Insurance models and risk-function premium principle

Aditya Challa
Supervisor : Prof. Vassili Kolokoltsov

July 2, 2012

Abstract

Insurance sector was developed based on the idea to protect people from random unforseen events. In this project we analyse the basic working of an insurance company and the various models used in this field. The insurance company is assumed to have a contract, called *insurance policy*, with the *policy holders*. The contract states that the insurance company is liable to pay for the losses due to certain random events. In return, the policy holder must pay a fixed amounts (premiums) in regular intervals of time. An insurance company is supposed to have such contracts with thousands of policy holders. Thus, from the point of view of the insurance company, we get a deterministic amount of money in each time interval, and *claims* at random times. Loosely, premiums are deterministic functions to compensate for the losses due to random events. Thus, premiums are most essential for the working of an insurance company. In this project we develop a new premium principle called *risk-function* premium principle which has the property that all other premium principles are special cases of this principle.

1 Introduction

Unforeseen circumstances could have a major impact on the society and life of the people in it, and how careful one might be there are a few things which man cannot control. It is thus important to think about such circumstances, and find out a way to deal with them. Now, the question arises as to how to deal with situations one might not know how and when they would arise. The theory of probability and statistics gives a reasonable tool for dealing with such situations. On the idea to protect people from the unforseen circumstances, the insurance sector was developed.

To ensure that the insurance company does not go bankrupt, we need to have a efficient model which is amenable to analysis. Insurance deals with situations which are highly complex. For example insurance against weather fluctuations, which the field of *Catastrophe modelling* deals with. Predicting weather is one of the questions which had made progress but still does not have an reasonable answer. Thus, when insurance company deals with this situation, it has huge amounts of data along with thousands of parameters. This is a typical example of a complex system in the field of insurance.

An insurance company would usually start with a *capital*, and start selling the policies. After a while it would start getting premiums for the policies. The company would save these premiums, or reinvest it for higher interests. Now, the company is liable to pay all those policy holders who would claim their losses from time to time. In this whole process, randomness could arise from:-

- The time at which the policy-holders would claim their losses.
- The size of the each claim.
- If the company would reinvest its gain from premiums in a risky asset, what is the chance that it would loose a portion of it?

There is also a small chance that the policy-holders would quit for some reasons, which can easily be ignored. Each of the above situations could be modelled in a number of ways, and the ideal model depends on the type of policy and the historic data.

Once the models are established, the company has to make few important decisions as, how much should be charged as premium?, where to invest the premiums?, what is the initial capital required to avoid bankruptcy?, etc. To answer such questions we need to have a criteria based on which we can answer such questions. For example, one of the criteria could be that the company chooses its premiums, investment strategies, and initial capital based on whether it would reach bankruptcy in finite time or not. That is, mathematically speaking, if τ is the time when the company would reach bankruptcy we need to calculate

$$\mathbb{P}\left(\tau < \infty | variables\right)$$

This is called the *Probability Of Ruin* and the decisions are made such that this probability is zero or as less as possible. Here, the variables stand for the decisions the company has to make. This is widely used in the industry. There could be other principles which are stronger or weaker, based on the company.

There are a lot of different kind of policies the insurance company offers. In theory, a pre-defined regular payments to the company and a huge payment in case an event occurs could be considered as a policy. It could also be that the company has to pay a series of payments after an event occurs, but all such payment could be combined to a single payment at time 0, using interest rates and depreciation. In general policies are divided into *life* and *non-life* insurance policies. In this project, we will be dealing majorly with the non-life insurance policies.

Apart from its obvious practical use, these models gives rise to rich mathematical and statistical theories, which in its own right is very interesting and challenging. Most of the results discussed here are widely used in the field of actuarial sciences. Thus we state only the most important results keeping in mind the practical application of the results. We refer the reader to the references for majority of the proofs. We start with discussing the basic models in insurance and few of the results. Most of these results are widely used and are borrowed from (Ramasubramanian, 2009) and (Rolski *et al.*, 2009). The new idea in this project is related to the *Premium Principles*. We propose a new premium principle of which all other premium principles are special cases.

2 Basic Models in Insurance

As mentioned earlier, there are sources of randomness in the process of the working of an insurance company, which needs to be modelled. In this section we look at few of the basic models for the *arrival times of the claims* and the *claim size* distributions.

2.1 Arrival Times of the Claims

We start with defining a stochastic process for the arrival times of the claims called *Claim Number Process*, $\{N(t) : t \geq 0\}$, where $N(t)$ denoted the number of claims before time t. The basic probability space is given by $\{\Omega, \mathcal{F}, P\}$. The immediate consequences of this being a model for arrival times are:-

A1: $N(0) = 0$ and $N(t) \geq 0, \forall t \geq 0$. This comes from the fact that we assume there are no claims at time $t = 0$, and since $N(t)$ denotes the number of claims, it must be a non-negative integer.

A2: If $0 \leq s < t$, then $N(s) \leq N(t)$, that is $N(t)$ is a non-decreasing process. This is because $N(t)$ denotes all the claims before time t and, all the claims before time s, are also before time t for $s < t$. Also, $N(t) - N(s)$ denotes the number of number of claims in $(s, t]$.

Note that $N(t)$ is a process which takes only integer values, and at some time a jump occurs. The jumps can once again only be positive integers. Corresponding to this claim number process we could define the n^{th} claim arrival time as

$$T_n = \inf\left\{t \geq 0 : N(t) = n\right\}$$

and time for the n^{th} claim is defined as

$$A_n = T_n - T_{n-1}$$

A_n are also called inter-arrival times. Note that the assumptions made above are the least possible assumptions, and we need to make further assumptions to get a quantifiable model. We now discuss two widely used models, and the extra assumptions they make.

2.1.1 Poisson Process

We begin with making few further assumptions:-

A3: The process $\{N(t) : t \geq 0\}$ has independent increments, that is, for all $0 \leq t_1 < t_2 < t_3$, $N(t_3) - N(t_2)$ is independent of $N(t_2) - N(t_1)$. This means that the claims arriving in two disjoint intervals are independent.

A4: Along with independent increments the process $\{N(t)\}$ also have stationary increments, that is for all $0 \leq s < t$ and $h > 0$, $N(t) - N(s)$ and $N(t + h) - N(s + h)$ have the same distribution.

A5: There could only be, at maximum, one claim arriving in a short interval of time. Mathematically stated this assumptions tells us

$$P(N(t+h) - N(t) \geq 2) = o(h) \quad \textit{as } h \downarrow 0$$

where $o(h)$ means that it reduces linearly as h. This is true for all t, because of the stationary increments assumption in $A4$.

A6: There exists $\lambda > 0$ such that

$$P(N(t+h) - N(t) = 1) = \lambda h + o(h) \quad \textit{as } h \downarrow 0$$

This number λ is called the claim arrival rate. Once again this is true for all t, because of the stationary increments assumption in $A4$.

Note that the assumption $A3$ is quite intuitive, since we think that claims arrive independently of other claims. This kind of thinking, however is negated to some extent in assumptions $A5$ and $A6$ where we assume that two claims cannot come at the same time. But, it is still a reasonable assumption in view of the company, since the chances of such a situation arising in real life are very small. For example, if we consider a situation where there is an accident or an earthquake, then it is possible that more than one claim could arrive at the same time. There are however other models to deal with such situations. Assumption $A4$ is also not completely realistic, but is a good first step to model insurance claims.

The assumptions $A5$ and $A6$ give information regarding how the process acts in small intervals, where we also used assumption $A4$. This along with the fact that $\{N(t)\}$ is a stocchastic process should, in theory, allow us to find the distribution of claims in a finite interval. This is infact true as is shown in the following theorem.

