Pricing European Barrier Options with Partial Differential Equations

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Abstract

Barrier options were first priced by Merton in 1973 using partial differential equation. In this work, we present a closed form formula for pricing European barrier option with a moving barrier that increases with time to expiration. We adopted a three-step approach which include; justifying that barrier options satisfy the Black-Scholes partial differential equation under certain conditions, partial differential equation transformation, and solution using Fourier Transform and method of images. We concluded that all barrier options satisfy the Black-Scholes partial differential equation under different domains, expiry conditions, and boundary conditions. And also that closed form solution for several versions of barrier option exists within the Black-Scholes framework and can be found using this approach.

1 Introduction and Literature Review

Barrier options are path dependent option with price barriers. They have been traded over the counter market since 1967 [2] and [3]. There are several ways in which barrier options differ from standard options. One is that, barrier option pay-offs match beliefs about the future behaviour of the market. Another is that it matches hedging needs more closely than standard options and also its premiums are generally lower than that of a standard option [4]. There are two main approach of pricing barrier options. They are the probability method, and the partial differential equation (pde) method. The probability method involves multiple use of reflection principle and the Girsanov theorem to estimate the barrier densities [10]. These densities are then integrated over the discounted payoff in the risk neutral framework. Rubinstein and Reiner [10] gave a list of pricing formulas for different versions of barrier options using the probability method. Rich [3] used this approach to price different versions of barrier options for both rebates and zero rebates features. Rebate are positive discount that options holders recieve if the barrier is (never) breached for an (in) out option. Other works on barrier options include: Gao et. al. [1], and Griesbsch [17]. They examined option contracts with both knock-out barrier and American exercise features, and barrrier option pricing uder the Heston model with Fourier transform respectively.

The pde method is based on the idea that all barrier options satisfy the Black-Scholes partial differential equation but with different domains, expiry conditions and boundary conditions. [13]. Merton [9], was the first to price barrier options using pde. He used the pde method to obtained the theoretical price of a down-and-out call option. This method basically involves the transformation of the Black-Scholes pde to heat equation over a semi infinite boundary. The equation is then solve using method of images [14] and [13].

This work is organized as follows. The general technique employed to value barrier options will be to prove that barrier options satisfy the Black-Scholes pde. Next, transform the Black-Scholes pde to heat equation by changing variables, and then solving the pde to obtain the price formula for barrier option. Thus, section 2 presents the introduction to options, barrier options and stochastic calculus. Section 3 presents the Black-Scholes model, its pde and pricing formula. Section 4 presents a brief introduction to the heat equation and the method of solution, in particular the method of images and the Fourier Transform. In Section 4, we show how barrier options satisfy the Black-Scholes pde. The valuation is presented in section 6. Section 7 is the conclusion.

2 General Background

2.1 Options

Derivative is a product obtained from one or more basic variables. A financial derivative is an instrument whose value is determined by an underlying asset. There exist various types of financial derivatives, the most common being options, futures and swaps.

A call option gives the holder the right but not the obligation to buy a particular number of the underlying assets at some future time for a pre-agreed price called strike price, on or before a given date, known as maturity time or expiration date. However, the writer of the call is obliged to sell the asset to the buyer if the latter decides to exercise the option. If the option to buy in the definition of a call is replaced by the option to sell, then the option is called a put option. In exchange for the option the buyer must pay a premium to the seller [15].

In general, options are either of European or American type. A European option can only be exercised at the maturity date of the option, otherwise the option simply expires. An American option is more flexible: it can be exercised at any time up to and including the maturity date. All the options treated in this discourse are European.

Let S be the price of the underlying asset and E its strike price where both S and E are non-negative real numbers. Then the pay-off of a vanilla(standard) call option is given as

$$\text{pay-off call option} = (S_T - E)^+$$

provided that $S_T > E$. Where S_T is the terminal value of the underlying asset.

