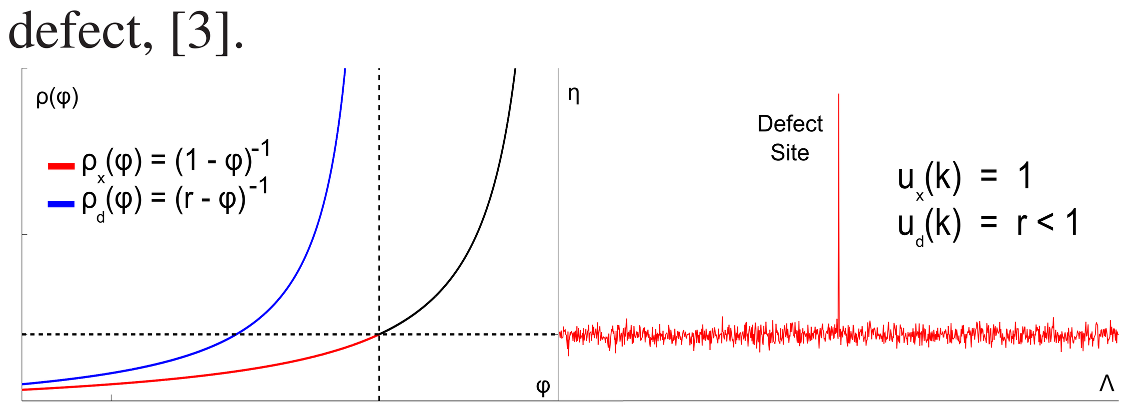
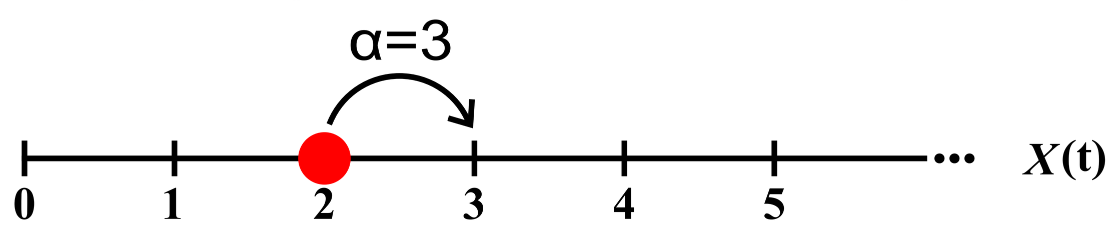


Figure. (Left) The density as a function of ϕ for a ZRP with a defect site. The defect site density diverges whilst non-defect sites have finite density. (Right) Example configuration in the condensed regime.

We use L independent birth-chains to grow stationary configurations for this condensing ZRP.



The birth-rates for the growth process are defined as follows.

$$\begin{aligned}\alpha_i^d(t) &= (i+1)h(t),\\ \alpha_i^x(t) &= (i+1) \quad \text{for all} \quad x \neq d.\end{aligned}$$

1. Solve the master equation of the birth chains to find $\mathbb{P}(X^i(t) = n)$.
2. Compare $\mathbb{P}(X^i(t) = n)$ and $\nu_\phi^i[n]$.
3. Calculate intensity function $H(t)$ by comparing time, t, and fugacity, ϕ.

$$H(t) = \int_0^t h(s)\,ds = -log\big(1 - r^{-1}(1 - e^{-t})\big).$$

The intensity function exhibits finite-time blow up. This implies the rate of adding particles to the defect sites diverges.

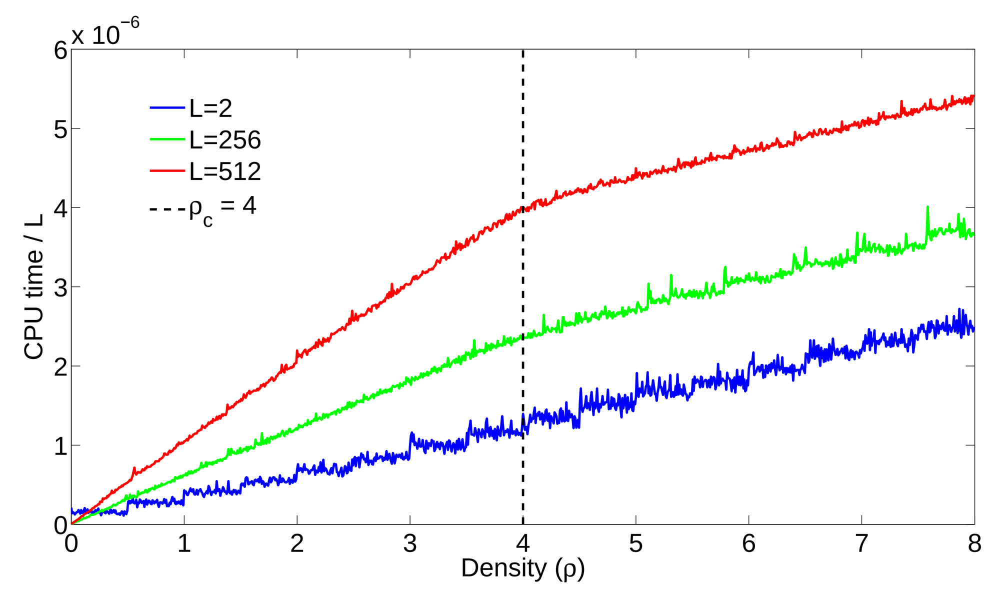


Figure. CPU time for the pure birth processes. CPU time scales linearly with density. However, the speed is slower below the critical density due to the binary search algorithm being implemented more often.