Theorem 2.1.1. *If there is a stochastic process $\{N(t) : t \geq 0\}$ which statisfies the assumptions $A1 - A6$, then for any $t, s \geq 0$ we have*

$$P(N(t+s) - N(t) = k) = P(N(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \tag{1}$$

Thus, $N(t)$ has the Poisson distribution with parameter λt.

See (Ramasubramanian, 2009) and (Rolski *et al.*, 2009) for the rigorous proof this theorem.

Definition 2.1.1. *A stochastic process $\{N(t) : t \geq 0\}$ which follows assumptions $A1 - A6$ and hence theorem 2.1.1 is applicable, is called time homogeneous Poisson process with a rate λ.*

An immediate extension is to allow for varying rates as a function of time, $\lambda(.)$. This would violate the assumption $A4$, and assumptions $A5$ and $A6$ have the r.h.s depending on time. This leads us to the following general definition

Definition 2.1.2. *Let $\lambda(.)$ be a non-negative measurable function, with finite integrals over all bounded sets in $[0, \infty)$. A stochastic process $\{N(t) : t \geq 0\}$ is said to be a Poisson Process with rate $\lambda(.)$ if*

- *Assumptions $A1 - A3$ hold*
- *The sample paths $N(., \omega)$ are continuous from the right.*
- *For $0 \leq s < t$, we have $N(t) - N(s)$ has a Poisson distribution with paramenter $\int_s^t \lambda(y)dy$.*

Note that the case $\lambda(.) =$ constant gives a homogeneous Poisson process. Also, the non-decreasing assumption $A2$ would give that the sample paths $N(., \omega)$ have left limits, thus giving us *cadlag* functions of time. The existence of such a process could be proved using Kolmogorov's consistency results and theory of weak convergence.

If $\lambda(.) = 1$ then the Poisson process is called a *standard homogeneous Poisson process.* It is the usual case that we prove results for the standard version and carry them onto the general Poisson processes. This is possible because of the following theorem

Theorem 2.1.2. *Let $\lambda(.)$ be a rate function of the Poisson process $N(t)$. Also define*

$$\mu(t) = \int_0^t \lambda(s)ds$$

Let $\tilde{N}(t)$ be the standard Poisson process. Then the following hold true *(i)* $\left\{\tilde{N}(\mu(t)) : t \geq 0\right\}$ *is a Poisson process with rate* $\lambda(.)$

(ii) *If* $\lambda(.) > 0$ *and*

$$\lim_{t\to\infty} \mu(t) = \infty$$

then, $\left\{N(\mu^{-}(t)) : t \geq 0\right\}$ *is the standard homogeneous Poisson process.*

Intuitively, the above theorem uses the re-normalisation of the time to jump between the standard and the general Poisson processes. $\mu(.)$ gives a different way to model the process. One could adjust this function to increase or decrease the number of claims arrived in a particular time interval, for example seasonal trends.

We have previously defined the inter-arrival times, A_n. In the model of Poisson process we can find the distribution of the inter-arrival times, as the next theorem demonstrates.

Theorem 2.1.3. *If* $\{N(t) : t \geq 0\}$ *denotes a time-homogeneous Poisson process with rate* $\lambda > 0$ *and* $\{T_i\}$ *denotes the claim-arrival times, then* $\{A_n\}$ *is a sequence if independent identically distributed random variables having exponential*(λ) *distribution.*

The proof of this theorem is quite standard and can be found in the references (Ramasubramanian, 2009) or (Rolski *et al.*, 2009).

The above theorem has a lot of implications. First of all, it gives us another way to define the Poisson process. We could construct the process by starting at time $t = 0$, and waiting for an exponentially distributed time before placing a claim at that time. Then we wait once again for an exponentially distributed time, independent of the previous time. This gives us a Poisson process according to the above theorem.

The above theorem also allows us to compute various quantities like the time till k^{th} claim which is nothing but sum of exponential distributions, and hence a gamma distribution. Also, the exponential distribution has a typical property called the *memoryless property* which gives a lot more information about the process. More importantly this way of describing the process using inter-arrival times allows us to extend the Poisson process by moving from the exponential waiting times to a general distributions. Such processes are very useful in the actuarial context and are called *Renewal Processes* which we would deal with in the next section.

The Poisson process model for arrival of claims is a very crude model for the arrival of claims. Practically, it has been found that the inter-arrival times do not always correspond to the exponential distributions. However, this provides a very good first step in understanding the claim arrival processes. Also, there are a number of ways one can define the Poisson process. See (Rolski *et al.*, 2009) for various ways to define a Poisson process.

2.1.2 Renewal Process

As mentioned earlier, we could extend the Poisson process by taking non-exponential inter-arrival times. This gives us the so called *Renewal Process.* We use the same notation $\{A_n\}$ for the inter-arrival times as before.

Definition 2.1.3. *Let* $\{A_n : n = 1, 2, \cdots\}$ *be the sequence of i.i.d non-negative random variables. Let* $T_0 = 0$, *and* $T_{n+1} = T_n + A_n$. *Thus the sequence* $\{T_i\}$ *would gives us times at which the* i^{th} *claim arrives. Then, we have that*

$$N(t) = max\,\{i \geq 0 : T_i \leq t\}$$

This process $\{N(t) : t \geq 0\}$ *is called the renewal process.*

We assume that

$$P(A_i = 0) < 1$$

that is, the random variables A_i are not degenerate at 0. This allows us to avoid trivial cases and also has the consequence that for any finite t

$$E(N(t)) < \infty$$

This result is quite intuitive. If we disregard the above assumption, we can add infinitely many A_i in a interval without changing its distribution. Note that this process still has independent increments, and even stationary increments if the inter-arrival distributions are the same.

We also define another process

$$N_0(t) = \sum_i \mathbb{I}_{[0,t]}(T_i) = N(t) + 1 \tag{2}$$

Here, we also counted the event happening at time $t = 0$. This has the advantage that $\{N_0(t)\}$ is a stopping time with respect to the natural filtration where as $\{N(t)\}$ is not.

Once we move away from exponential inter-arrival times, its tough to obtain results for finite times. Thus, many nice properties of the Poisson process are lost. We can however get some asymptotic results on the claim-number process. Also, the exact distribution of $N(.)$ is tough to find in a renewal process. We get a result similar to the central limit theorem, which gives us a nice approximation. Such asymptotic results may not help us to find the distributions for finite times, but are useful for calculating *ruin probabilities*.

Theorem 2.1.4 (Elementary Renewal Theorem)**.** *Assume* $P(A_i = 0) < 1$ *and* $E(A_i) = \dfrac{1}{\lambda} < \infty$*, then*

$$\lim_{t\to\infty} \frac{1}{t} N(t) = \lambda \tag{3}$$

This theorem suggests that the expected number of claims would linearly increase with time, for large times.

Theorem 2.1.5 (Central Limit Theorem)**.** *Assume that* $0 < Var(A_i) < \infty$*,* $E(A_i) = \dfrac{1}{\lambda}$ *and* $c = \lambda^3 Var(A_i)$*, then*

$$\lim_{t\to\infty} P\left(\frac{N(t) - \lambda t}{\sqrt{ct}} < x\right) = \Phi(x)$$

Where $\Phi(,)$ *is the distribution function of the standard normal. Also, the convergence is uniform over all* x*.*

The proofs of the above results are fairly standard and can be found in any of the standard textbooks. See (Ramasubramanian, 2009) and (Rolski *et al.*, 2009).

As mentioned before, it is tough to get the distributions for $N(t)$ for some time t. The previous results suggest that, it is easy to look at $E(N(t))$ instead, since this is amenable to manipulation. This gives the motivation to define a *Renewal Function.*

Definition 2.1.4. *The function*

$$H(t) = E(N_0(t)) = E(N(t)) + 1$$

is called the renewal function corresponding to the renewal process $N(t)$*.*

The next theorem gives us a way to connect the renewal function with the inter-arrival times, and also show that the renewal function has all the information about the inter-arrival times.