The pay-off of a vanilla put option is given as:

$$\text{pay-off put option} = (E - S_T)^+$$

Remark. *The pay-off of a vanilla option depends only on the terminal value of the underlying asset.*

2.2 Barrier Options

Barrier options are path dependent options with price barriers. They are path dependent because their pay-off depends on the whole price process of the underlying asset. This is the major difference between barrier options and standard options. Another significant difference is the possibility of a rebate. Rebates are positive discounts that barrier option holders receive if the barrier is breached for a knock-out feature or if the barrier is never breached for a knock-in feature. Generally for barrier options, a **knock-in** feature activates the option only if the underlying asset price first hits the barrier while a **knock-out** feature deactivates the option immediately the underlying asset price hits the barrier. There are eight different types of barrier options which are:

- **down-and-out** call and put option
- **up-and-out** call and put option
- **down-and-in** call and put option
- **up-and-in** call and put option

The pay-off of a barrier option, for example a **down-and-out** call option is given as

$$\text{pay-off} = \begin{cases} S_T - E & \text{if} \quad S_T > B \quad \forall\, t \in [0, T) \\ 0 & \text{if} \quad S_T \leq B \quad \text{for at least one } t \leq T \end{cases}$$

The pay-off for other versions of barrier options are similar to the above.

The In-Out Parity

The in-out parity for European barrier option explains the relationship between an in-out option and a plain vanilla option. Generally,

$$\text{Plain vanilla option} = \text{out-option} + \text{in-option}$$

This relationship is useful in pricing barrier options.

2.3 Geometric Brownian Motion and It$\hat{o}$ Calculus

In mathematical finance, prices of stock are modelled as Geometric Brownian Motion. A stochastic process $S(t)$ is said to follow a Geometric Brownian Motion if it satisfies the following stochastic differential equation:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \tag{2.1}$$

where, $W(t)$ is a Wiener process and μ, σ are percentage drift and percentage volatility respectively. The analytical solution to (2.1) given as $S(t) = S(0)\exp((\mu - \sigma^2/2)t + \sigma W(t)$ is the model for stock prices. It explains the dynamics of stock prices in time evolution.

It$\hat{o}$'s formula. Let $f(W(t)) : \mathbb{R} \to \mathbb{R}_+$ be a twice differentiable function and $W(t)$ a stochastic process. Then by applying It$\hat{o}$'s formula we obtained the differential of $f(W(t))$ as given below:

$$df(W(t)) = f'(W(t))dW(t) + 1/2f''(W(t))dt$$

Generally, It$\hat{o}$'s formula is an identity used in stochastic calculus to find the differential of a stochastic process. It gives the martingale and finite variation part of the stochastic process $f(W(t))$ which infact is a semi-martingale.

3 The Black-Scholes Model

The Black and Scholes model was first published in 1973 by Fischer Black and Myrion Scholes in their seminal paper on pricing of options and corporate liabilities [6]. It is a model for pricing European options. From the model we have the Black-Scholes partial differential equation (3.1) which when solved gives the theoretical price of European call options as given by (3.2) .

Unlike some other works on valuations of options, the Black-Scholes formula expresses the valuation of options in terms of price of stock. The model is based on the assumptions that stock price follows a geometric Brownian motion, and that the distribution of any stock prices over finite interval is log-normal.

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} - \frac{\partial f}{\partial t} - rf = 0 \tag{3.1}$$

$$f(S,t) = S\ N(d_1) - Ke^{-rT}\ N(d_2) \tag{3.2}$$

where

S = stock price
K = strike price
r = risk free rate
T = time to maturity
σ = volatility of the stock
$N(.)$ = cumulative distribution function

$$d_1 = [\ln[S/K] + (r + \sigma^2/2)T]/\sigma\sqrt{T}$$
$$d_2 = [\ln[S/K] + (r - \sigma^2/2)T]/\sigma\sqrt{T}$$

4 The Mathematics of Heat Equations

4.1 Heat Equation and Semi-infinite Boundary

The problem of valuing simple options is related to heat flow in a semi-infinite bar whose end (say $x = 0$) is held at zero temperature [14]. This section gives a brief introduction to heat equation on a semi-infinite boundary and its solution.

Consider the heat equation on an infinite rod $\quad -\infty < x < \infty,\ \ t > 0$

$$k\frac{\partial^2 H}{\partial x^2} - \frac{\partial H}{\partial t} = 0 \tag{4.1}$$

subject to

$$H(x,0) = f(x) \tag{4.2}$$

The Fourier Transform solution to the above is given as

$$H(x,t) = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} H_0(s)\exp\left(-\frac{(x-s)^2}{2t}\right) ds \quad t > 0, \quad x \in \mathbb{R} \tag{4.3}$$