Theorem 2.1.6. *Let the inter-arrival times satisfy* $P(A_i = 0) < 1$*. Then the renewal function is (1) right continuous, (2) non-decreasing and (3)* $H(0-) = 0$*,* $H(t) < \infty, \forall t \geq 0$*. The measure* μ_H *induced by* $H(t)$ *is* $\sigma - finite$ *and is supported on* $[0.\infty)$*. Also, If* F *denotes the distribution of the inter-arrival times and* μ_F *the measure induced by it, then*

$$H = \sum_{n=0}^{\infty} F^{*(n)} \qquad \mu_H = \sum_{n=0}^{\infty} \mu_F^{*(n)} \tag{4}$$

Where $F^{*(n)}$ *is the* n *times convoluted distribution. Also, if* $\hat{\ell}_H(s)$ *denote the Laplace transform of* H *and if* $\hat{\ell}_F(s)$ *denote the Laplace transform of* F*, then*

$$\hat{\ell}_H(s) = \frac{1}{1 - \hat{\ell}_F(s)} \tag{5}$$

Thus, using the uniqueness property of the Laplace-transforms we can find the distribution of the inter-arrival times from the renewal function.

The proof can be found in (Ramasubramanian, 2009) and (Rolski *et al.*, 2009).

Consider the following calculation:

$$\begin{aligned}
H(t) &= \sum_{n=0}^{\infty} F^{*(n)}(t) \\
&= F^{*(0)}(t) + \sum_{n=1}^{\infty} F^{*(n)}(t) \\
&= \mathbb{I}_{[0.\infty)}(t) + \sum_{n=1}^{\infty} \int_0^t F^{*(n-1)}(t-s)dF(s) \\
&= \mathbb{I}_{[0.\infty)}(t) + \int_0^t \sum_{n=0}^{\infty} F^{*(n)}(t-s)dF(s) \\
&= \mathbb{I}_{[0.\infty)}(t) + \int_0^t H(t-s)dF(s)
\end{aligned}$$

Note that we used the fact that $t \geq 0$ in the second step, and *monotone convergence* to exchange the summation and the integral. The above calculation motivates us for the following definition.

Definition 2.1.5. *A convolution equation of the form*

$$U(t) = u(t) + \int_0^t U(t-s)dF(s)$$

is called the renewal equation. Here, F is the distribution for the non-negative random variables, and $u(.)$ is a known function. $U(.)$ is the unknown function. Clearly $H(,)$ satisfies the renewal equation with $u(.) = \mathbb{I}_{[0.\infty)}(.)$ from the previous calculation.

The following theorem gives a general solution to the renewal equation.

Theorem 2.1.7. *Let the renewal equation*

$$U(t) = u(t) + \int_0^t U(t-s)dF(s)$$

hold for a given $u(.)$ which is locally bounded(bounded on bounded sets) and a non-degenerate distribution F. Define the renewal function(measure) by

$$H(t) = \sum_{n=0}^{\infty} F^{*(n)} \qquad \left(\mu_H(t) = \sum_{n=0}^{\infty} \mu_F^{*(n)}\right)$$

Then, the unique solution in the class of locally bounded measurable functions on $[0, \infty)$ is given by

$$U(t) = \int_0^t u(t-s)dH(s)$$

The proof of this theorem can be found in (Ramasubramanian, 2009). Note that the above theorem concludes the existence of a solution to the renewal equation and also gives a way to construct it. Now that the existence of solution has been established, we are interested in the properties of such a solution. Renewal theorems gives some asymptotic properties of the solutions to the renewal equations.

Before we state the renewal theorems, note that we previously considered $u(.)$ is in the class of locally bounded functions. But the renewal theorems here hold for a different class of functions called the *Directly Reimann Integrable* functions. Few of the classes of *Directly Reimann Integrable* functions are

- Continuous functions with support on a bounded set.
- Indicator functions on finite intervals.
- All Reimann integrable functions with support on a bounded set
- $u(t) = F(t+\zeta) - F(t)$ and $u(t) = F(t) - F(t-\zeta)$ where, F is a distribution of the probability measure and $\zeta > 0$ is fixed.

Now we state the renewal theorems.

Theorem 2.1.8. *Let F denote the distribution of the inter-arrival times supported on $[0.\infty)$, and having a finite expectation $\frac{1}{\lambda}$, Then*

- *(Blackwell Renewal Theorem) If H is the corresponding renewal function, then for any $h > 0$*

$$\lim_{t\to\infty}[H(t+h) - H(t)] = \lambda h$$

- *(Key Renewal Theorem) Let $u(.)$ is a directly reimann integrable function and $U(.)$ be the solution to the general renewal equation in the class of locally bounded function. Then*

$$\lim_{t\to\infty} U(t) = \lambda \int_0^\infty u(s)ds.$$

To appreciate the importance of the theorem, we give the distribution for the *Residual Life Time*, that is the time to the next claim. The residual life time at time $t > 0$ is defined as

$$Q(t,\zeta) = P(T_{N(t)+1} - t \leq \zeta)$$

Then, If F denotes the distribution of the inter-arrival times, then we find that $Q(.,\zeta)$ is the solution to the renewal equation

$$U(t) = (F(t+\zeta) - F(t)) + \int_0^t U(t-s)dF(s)$$

and hence from the key renewal theorem we have

$$\lim_{t\to\infty} Q(t,\zeta) = \lambda \int_0^\infty (F(t+\zeta) - F(t))dt = \lambda \int_0^\zeta (1 - F(t))dt$$

This distribution is also called the integrated tail distribution of $F(.)$.

2.1.3 Conclusion

In this subsection we looked at two important models for the claim arrivals. One of the main assumptions we consider is that od the stationary increments. A general theory can be developed to some extent without this assumption. All these models come under the category of point processes. See (Rolski *et al.*, 2009) for more on point processes.

One other inherent assumption we made is that, we counted each claim as a single unit. This, in reality is not true. Each claim is of different amount. Thus, instead of a jump of size one, we can consider randomly distributed jumps.A Poisson process with random jumps is called a *Compound Poisson Process* and usually goes along with the normal Poisson process. In this report, we have taken a different route. We characterise each source of randomness separately and combine them later on in the subsection on *Ruin Probabilities*.

2.2 Claim-Size distributions

The next source of randomness in a insurance model comes from the sizes of the claims. The exact distrubution depends mainly on the insurance policy. For example, in case of life insurance policies the amount is fixed and known before hand, and hence there is nothing random in it. On the other hand, the non-life insurance policies like accident coverage or automobile insurance has a random element in the claim sizes.

In theory any probability distribution supported on $[0,\infty)$ could be a claim-size distribution. But few distributions could be dangerous (the ones with infinite variance). In this subsection we discuss the various types of claim size distributions and their impact.

2.2.1 Light-Tailed Distributions

As an insurer one would like to reduce the probability of high claims. Mathematically stated, this means we would like to have the claim-size distributions whose tail probabilities reduce, preferably reduce exponentially. Such distributions are called "Light-tailed" distributions defined as follows.

Definition 2.2.1. *Let F be the claim-size distribution with its support on $[0.\infty)$. We say that F is " Light-tailed" if there exists a λ and C such that*

$$1 - F(x) \leq Ce^{-\lambda x} \qquad for\ all\ x \geq 0$$

An equalent condition is the existence of moment generating function defined by

$$\hat{m}_F(s) = E(e^{sX}) = \int_0^{\infty} e^{sx} dF(x)$$

for $s \in (-\epsilon, \epsilon)$, for some ϵ. Also, if we define

$$s_F = \sup\{s \geq 0 : \hat{m}_F(s) < \infty\}$$

then we have that $\lambda < s_F$. Any distribution which is not light-tailed is called heavy-tailed distribution.

As mentioned earlier such distributions are " nice" in the sense that they are amenable to analysis and would not pose a great danger to the insurance company. Few examples of the light-tailed distributions are

- Exponential distribution with parameter λ
- Gamma distribution,
- Any distribution with compact support
- Truncated Standard normal, defined as

$$TN(x) = P(|N(0,1)| \leq x) = 2\phi(x) - 1$$

  where $\phi(.)$ is the distribution of the standard normal.

An insurance company cannot directly control the claim-size distributions. It can, however, remove those policies which have higher risk. But that is not the only criteria which controls the claim-size distribution. It is thus important to look at heavy-tailed distributions as well. We now define an important subclass of the heavy-tailed distributions useful in actuarial context. For other subclasses of the heavy tailed distributions see appendix A.1.

2.2.2 Sub-Exponential distributions

There are various ways to define sub-exponential distributions. The following definition is the most straightforward one, from preceding discussion.

Definition 2.2.2. *Let, $X_1, X_2, \cdots,$ be a set of i.i.d random variables from a distribution F, and let $S_n = X_1 + X_2 + \cdots + X_n$, and $M_n = max\{X_1, X_2, \cdots, X_n\}$, If,*

$$P(S_n > t) = P(M_n > t)[1 + o(1)], \qquad for\ all\ n \geq 2 \tag{6}$$

then F is said to be sub-exponential. The class of sub-exponential distributions os denoted by $\mathcal{S}$

Note that the following are equivalent.

$$\begin{aligned}
& P(S_n > t) = P(M_n > t)[1 + o(1)], \qquad for\ all\ n \geq 2 \\
\iff & \lim_{t\to\infty} \frac{P(S_n > t)}{P(M_n > t)} = 1 \qquad for\ all\ n \geq 2 \\
\iff & \lim_{t\to\infty} \frac{1 - F^{*n}(t)}{1 - F^n(t)} = 1 \qquad for\ all\ n \geq 2 \\
\iff & \lim_{t\to\infty} \frac{1 - F^{*n}(t)}{1 - F(t)} = n \qquad for\ all\ n \geq 2 \\
\iff & \lim_{t\to\infty} \frac{1 - F^{*2}(t)}{1 - F(t)} = 2
\end{aligned}$$

Thus, any of the above conditions could be taken as as standard definition.The following theorem is useful in the actuarial context.

Theorem 2.2.1. *Let $F \in \mathcal{S}$, and a discrete distribution $\{p_1, p_2, \cdots\}$, such that $\sum_n p_n(1+\epsilon)^n < \infty$, for some $\epsilon > 0$, then, if $G(x) = \sum_n p_n F^{*n}(x)$, then*

$$\lim_{x\to\infty} \frac{1-G(x)}{1-F(x)} = \sum_n np_n$$

In actuarial context, suppose we are interested in the total claim amount till time t. As before, let $N(t)$ be the number of claims till time t, and X with distribution F denote the claim size. Then the total claim is given by

$$S(t) = \sum_{k=1}^{N(t)} X_k$$

Where, X_k is the k^{th} claim amount. The above theorem then tells us that the tail distributions of the $S(t)$ is almost the same as F.

2.2.3 Conclusion

In this sub-section, we have seen various types of claim-size distributions, from light-tailed, to regularly varying, and sub-exponential which are subclasses of heavy-tailed distributions. Our next aim, to use these claim size distributions, with the point processes discussed before, and calculate the ruin probability, that is the probability that the company goes bankrupt in finite time.

2.3 Ruin-Probabilities

In this subsection, we shall look at the ruin probabilities of a company. Ruin Probabilities give a good idea of the health of an insurance company. Also, this quantity is important in making decisions about the premium rates, and other business strategies to keep the company making profits. The definition of ruin-probability, a claim number process, and claim size distribution are already discussed above. Along with these we define *Claim amount Process* and the *Surplus Process*. The notation is the same as above.

Definition 2.3.1. *If $N(t)$, and X_i, be as defines above.*

- *Define*

$$S(t) = \sum_{k=1}^{N(t)} X_i \tag{7}$$

  Where, $N(t)$ gives the number of claims till time t, and X_i is the i^{th} claim. $\{S(t)\}$ is called the Total Claim Amount Process.

- *If we assume that the premiums is a deterministic function, $p(.)$, and moreover assume $p(t) = ct$, where c is called the premium rate. Also, take that the initial capital to be x. Define*

$$R(t) = x + ct - S(t) \qquad \forall t \geq 0 \tag{8}$$

  Then $\{R(.)\}$ is called the surplus process.

- *Then the ruin is defined by the event $\{R(t) < 0 \ for\ some\ t \geq 0\}$, and the ruin time is given by*

$$\tau_0 = \inf\{t \geq 0 : R(t) < 0\}$$

  and the ruin probability then becomes,

$$\phi(x) = P(\tau_0 < \infty | R(0) = x) = P_x(\tau_0 < \infty)$$

The aim is to get expressions for the ruin probability with a initial capital x. One can look at the Risk Process as a random walk on the real line. To simplify this a little, note that the ruin can occur only when a claim takes place, that is at times $\{T_1, T_2, \cdots\}$. Hence, we can discretize the risk process to $\{R(T_n) : n \in \mathbb{N}\}$. This is known as the *Skeleton Process.*

Also, If we write

$$Z_n = X_n - cA_n$$

Where A_n is the inter-arrival times, defined before. These $\{Z_n\}$ would give the loss to the company in each time interval between two claims. Then if we write

$$S_n = \sum_{k=1}^{n} Z_i$$

then $\{S_n\}$gives the total loss for the company in the first n claims, and the ruin probability can be written as

$$\phi(x) = P\left\{\sup_{n\geq 0} S_n > x\right\}$$

Moreover $\{S_n\}$ gives a discrete random walk on real line, with jumps forming an *i.i.d* sequence $\{Z_n\}$. The following theorem gives a basic result on this random walk.

Theorem 2.3.1. *With the notation as fixed,*

- *If* $E(Z_1) < 0$ *then* $\lim_{n\to\infty} S_n = -\infty$*, and hence the ruin probability is less than* 1.
- *If* $E(Z_1) > 0$ *then* $\lim_{n\to\infty} S_n = \infty$*, and hence the ruin probability is equal to* 1*, that is the ruin is certain.*
- *If* $E(Z_1) = 0$*, then* $\limsup_{n\to\infty} S_n = \infty$ *and* $\liminf_{n\to infty} S_n = -\infty$*. Hence even in this case, the ruin is certain.*

From the above theorem it is certain that, we need that $E(Z_1) < 0$, for an insurance policy to keep the company afloat. This is known as the *Net Profit Condition.* However, this only can guarantee that $\phi(x) < 1$, and does not remove the possibility of ruin completely. With further information, one can find upper bound on the ruin probabilities as we shall see now.

2.3.1 Sparre-Anderson Model

A *Sparre-Anderson Model* takes the claim number process to be a renewal process, and the claim sizes are independent of the claim number process. The independence of the processes are not always true in real life, but it still gives us a good model to work with. The first basic results one can obtain are are the analouges of the *SLLN*, *Elementary renewal Theorem*, *Blackwell's renewal theorem*, and *Central Limit theorem* concisely put in the following theorem.