If we include an additional condition say $H(0,t) = 0$. Then the problem becomes

$$k\frac{\partial^2 H}{\partial x^2} - \frac{\partial H}{\partial t} = 0 \tag{4.4}$$

subject to

$$H(x,0) = f(x) \qquad\qquad x \in \mathbb{R} \tag{4.5}$$

and

$$H(0,t) = 0, \qquad\qquad t > 0 \tag{4.6}$$

In other to account for the new boundary condition, we used the method of images. In the method of images, a semi-infinite problem is solved by first solving two infinite problems with equal and opposite initial temperature distributions so as to have a net effect of zero temperature at the joint. To apply this method to $(4.4)-(4.6)$, we need to reflect the initial condition $H(x,0)$ about the point $x = 0$ while changing its sign. This guarantees that $x = 0$. Since equation (4.4) is invariant under reflection, if $H(x,t)$ is a solution so is $H(-x,t)$. Thus if 4.3 gives the solution to the problem (4.1) – (4.2). Then

$$H(x,t) = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} \bar{H}_0(s)\text{exp}\left(-\frac{(x-s)^2}{2t}\right) ds \tag{4.7}$$

gives the solution to (4.4) – (4.6) where $\bar{H}_0(s)$ is the odd extension of $H_0(s)$, that is,

$$\bar{H}_0(s) = \begin{cases} H_0(s) & s > 0 \\ -H_0(-s) & s < 0 \end{cases}$$

This result will be used in section (6) to value barrier options. For further Reading see [14].

4.2 The Fourier Transform and The Convolution Theorem

The Fourier Transform $F[.]$ of a function $H(x,t)$ is defined as

$$F[H] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} H(x,t)e^{-2\pi ixf}dx \tag{4.8}$$

For convenience we sometimes use $\zeta(f,t)$ instead of $F[H]$ in our notations.

In particular, the Fourier Transform of the heat equation (4.1) subject to the condition (4.2) is given as

$$\begin{aligned}
F[H_2] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial H}{\partial t}(x,t)e^{-2\pi ixf}dx \\
&= \frac{1}{\sqrt{2\pi}}\left[e^{-2\pi ixf}H(x,t)\right]_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} H(x,t)(2\pi if)e^{-2\pi ixf}dx \\
&= -2\pi if\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} H(x,t)(2\pi if)e^{-2\pi ixf}dx \\
&= -2\pi ifT(f,t)
\end{aligned} \tag{4.9}$$

and

$$F[H_{11}] = k(2\pi i f)^2 \zeta(f,t) \tag{4.10}$$

Equating equations (4.9) and (5.9) gives

$$\frac{\partial \zeta(f,t)}{\partial t} = -4k\pi^2 f^2 \zeta(f,t) \tag{4.11}$$

which is a first order ordinary differential equation with general solution

$$\zeta(f,t) = \zeta(f,0)e^{-2k\pi^2 f^2 t} \tag{4.12}$$

where

$$\zeta(f,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H_0(s) e^{-2\pi i x f} dx$$

Lastly, $H(x,t)$ can be obtained by taking the inverse Fourier Transform of (4.12).

$$\begin{aligned} H(x,t) &= F^{-1}\left[H_0(f)e^{-2k\pi^2 f^2 t}\right] \\ &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} H_0(s) \text{exp}\left(-\frac{(x-s)^2}{2t}\right) ds \end{aligned}$$

where we have used the definition of convolution theorem which we will now state. Let $F(f)$ and $G(f)$ be the Fourier Transforms of $f(x)$ and $g(x)$ respectively and let $H(f) = F.G$. Then, the first and second integral in the inverse Fourier Transform of $H(f)$ as given below

$$h(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)g(x-s)ds = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)f(x-s)ds \tag{4.13}$$

are the convolution of $f(x)$ and $g(x)$ respectively. See [12] for further reading.

5 Barrier Options and the Black-Scholes

Starting from the Black-Scholes world where stock price is assumed to follow a geometric Brownian motion and at such satisfying the stochastic differential equation (2.1)

Theorem 5.1. *Let $f(S,t)$ be the value of an European " down- and-out" call option at time t under the assumption that their has been no knock out prior to t and that $S(t) = x$. Then $f(t,x)$ satisfies the Black-Scholes pde (3.1) in the domain $\{(t,x) : 0 \leq t < T, B \leq x < \infty\}$ for some barrier B*

subject to the boundary conditions,

$$f(B,\tau;E) = 0 \tag{5.1}$$

$$f(x,0;E) = \text{Max}[0, x-E] \qquad x > B \tag{5.2}$$

Proof. The boundary condition (5.1) follows from the fact that when the geometric Brownian $S(t)$ hits the barrier level B, it immediately rises and falls along B due to its non-zero quadratic variation. Shreve [16] highlighted the following steps to proof that barrier options satisfy the Black-Scholes pde:

- find the martingale part

- apply Itô's theorem
- set the finite variation term (dt) equal to zero

To find the martingale part, let F be a pay-off of an option. Since the stock price is a Markov process and the pay-off depends only on the stock price there must be a function $f(t,x)$ such that

$$F(t) = f(t, S(t)) \tag{5.3}$$

$F(t)$ is the value of the option without any assumption and $f(t)$ is the value of the option under the assumption that it has not knock-out prior to t. If $S(t)$ rises above the barrier and then returns below it by time t, $F(t)$ will be zero but $f(t)$ will be strictly positive since it is based on the assumption that there has been no knock-out prior to t. This discrepancy, is eliminated by finding the first time(stoppage time) at which the asset price reaches the barrier. Let ρ be the stoppage time such that $S(t) > B$ for $0 \leq t \leq \rho$ and $S(\rho) = B$ by optional stopping theorem, a martingale stopped at a stopping time is still a martingale and the theorem also holds for continuous time. Hence, the process

$$e^{-rt} f(t, S(t)) \tag{5.4}$$

is a $\mathbb{P}$ martingale up to ρ where e^{-rt} is a discount factor. The differential of (5.4) follows immediately by using Itô's formula, as given below

$$\begin{aligned} d(e^{-rt} f(t, S_t)) = e^{-rt}[&-rf(t, S_t) + f_t(t, S_t) + rS_t f_x(t, S_t) \\ &+\frac{1}{2}\sigma^2 S_t^2 f_{xx}(t, S_t)]dt + e^{-rt}\sigma S_t f_x(t, S_t) dW_t. \end{aligned} \tag{5.5}$$

Lastly, the finite variation term dt must be zero for $0 \leq t \leq \rho$ so that

$$rf(t, S_t) = f_t(t, S_t) + rf_t c_x(t, S_t) + S_t f_x(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 f_{xx}(t, S_t) \qquad \forall\ t \in [0, T) \tag{5.6}$$

Since $(t, S(t))$ can reach any point in $\{(t, x) : 0 \leq t < T, x \geq B\}$ before the option knocks out, (4.10) must hold for every $t \in [0, T)$ and $x \in [B, \infty)$. See [16] for further reading. $\square$

6 Barrier Option Valuation

This section presents the valuation formulas for barrier options. It has been shown in the previous section that barrier options satisfy the Black-Scholes pde under certain conditions. All we need to do to obtain our valuation formula is to solve the pde.

6.1 Notation and Results

The following results and additional notations will be used in this section.

$B[\tau] = bE\exp[-\eta\tau]$,
$x = \ln[S/B(\tau)]$,
$x' = \ln[B(\tau)/E] - \ln[S/E]$
$T = \sigma^2\tau$,
$H = \exp[ax + \gamma\tau] f(S, \tau; E)/E$,
$a = [r - \eta - \sigma^2/2]/\sigma^2$,
$\gamma = r + a^2\sigma^2/2$,

$\delta = 2(r - \eta)/\sigma^2$,
$s = x + z\sqrt{2T}$,
$z := -[\ln b + x]/\sqrt{2T}$,
$x' = \ln[B(\tau)/E] - \ln[S/E]$,
$h_1 = -\left[\ln(S/E) + (r + \sigma^2)\tau\right]/\sqrt{2\sigma^2\tau}$,
$h_2 = -\left[\ln(S/E) + (r - \sigma^2)\tau\right]/\sqrt{2\sigma^2\tau}$,
$h_3 = -\left[2\ln(B[\tau]/E) - \ln(S/E) + (r + \sigma^2)\tau\right]/\sqrt{2\sigma^2\tau}$,
$h_4 = -\left[2\ln(B[\tau]/E) - \ln(S/E) + (r - \sigma^2)\tau\right]/\sqrt{2\sigma^2\tau}$,

where S is the price of the underlying, E is the strike price, σ its volatility and r its risk-free rate. Also, $\eta \geqq 0$ and $0 \leqq b \leqq 1$ and T is the time to maturity.

6.2 Down-and-Out Option with Zero Rebate and a Moving Barrier

The problem of a down-and-out call option with a zero rebate and a moving barrier as posed by Merton [9] is similar to theorem (5.1) which has been proven to satisfy the Black-Scholes pde. Here, we shall proceed with the valuation.