Theorem 2.3.2. *Consider the Sparre-Anderson model, with the notation as before. Then,*

- $\lim_{t\to\infty} \frac{1}{t} S(t) = \frac{E(X_1)}{E(A_1)}$ *a.s.*
- $\lim_{t\to\infty} \frac{1}{t} S(t) = \frac{E(X_1)}{E(A_1)}$
- *for any* $h > 0$, $\lim_{t\to\infty} E(S(t+h)) - E(S(t)) = \frac{E(X_1)}{E(A_1)} h$
- *Assuming finite second moments,*

$$\frac{1}{(\sigma_{X_1}^2 + \lambda^2\mu^2\sigma_{A_1}^2)^{\frac{1}{2}}} \frac{1}{\sqrt{\lambda t}} (S(t) - \lambda\mu t) \xrightarrow{d} N(0,1)$$

as $t \to \infty$ *, where,* $\sigma_{X_1}^2$ *is the variance of* X_1*,* $\sigma_{A_1}^2$ *is the variance of* A_1*,* $\lambda = E(A_1)^{-1}$ *and* $\mu = E(X_1)$.

If the claim sizes are light tailed, then we can find a solution $h = r$ for

$$E\left(e^{hZ_1}\right) = 1$$

If $E(Z_1) < 0$, where Z_1 is as previously defined. This solution is called the *Lundberg coefficient.* This leads us to get an exponentially decreasing bound on the ruin probabilities.

Theorem 2.3.3 (Lundberg Bound)**.** *Assuming that Lundberg coefficient ,* $r > 0$*, exists, we have*

$$\phi(x) \leq e^{-rx}$$

The proof of the above theorem can be seen by noting that $\left\{e^{rS_n} : n \in \mathbb{N}\right\}$ forms a martingale, and using optional sampling theorem.

This is an important result, as it tells us how much should our initial capital be, given the risk in each of our time steps. It is clear that the rate at which the ruin probabilities reduce with the initial capital depends on the Lundberg coefficient, which in-turn depends on the premium rate c. As, c increases, $E(Z_1)$ reduces, thus giving a high value of r and hence reducing the ruin probabilities. So, for an insurance company, one can fix the ruin probability, and then decide on the (c, x) depending on constraints.

2.3.2 Cramer-Lundberg Model

As we have seen previously, one special case of the renewal model for claim number process is the Poisson process. A model where the claim number distribution is given by the Poisson process, and the claim sizes are independent of the claim number process, is known as the *Cramer-Lundberg Model.* With this extra information, one can get a renewal equation for the ruin probabilities as given by the next theorem.

Theorem 2.3.4. *Assuming that $P(X_1 > 0) = 1$, and defining, $\psi(t) = 1 - \phi(t)$, we have*

$$\psi(t) = \frac{\rho}{1+\rho} + \frac{1}{1+\rho}\int_0^t \psi(t-x)dF_{X,I}(x)$$

where

$$\rho = \left(\frac{c}{\lambda\mu} - 1\right)$$

and c is the premium rate, $\mu = E(X_1)$, and λ is the rate of Poisson process. $F_{X,I}(.)$ is the integrated tail distribution. This is called a defective renewal equation, since $F_{X,I}(.)$ is not a total probability distribution. Also, the solution to the above renewal equation is given by

$$\psi(t) = \frac{\rho}{1+\rho}\sum_0^\infty \frac{1}{(1+\rho)^n} F_{X,I}^{*(n)}(t)$$

This is called the Pollaczek-Khinchin fromula.

Also, in this special case one can find the exact asymptotic results. The major advantage of this is that, we need no longer assume that the claim sizes are light tailed. This is very important in practical cases. We get the following result.

Theorem 2.3.5. *Assume that the net profit condition holds, and $\rho = \left(\frac{c}{\lambda\mu} - 1\right)$, then*

- *In the case where claim sizes are light tailed, and if we assume the so called cramer-Lundberg condition*

  $$C = \frac{r}{\rho\mu}\int_0^\infty xe^{rx}(1 - F_X(x))dx < \infty$$

  where r is the Lundberg coefficient., then

  $$\lim_{t\to\infty} e^{rt}\phi(t) = \frac{1}{C}$$

  That is the ruin probability decays exponentially.

- *In the case where the claim sizes are sub-exponential, then*

  $$\lim_{t\to\infty} \frac{\phi(t)}{1 - F_{X,I}(t)} = \frac{1}{\rho}$$

The above theorem is quite powerful in real world applications, since the claim sizes are usually sub-exponential, like Pareto, log-normal, or heavy-tailed Weibull distributions. In these cases we do not have that the ruin probabilities reduce exponentially, but as the integrated tails of the claim size distributions. This is expected because, we have seen that the light-tailed distributions have tails which are exponentially reducing thus giving an exponential decay for the ruin probabilities.

2.3.3 Infinite Profit Probabilities

We have earlier seen that the probability of ruin, with initial capital defined as

$$\phi(x) = P(\tau < \infty | R(T_0) = x) = 1 - P(\{R(T_n) > 0, \forall n\} | R(T_0) = x)$$

One the same lines one can also define *probability of infinite profits* as

$$\chi(x) = P(\{R(T_n) \to \infty\} | R(T_0) = x)$$

As an insurer, one would be interested in the latter probability since this ensures that the company makes infinite profits. As the following theorem shows, under some situations, we have

$$\chi(x) = 1 - \phi(x)$$

Theorem 2.3.6. *Suppose $\{Z_n\}$ are all bounded, that is the aggregate loss in a time interval $(T_n, T_{n+1}]$ is bounded. Then*

$$\{R(T_n) > 0, \forall n\} \Rightarrow \{R(T_n) \to \infty\}$$

Proof. Note that

$$R(T_n) = x + p(T_n) - S(T_n) = R(T_0) - \sum_{i=1}^{n} Z_i$$

We can rewrite this as

$$R(T_n) - R(T_0) = -\left(\sum_{i=1}^{n}(Z_i - E[Z_i]) + \sum_{i=1}^{n} E[Z_i]\right)$$

If $Y_i = -(Z_i - E[Z_i])$, then $\left(\sum_{i=1}^{n} Y_i\right)$ is a martingale with bounded increments, and since $\{R(T_n) > 0, \forall n\}$, we have $\left(\sum_{i=1}^{n} Y_i\right)$ is bounded below. Hence,

$$\left(\sum_{i=1}^{n} Y_i\right) \to Y_\infty \qquad as\ n \to \infty$$

Also, we have that $E[-Z_i] > \delta > 0, \forall i$. Hence

$$\sum_{i=1}^{n} E[-Z_i] \to \infty \qquad as\ n \to \infty$$

Hence, we have

$$\{R(T_n) > 0, \forall n\} \Rightarrow \{R(T_n) \to \infty\}$$

□

The above theorem gives a way to connect the probability of no ruin to probability of infinite profits. if we also include the assumption that the insurer is able to invest the premiums, then one can obtain a similar result. See lemma 5 in reference (Gaier *et al.*, 2003) for details.

2.4 Premium Principles

We previously saw the effect of initial capital on the ruin probabilities. One of the other key control parameter of an insurer is the premium amount. A premium is a series of regular payments from the policy holder to the insurer in exchange for the protection against random events. Thus, stating a premium principles involves assessing the risk of a random event and framing a suitable deterministic function to compensate for the losses. The amount of premium collected till time t by the insurer is denoted by $p(t)$. Also, if we assume that there are no further policies taken after the start, then we have $p(t) = ct$, where c is the rate of premiums.

It is intuitively clear that the amount of premium collected till time t, must be greater than or equal to the total claim till time t, and should depend on the total claim till time t, $S(t)$. A premium principle is denoted by $\Pi(S(t))$. There is, however, no unique method to do so. Thus, there are a lot of ways to state a premium principle. All the premium principles can be broadly categorised into three methods - the *Ad-hoc* method, *Characterisation* method, and the *Economic* method. In this section we review the basic ideas of the premium principles, and also few of the principles used. See (Landsman & Sherris, 2001),(Wang & Dhaene, 1998),(Goovaerts *et al.*, 2010),(Young, 2006),(Wang *et al.*, 1997),(Yaari, 1987) for details.