As a first step, the pde (6.1) is reduced to the heat equation by changing of variables using: $x = \ln[S/B(\tau)]$ and $T = \sigma^2\tau$ we have that

$$\frac{\partial f}{\partial S} = \frac{1}{S}\frac{\partial f}{\partial x},$$

$$\frac{\partial^2 f}{\partial S^2} = \frac{1}{S^2}\frac{\partial^2 f}{\partial x^2} - \frac{1}{S^2}\frac{\partial f}{\partial x}$$

and

$$\frac{\partial f}{\partial \tau} = \sigma^2\frac{\partial f}{\partial T} + \eta\frac{\partial f}{\partial x}$$

. Substituting back into (6.1) gives

$$\frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2} + [r - \eta - 1/2\sigma^2]\frac{\partial f}{\partial x} - rf - \sigma^2\frac{\partial f}{\partial T} = 0 \tag{6.1}$$

and lastly, the following change of variables
$f(S, \tau; E) = H(x, T)e^{-ax-\gamma\tau}E$
$\dfrac{\partial f}{\partial x} = -aEe^{-ax-\gamma T/\sigma^2}H + Ee^{-ax-\gamma T/\sigma^2}\dfrac{\partial H}{\partial x}$
$\dfrac{\partial^2 f}{\partial x^2} = Ea^2e^{-ax-\gamma T/\sigma^2}H - 2aEe^{-ax-\gamma T/\sigma^2}\dfrac{\partial H}{\partial x} + Ee^{-ax-\gamma T/\sigma^2}\dfrac{\partial^2 H}{\partial x^2}$
$\dfrac{\partial f}{\partial T} = -\gamma/\sigma^2 Ee^{-ax-\gamma T/\sigma^2}H + Ee^{-ax-\gamma T/\sigma^2}\dfrac{\partial H}{\partial T}$

give $a = [r - \eta - \sigma^2/2]/\sigma^2$, $\gamma = r + a^2\sigma^2/2$ and the heat equation (4.4) for $k = 1/2$ with boundary conditions $H(0, T) = 0$ and $H(x, 0) = e^{ax}\mathrm{Max}[0, be^x - 1]$ on the interval $B(\tau) \leq S < \infty$.

This is the standard heat equation on a semi-infinite boundary discussed in section (4). Taking $f(x) = e^{ax}\mathrm{Max}[0, be^x - 1]$, the solution is given as

$$H(x, T) = \frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{\infty}\bar{H}_0(s)\exp\left(-\frac{(x - s)^2}{2T}\right)\mathrm{d}s \tag{6.2}$$

with

$$\bar{H}_0(s) = \begin{cases} e^{as}\text{Max}[0, be^s - 1] & s > 0 \\ -e^{-as}\text{Max}[0, be^{-s} - 1] & s < 0 \end{cases}$$

Let A and B represent the part $s > 0$ and $s < 0$ respectively. Then, $H(x,T) = A + B$, and after several mathematical steps which are given in appendix A we arrived at the following expressions for A and B:

$$A = [be^{(a+1)x+1/2(a+1)^2T} erfc(h_1) - e^{ax+1/2a^2T} erfc(h_2)]/2$$
$$B = [be^{(a+1)x'+1/2(a+1)^2T} erfc(h_3) - e^{ax'+1/2a^2T} erfc(h_4)]/2$$

Lastly we reverse the transformation by writing $f(S,\tau;E) = H(x,T)e^{-ax-\gamma\tau}$ and substituting for a, x, x' to obtain the price of a **down-and-out call** option with zero rebate and a moving barrier as given below:

$$\begin{aligned} f_{DO}(S,\tau;E) &= [S\ erfc(h_1) - Ee^{-r\tau} erfc(h_2)]/2 \\ &-(S/B[\tau])^{-\delta}[B[\tau]erfc(h_3) - (S/B[\tau])Ee^{-r\tau} erfc(h_4)]/2 \end{aligned} \tag{6.3}$$

or

$$f_{DO}(S,\tau;E) = f_p(S,\tau;E) - (S/B[\tau])^{1-\delta} f_p(B(\tau)^2/S,\tau;E) \tag{6.4}$$

where the first term of (6.4) is the price of a plain vanilla option and the negative term is a discount due to the barrier feature.

The down-and-in

The **down-and-in** call option corresponding to (6.4) follows immediately by using the **in-out parity** in section (2).

$$f_{DI}(S,\tau;E) = S/B[\tau])^{1-\delta} f(B(\tau)^2/S,\tau;E) \tag{6.5}$$

7 Conclusion and Future Work

7.1 Conclusion

All barrier option prices satisfy the B-S pde under different domains, expiry conditions and boundary conditions. The method of valuation discussed in this work can be applied to value other versions of barrier options.

7.2 Further Work

For future work we suggest the valuation of double barrier option. A double barrier option is a combination of two dependent knock-in or knock-out options. If one of the barriers are reached in a double knock-out option, the option is killed. If one of the barriers are reached in a double knock-in option, the option comes alive [8].

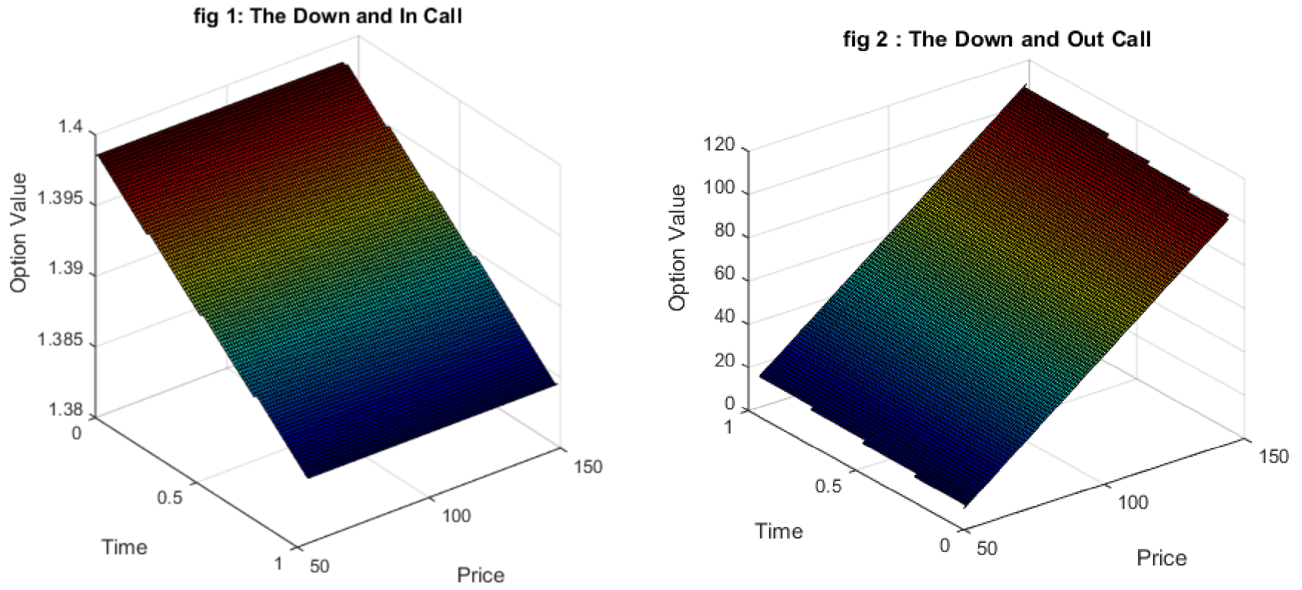

Figure 6.1: fig 1 and 2 show the value of **down-and-in** and **down-and-out** call option with strike $E = 40$, barrier level $B = 50$ and volatility 35%. The option matures at time $T = 1$. It is clear from the figure that whenever the strike price is placed above the barrier, the values of a **down-and-in** and **down-and-out** call option increase with respect to the price of the underlying asset.

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A

Appendix A: The down-and-out call option with zero rebate and a moving barrier

$$\frac{1}{2}H_{11} - H_2 = 0 \tag{A.1}$$

subject to

$$H(0,T) = 0 \tag{A.2}$$
$$H(x,0) = e^{ax}\text{Max}[0, be^x - 1] \tag{A.3}$$

Fourier Transform

$$\begin{aligned}
F\left[\frac{\partial^2 H}{\partial x^2}\right] &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{\partial^2 H}{\partial x^2}e^{-2\pi i f x}dx \\
&= \frac{1}{2}(2\pi i f)^2\int_{-\infty}^{\infty}H(x,t)e^{-2\pi i f x}dx \\
&= -2\pi^2 f^2\zeta(f,t) \\
F\left[\frac{\partial H}{\partial t}\right] &= \int_{-\infty}^{\infty}\frac{\partial H}{\partial t}e^{-2\pi i f x}dx \\
&= \frac{\partial}{\partial t}\int_{-\infty}^{\infty}H(x,t)e^{-2\pi i f x}dx \\
&= \frac{\partial}{\partial t}\zeta(f,t) \\
\frac{\partial\zeta(f,t)}{\partial t} &= -2\pi^2 f^2\zeta(f,t)
\end{aligned}$$