The ad-hoc method involves framing a premium principle intuitively, and looking at its properties. For example the *Net Premium Principle* stated below. The characterisation method needs the insurer to state his properties, and find a premium principle to satisfy these properties. In the Economic method, the constraints of a premium principle are stated as a economic condition. For example the *principle of Zero-utility.* Although, the methods stated above are different, a particular premium principle can be derived from more than one method. One such principle is the *Proportional Hazard's Premium Principle.*

The advantages and disadvantages of a premium principle are characterised by the properties of the premium principle. We state a few of the common properties an actuary is interested in.

- **Independence:** If a principle satisfies that $\Pi(S(t))$ depends only on the distribution of $S(t)$, $F_{S(t)}$, then the principle has the property of independence. All the principles stated in this article have this property.
- **Risk Loading:** If we have that $\Pi(S(t)) \geq E[S(t)]$, then the principle is said to have risk loading. In case the equality holds, for all constant risks, then the principle has *no unjustified risk loading.*
- **Invariance:** One of the properties actuaries are interested in are the scale and translational invariance. Scale invariance states that $\Pi(aS(t)) = a\Pi(S(t))$, and translational invariance states that that $\Pi(S(t) + a) = \Pi(S(t)) + a$.
- **Additivity:** If a principle satisfies $\Pi(S(t_1) + S(t_2)) = \Pi(S(t_1)) + \Pi(S(t_2))$, then it is said to be additive. Similarly, if $\Pi(S(t_1) + S(t_2)) \leq \Pi(S(t_1)) + \Pi(S(t_2))$, then it is said to be sub-additive, and if $\Pi(S(t_1) + S(t_2)) \geq \Pi(S(t_1)) + \Pi(S(t_2))$, then it is said to be super-additive.
- **Preserving Stochastic order:** A principle is said to preserve first stochastic dominance if, $F_{S(t_1)}(x) \leq F_{S(t_2)}(x), \forall x$, then $\Pi(S(t_1)) \leq \Pi(S(t_2))$. Similarly, a principle is said to preserve stop-loss order if, $E[S(t_1) - d]_+ \leq E[S(t_2) - d]_+, \forall d \geq 0$, then $\Pi(S(t_1)) \leq \Pi(S(t_2))$.

Apart from these properties, an actuary might be interested in other specific properties like *continuity*, which arise depending on the situation. See reference (Young, 2006) for more details. We now state a few of the commonly used premium principle, along with the properties. Before that, note that if there is no risk loading, that is $\Pi(S(t)) = E[S(t)]$ then the insurer almost always ends up in bankruptcy.
Thus, the premium principle must be such that the $E[Z_i] < 0$, to avoid bankruptcy. In the case of the Net premium principle, we have $E[Z_i] = 0$. This is known as the *Net Profit Condition (NPC).*

Our aim in this section is to combine all the existing premium principles into a single principle. Since, checking the result for all the possible premium principles is impossible, we take a few principles covering all the methods - the *Ad-hoc* method, *Characterisation* method, and the *Economic* method, and show it for these principles.

- **Expected Value Premium Principle :** It states that
$$\Pi(S(t)) = (1 + a)E[S(t)]$$
It is the most widely studied premium principle. In case $a = 0$, we get the *Net Premium Principle.*
- **Variance Premium Principle:** This principle states
$$\Pi(S(t)) = E[S(t)] + \alpha Var[S(t)]$$
If we replace $Var[S(t)]$ by the standard deviation $\sqrt{Var[S(t)]}$, we get the *Standard deviation Premium Principle.* It is shown that this principle approximates a premium principle that one obtains from the economic method.
- **Exponential Premium Principle:** This principle states that
$$\Pi(s(t)) = \frac{1}{a} log E[e^{aS(t)}]$$
This principle can also be derived from the principle of zero-utility, by taking the utility function to be exponential. (See 35,36).
- **Wang's Premium Principle:** This principle states that
$$\Pi(S(t)) = \int_0^\infty g(1 - F_{S(t)}(x))dx$$
Where $g(.)$ is a concave, increasing function mapping [0 1] $\rightarrow$ [0 1]. $g(1 - F_{S(t)}(x))$ is called the distorted tail. If we take $g(s) = s^c$ for $0 < c < 1$, then we get the *Proportional Hazards Premium Principle.* This principle can be derived using the characterisation method.
- **Principle of Equivalent Utility:** It states that, the premium $\Pi(S(t))$ solves
$$u(0) = E[u(\Pi(S(t)) - S(t))]$$
Where $u(.)$ is a utility function. This can be derived from the economic method by using theory of expected utilities.

The above list is by no means complete, but it covers all the basic types of premium principles. Our aim is to show that all these premium principles are special cases of the risk function premium principle.

2.4.1 Premium principle using risk functions

As mentioned before, a premium principle is fundamentally assessing the risk of a random event and giving a suitable deterministic form to it, to avoid losses. Here we propose a new method using *risk functions*. Risk functions are functions which assess the risk of a particular amount. In classical statistics these are referred to as loss functions, but we use a more appropriate term risk functions here.

Some of the typical properties of the risk functions are

- The first thing to note is that $r(.)$ is strictly increasing. This is reasonable because a higher amount would draw more risk than a lower amount.
- Every single unit of currency would give a higher risk than the previous unit of currency. Thus, the risk of 1 pound after 10 pounds is less than or equal to 1 pound after 100 pounds. This tells us that the risk functions are convex.
- The risk of zero should be zero, that is $r(0) = 0$. This is a intuitive assumption.
- The risk function depends on the insurer. If the insurer can take large risks, then the risk function has a smaller second derivative.

It is not always the case that risk functions have all the above properties. For example in case of re-insurance, where the insurance company is insured for all claims above a certain amount $M(say)$, we have that the risk function is actually zero for values greater than M. The following is the main assumption in this section

> Every insurer has risk function $r(.)$, which would depend only on the amount of possible loss and not on the probability of the loss.

We now state the risk-function premium principle.

Definition 2.4.1 (Risk Function Premium Principle)**.** *Given the total claim amount till time t to be $S(t)$, and a suitable risk function $r(.)$ for the insurer, the Risk function premium principle states*

$$\Pi(S(t)) = E[r(S(t))]$$

Thus, we have

$$p(t) = E[r(S(t))] \tag{9}$$

The most important observation is to note that all the previously stated premium principles are special cases of the risk function premium principle.

- (Expected Premium Principle) If we consider the risk function $r(x) = (1 + a)x$, then we have

  $$E[r(S(t))] = E[(1 + a)S(t)] = (1 + a)E[S(t)]$$

  Thus, giving the expected premium principle.

- (Variance Principle) Consider the risk function of the form

  $$r_t(x) = \alpha x^2 + (1 - \alpha E[S(t)])x$$

  This risk function gives us the variance principle. Note that, unlike the expected principle, the risk function here also changes with time. But this is still deterministic.

- (Exponential Principle) The exponential principle states that

  $$\Pi(S(t)) = \frac{1}{a} log(E[e^{aS(t)}])$$

  Note that If, $E[e^{aS(t)}]$ is less than 1, then the premiums are negative, which is absurd. Thus, this principle is only valid in the region where $E[e^{aS(t)}] > 1$. The corresponding risk function is given by

  $$r(x) = \frac{1}{a} \frac{e^{ax}}{\int_0^x e^{ax} dF_{S(t)}(x)} \mathbb{I}_{[x_0 \ \infty)}$$

  Where $F_{S(t)}(.)$ is the distribution function of $S(t)$ and where x_0 is such that

  $$\int_0^{x_0} e^{ax} dF_{S(t)}(x) = 1$$

  Note that, as in the variance principle the risk function depends on time, but is still deterministic. Also, $\displaystyle\int_0^x e^{ax} dF_{S(t)}(x)$ should exist and be non-zero for all x for the above formula to make sense.