Integrating

$$\zeta(f,t) = \zeta(f,0)e^{-2\pi^2 f^2 T} = H_0(f)e^{-2\pi^2 f^2 T}$$

Applying inverse Fourier Transform and the convolution theorem

$$F^{-1}\left[e^{-2\pi^2 f^2 T}\right] = \frac{1}{\sqrt{2\pi T}}\exp\left(-\frac{x^2}{2T}\right)$$

$$\begin{aligned}
H(x,t) &= F^{-1}\left[H_0(f)e^{-2\pi^2 f^2 T}\right] \\
&= \frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{\infty}H_0(s)\exp\left(-\frac{(x-s)^2}{2T}\right)ds
\end{aligned}$$

Changing of variables

$$\frac{x-s}{\sqrt{2T}} = -z$$
$$s = x + z\sqrt{2T}$$
$$ds = dz\sqrt{2T}$$

$$H(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} H_0(x + z\sqrt{2T}) e^{-z^2} dz$$

$$H(x,t) = A + B$$

For A

$$\begin{aligned} H_0(s) &= e^{as}\text{Max}[0, be^s - 1] \\ &= e^{a(x+z\sqrt{2T})}\text{Max}[0, be^{x+z\sqrt{2T}} - 1] \end{aligned}$$

$$H_0(s) > 0$$

$$z > \frac{-\ln b - x}{\sqrt{2T}}$$

Set

$$Z := \frac{-\ln b - x}{\sqrt{2T}}$$

Then we have

$$\begin{aligned} A &= \frac{1}{\sqrt{\pi}} \int_Z^{\infty} be^{(a+1)(x+z\sqrt{2T})} e^{-z^2} dz - \frac{1}{\sqrt{\pi}} \int_Z^{\infty} e^{a(x+z\sqrt{2T}} e^{-z^2} dz \\ &= I_1 - I_2 \\ I_1 &= \frac{be^{(a+1)x}}{\sqrt{\pi}} \int_z^{\infty} e^{(a+1)z\sqrt{2T} - z^2} dz \\ I_1 &= \frac{be^{(a+1)x + 1/2(a+1)^2 T}}{\sqrt{\pi}} \int_z^{\infty} e^{-(z - 1/2(a+1)\sqrt{2T})^2} dz \end{aligned}$$

$$y = z - \frac{(a+1)}{2}\sqrt{2T} = \frac{\ln b - x}{\sqrt{2T}} - \frac{(a+1)}{2}\sqrt{2T}, \qquad dy = dz$$

$$\begin{aligned} I_1 &= \frac{1}{2} be^{(a+1)x + 1/2(a+1)^2 T} \frac{2}{\sqrt{\pi}} \int_{h_1}^{\infty} e^{-y^2} dy \\ I_1 &= \frac{1}{2} be^{(a+1)x + 1/2(a+1)^2 t} erfc(h_1) \end{aligned}$$

$$h_1 = -\left[\frac{\ln b + x}{\sqrt{2T}} + \frac{(a+1)}{2}\sqrt{2T}\right], \qquad erfc(t) = \frac{2}{\sqrt{\pi}} \int_t^{\infty} e^{-x^2} dx$$

Similarly,

$$\begin{aligned} I_2 &= \frac{1}{\sqrt{\pi}} \int_z^{\infty} e^{a(x+z\sqrt{2T}) - z^2} dz = \frac{1}{2} e^{ax + 1/2a^2 T} erfc(h_2) \\ h_2 &= -\left[\frac{\ln b + x}{\sqrt{2T}} + \frac{a}{2}\sqrt{2T}\right] \end{aligned}$$

$$A = [be^{(a+1)x + 1/2(a+1)^2 T} erfc(h_1) - e^{ax + 1/2a^2 T} erfc(h_2)]/2$$

B is obtained is a similar way by replacing x with x'. Note that $x' = -x$

$$B = I_3 - I_4$$

$$B = [be^{(a+1)x'+1/2(a+1)^2T}erfc(h_3) - e^{ax'+1/2a^2T}erfc(h_4)]/2$$

$$h_3 = -\left[\frac{\ln b + x'}{\sqrt{2T}} + \frac{(a+1)}{2}\sqrt{2T}\right], \qquad h_4 = -\left[\frac{\ln b + x'}{\sqrt{2T}} + \frac{a}{2}\sqrt{2T}\right]$$

Lastly,

$$\begin{aligned} f(S,\tau;E) &= EH(x,T)e^{-ax-\gamma\tau} \\ &= E[A+B]e^{-ax-\gamma\tau} \\ &= EAe^{-ax-\gamma\tau} + EBe^{-ax-\gamma\tau} \\ &= C + D \end{aligned} \tag{A.4}$$

$$C = 1/2Ebe^{(a+1)x+1/2(a+1)^2T}erfc(h_1) - 1/2Ee^{ax+1/2a^2T}erfc(h_2)$$

$$x = \ln[S/B[\tau]] \Rightarrow e^x = S/B[\tau], \quad B[\tau] = bEe^{-\eta\tau} \Rightarrow bE = B[\tau]e^{\eta\tau}$$
$$\gamma = r + a^2\sigma^2/2, \quad a = [r - \eta - \sigma^2/2]/2, \quad T = \sigma^2\tau$$

$$C = [S\ erfc(h_1) - Ee^{-r\tau}erfc(h_2)]/2$$

D is obtained in a similar way by using x' and $\delta = 2(r-\eta)/\sigma^2$

$$D = -(S/B[\tau])^{-\delta}[B[\tau]erfc(h_3) - (S/B[\tau])Ee^{-r\tau}erfc(h_4)]/2$$

putting C and D into $(A.4)$ completes the solution.

B

Appendix B: The down-and-out call option with zero rebate and a moving barrier

Here we provide a faster approach to arrive at (6.4). Let $f_p(S,\tau;E)$ be a plain vanilla call with same expiration time and strike price as our **down-and-out** call, and let $f_{DO}(x,\tau;E)$ be a **down-and-out** call. Also, let $H_p(x,T)$ be the corresponding solution to the heat equation (4.4). Since the pay-off of call option is zero for all S below the strike price we have that:

$$H_p(x,0)\forall x < \ln[E/B(\tau)]$$

. Also,

$$B(\tau) > E \Rightarrow \ln(E/B(\tau)) > 0$$

,and by extending the pay-off of our **down-and-out** call into $x < 0$ then we have that $H(x)$ is equal to $H_p(x)$ and we can write

$$H(x,0) = H_p(x) - H_p(-x) \quad \forall x.$$

Thus

$$H(x,T) = H_p(x,T) - H_p(-x,T),$$

since both side satisfies the heat equation and vanishes at $x = 0$.
Recall that $x = \ln[S/B(\tau)]$. This implies that $B(\tau)e^x = S$ and so,

$$f_{DO}(S,\tau;E) = f_{DO}(B(\tau)e^x,\tau;E) = Ee^{-ax-\gamma\tau}H_p(x,T)$$

shows that

$$H_p(x,T) = e^{ax+\gamma\tau}f_p(B(\tau)e^x,t(\tau);k)/E$$

and

$$H_p(-x,T) = e^{-ax+\gamma\tau}f_p(B(\tau)e^{-x},t(\tau);k)/E$$

Thus the value of the **down-and-out** call option is

$$\begin{aligned}
f_{DO}(S,\tau;E) &= Ee^{-ax-\gamma\tau}H(x,T) \\
&= Ee^{-ax-\gamma\tau}[H_p(x,T) - H_p(-x,T)] \\
&= Ee^{-ax-\gamma\tau}[e^{ax+\gamma\tau}f_p(B(\tau)e^x,t(\tau);k)/E - e^{-ax+\gamma\tau}f_p(B(\tau)e^{-x},t(\tau);k)/E] \\
&= f_p(B(\tau)e^x,\tau;E) - e^{-2ax}f_p(B(\tau)e^{-x},\tau;E) \\
&= f_p(S,\tau;E) - (S/B)^{1-\delta}f_p(B(\tau)^2/S,\tau;E), \qquad \text{(B.1)}
\end{aligned}$$

as expected. This method can be used to obtain other versions of barrier options without going through the process of transformation and integration. Detailed explanation can be found in [14] and [5].