- (Wang's Premium Principle) Wang's premium principle states that

$$\Pi(S(t)) = \int_0^\infty g(1 - F_{S(t)}(x))dx$$

where $g(.)$ is a concave function. If we assume that there exists a density, $f(.)$ for the distribution of $S(t)$, and that $f(.)$ is non-zero on the real line, then

$$r(x) = \frac{g(1 - F_{S(t)}(x))}{f(x)}$$

gives the corresponding risk function.

- (Principle of Equivalent utility) This states that, given a utility function $u(.)$ of the insurer, $\Pi(S(t))$ is a solution to

$$u(0) = E[u(\Pi(S(t)) - S(t))]$$

Usually it is not possible to find the exact closed form for the risk function, for a general $u(.)$. However, note that in the special case where

$$u(x) = \frac{1}{a}1 - e^{-ax}$$

gives the exponential utility principle, for which we already found a corresponding risk function. Also, we have that the variance principle approximates approximates the utility premium principle for small risks. It is thus reasonable to say that we can find a suitable risk function corresponding to the principle of equivalent utility.

Thus, risk functions generalize the notion of existing premium principles. Rephrasing all the existing premium principles as risk functions allows us to compare various premium principles and efficiency of each of those principles.

The following theorem provides a way to calculate the probability of ruin based on the premium principle.

Theorem 2.4.1. *Given a risk function $r(.)$, and assuming the premiums are calculated using the risk functon premium principle, we have*

$$P(\{R(T_n) > 0\}|R(T_0) = x) \geq 1 - \frac{E[S(T_n)]}{r(E[S(T_n)]) + x} \tag{10}$$

Proof. Firstly note that,

$$R(T_n) = x + p(T_n) - S(T_n)$$

which, when using the risk functio premium principle reduces to

$$R(T_n) = x + E[r(S(T_n))] - S(T_n)$$

So,

$$\begin{aligned}
P(\{R(T_n) > 0\}) &= P(\{x + E[r(S(T_n))] - S(T_n) > 0\}) \\
&= P(\{S(T_n) < x + E[r(S(T_n))]\}) \quad (\because S(T_n) > 0, \textit{by markov inequality}) \\
&\geq 1 - \frac{E[S(T_n)]}{E[r(S(T_n))] + x} \quad (\because r(.) \textit{ is convex}) \\
&\geq 1 - \frac{E[S(T_n)]}{r(E[S(T_n)]) + x}
\end{aligned}$$

□

The above gives a bound on the probability of ruin in a particular interval $[T_n, T_{n+1})$ in terms of the risk function. For example, if the risk function is order n, then the ruin in time interval $[T_n, T_{n+1})$ reduces as $o(E[S(t)]^{n-1})$. This is expected, since higher order risk function gives higher premiums and hence lower risk of bankruptcy.

The power of the risk function premium principle comes from the fact that many known premium principles are special cases of the risk function premium principle. In this subsection we mention few of the advantages of this new method.

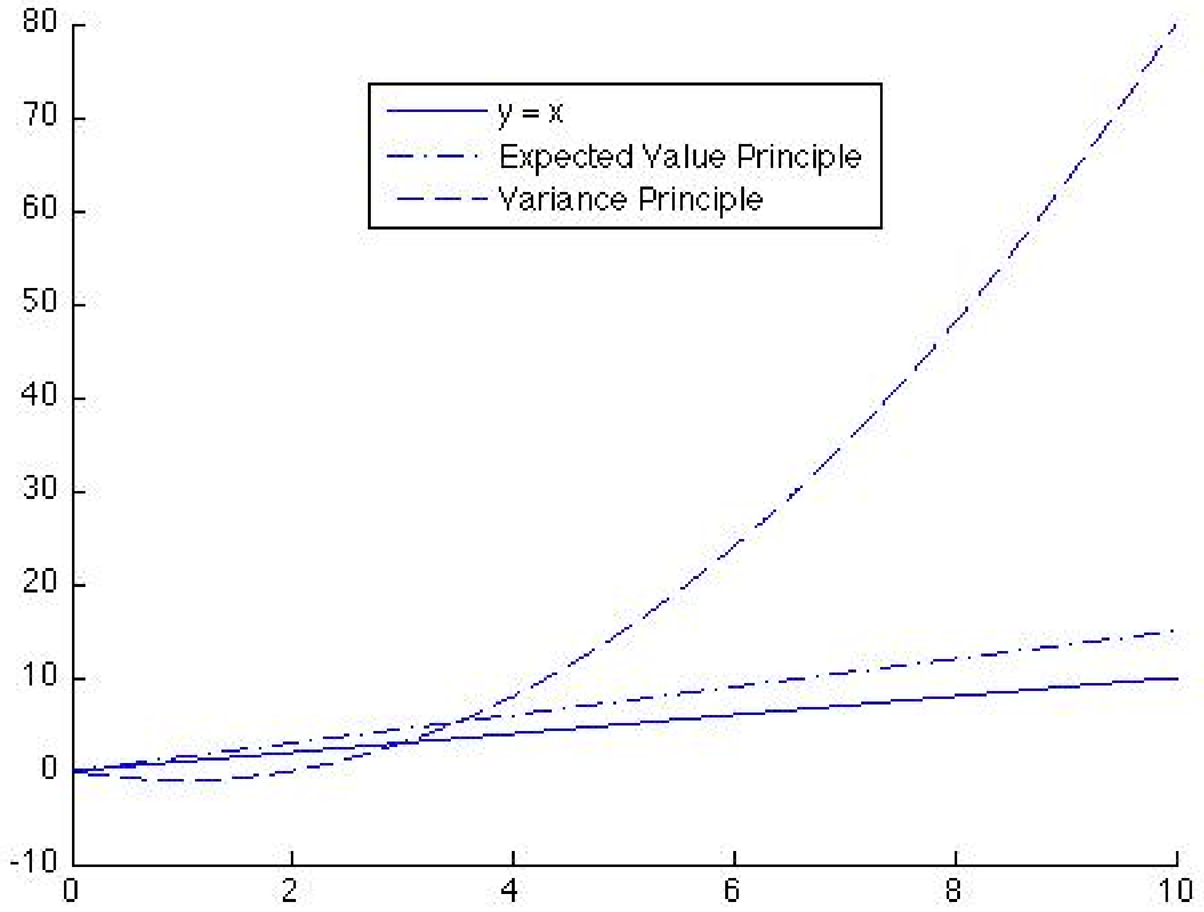

Figure 1: Risk Functions For Some Premium Principles

The first major advantage of this principle is that it offers a way to compare the premium principles. For instance, in figure 1 we plotted the sample graphs for the risk functions corresponding to the variance principle and the expected principle. One observation we can make is, if the distribution of $S(t)$ is heavy-tailed then the variance principle gives high premiums than the expected principle. Thus, risk functions offer a common ground to compare various premium principles.

In a competitive market, the insurers fight to offer lower premiums. Sticking to the usual rules, the limited number of premium principles do not offer a wide range of possibilities for premiums, which, the use of risk functions can solve. To offer lower premiums, all the insurer has to do is to make sure his risk function is less than his competitors. Thus, the risk function premium principle offers more choices for framing premium principles.

Also, since we now can compare the premium principles on a common ground, the question of optimal premiums can be answered. The optimal premium depends on the risk function of the insurer.

3 Multi-dimensional models

In the previous section we looked at the basic models of insurance. Note that all of them assume a single type of policy, and hence one-dimensional models were sufficient, In general, a insurance company would deal with multiple kind of insurance policies. Also, sometimes there could be treaties between insurance companies to reduce the ruin probabilities. All these situations require a more complicated multi-dimensional models.

One of the way to deal with multiplt kind of insurance policies is by looking at *Multivariate Poisson Process with interacting intensities*. Here the interactions model the correlations between various lines of business. See section A.2 in the appendix for more details.

In some situations, an group of insurance companies would have a contract to help the company on verge of bankruptcy by loaning an amount at a lower interest rate compared to the market rate. These are called *risk reducing treaties*. Multidimensional models are apt for such situations. It is intuitively clear that such treaties reduce the risk of bankruptcy.A practical question which arises in such situations is how much must each of the companies agree to lend? Every company wants to reduce the amount it pays, but maximize its benefits. This would lead to a *nash equilibrium*. A suitable formulation of this problem is in terms of the *skorokhod problem*. See (Ramasubramanian, 2006) and (Ramasubramanian, 2007) for details.

4 Reinvestment in Insurance models

The final source of randomness is when the insurance companies decide to invest the excess amount obtained from premiums in the market. It is important for the insurance company to invest, since otherwise there is not much reason as to why the policy holder must buy a insurance policy from the insurer. It is clear that the better investment strategies an insurer has, the lower rate of premiums he can offer and attract a huge portion of the market. This topic has attracted a lot of researchers in recent years. There are a lot of ways, one can model the investment strategies, and the optimal approach depends too much on the exact situation.

In general investment strategies are usually very complex, and the optimal strategy is tough to find. Investment strategies usually aim at making as much profits as possible. However in case of investment for insurance companies , the main aim is to reduce its ruin probabilities, and remain capable of paying the claims. There are a huge number of questions one can investigate in this area.

- How much fraction of the premium amount must be invested in the risk-free bonds, and how much must be invested in the risky assets?
- What kind of assets are suitable for investing for an insurance company?
- How can one mathematically formulate the problem of reinvesting in general?
- How are investment strategies effected by the kind of policies the insurers offers?
- How exactly are ruin probabilities effected by the investment strategies?

These are but a few of the many questions of practical importance to the insurance company. Our main aim in this section is to try and answer the above questions, and review few of the ways adopted for answering these questions.

Before we begin answering the questions above, it is important to have a model for the investment for risky assets. The model we consider is the Black-Scholes model which is very popular in the area of finance today. See (Hull, 2011) for a general overview of the financial markets. See (Shreve, 2004) and (Shreve, 2005) for more mathematical aspects of financial modelling. The model states that, suppose $X(t)$ denotes the price of an asset at time t, then the future prices can be modelled by a stochastic differential equation

$$\frac{dX(t)}{X(t)} = adt + bdW(t) \tag{11}$$

Where $W(t)$ is the standard weiner process. For more details on the aspects of stochastic differential equations, refer (Øksendal, 2003).

Intuitively from the point of view of the insurer, we expect to invest in such a way that the risk from the policy is reduced by the risky asset we invest in. In other words, we would like to hedge the risk from the policy. Since the risk from the policy usually arises from a event like an accident, or a epidemic, it is tough to find the right asset for hedging our risk. None the less, we can investigate the extent of offset of the risk using this understanding. It is equally difficult to find two assets whose risks are purely uncorrelated as much as to find two assets whose risks are completely correlated. Thus, investing in a risky asset will reduce the ruin probabilities. This is also observed in various models. For example, in reference (Gaier *et al.*, 2003) the authors observe that with investment the ruin probabilities reduce with a faster rate than from non-investing in a Cramer-Lundberg model. They also find that, in this case, the optimal strategy is to invest a constant proportion of money at all times in the risky asset. In reference (Hipp & Plum, 2000) the authors, solve an optimal control problem where a risky asset is used for investment. The solution to this is used to find an optimal investment strategy in case of a compound Poisson process. For more on stochastic optimal control in insurance see (Schmidli, 2008).

It is usual for insurers to consider reinsurance, especially if the insurance company is small. There are various methods of reinsurance. For example a proportional reinsurance contract states that the reinsurer pays a proportion of the claim, and a excess loss reinsurance contract states that the reinsurer pays all the claims above a certain threshold. For more on the types of reinsurance contracts, see (Rolski *et al.*, 2009). It is intuitively clear that reinsurance definitely reduces the ruin probabilities. However, it is not clear on how to make a choice between reinsurance and reinvestment. The author of (Schmidli, 2002) consider this question in detail.

In addition to these, there are various extensions and different models investigated, This is a area of current research. See (Liu & Yang, 2004) for a particular extension of the model presented in (Hipp & Plum, 2000).

5 Summary

Insurance plays a crucial role in protecting the society from random events. In the working of an insurance company there are three sources of randomness- The times of arrival of claims, the claim sizes, and the returns on the investment of premiums. We have seen that the arrival times are modelled by a Poisson process, or a Renewal process, and examined them in detail. The claim sizes could either be light-tailed or heavy tailed. Light tailed distributions are ideal for mathematical treatment, but not always practical. We identified a subclass of the heavy-tailed distributions- sub-exponential distributions, which are observed in practise and also offer a advantage of mathematical treatment to a certain extent.

The main aim of a insurance company is to avoid bankruptcy. We introduced a notion of the probability of ruin. We combined the models for the claim sizes and arrival times, to have two basic models- Cramer-Lundberg Model and Sparre-Anderson Model, to obtain a working model for the insurance company. The probability of ruin is calculated in both these cases. One of the new results here is to see that the probability of no ruin is infact probability of infinite profits under some conditions.

One of the key control factors of insurance is the premiums. We have seen a few methods used to calculate the premiums. It is observed that all these methods could be combined into a single principle using risk functions known as "risk function premium principle". We used this principle to give bounds on the ruin probabilities in a time interval between two claims.

There are some situations in real life not covered by the basic models. It is thus important to consider a bit more complicated models. One of the ways is to consider the multi-dimensional models. We discussed in detail a multi-dimensional model for interacting intensities. Another way to reduce the ruin probabilities is by having a risk-reducing treaties between many companies. We mentioned briefly about the risk-reducing treaties and the ways to model such situations.The third source of randomness- returns from the investment of premiums, is a currently active field of research. There is no one optimal approach to this problem, and forces us to consider different situations separately. We mentioned briefly the active research in these areas in recent times.

On the whole, insurance is a typical example of a complex system, where many processes act on their own, and also have complex interactions with other processes. It is a very practical subject, and also a source of rich mathematics. The natural randomness makes probability and statistics an invaluable tool in analysing the working of the insurance companies.

Acknowledgements

I thank my supervisor Prof. Vassili Kolokoltsov for his help and support during the whole project. I also thank the European Commission(EPSRC) for the Erasmus Mundus Scholarship, and an excellent chance to study in European Union.

References

Bauerle, Nicole, & Grubel, Rudolf. 2005. Multivariate Counting Processes: Copulas and Beyond. *ASTIN Bulletin*, **35**(2), 379–408.

Bauerle, Nicole, & Grubel, Rudolf. 2008. Multivariate risk processes with interacting intensities. *Adv. in Appl. Probab.*, **40**(2), 578–601.

Cousin, A., Crépey, S., Guéant, O., Hobson, D., Jeanblanc, M., & Lasry, J.M. 2010. *Paris-Princeton Lectures on Mathematical Finance 2010*. Lecture Notes in Mathematics, no. v. 4. Springer.

Foster, Dean P., & Hart, Sergiu. 2009. An Operational Measure of Riskiness. *Journal of Political Economy*, **115**(5), 785–814.

Gaier, J., Grandits, P., & Schachermayer, W. 2003. Asymptotic ruin probabilities and optimal investment. *Ann. Appl. Probab.*, **13**(3), 1054–1076.

Goovaerts, Marc J., Dhaene, Jan, & Rachedi, Omar. 2010. *Actuarial Premium Principles*. John Wiley and Sons, Ltd.

Hipp, C., & Plum, M. 2000. Optimal Investment for Insurers. *Insurance: Mathematics and Economics*, **27**, 215–228.

Hipp, C., & Plum, M. 2003. Optimal Investment for investors for state dependent income, and for insurers. *Finance and Stochastics*, **7**, 299–321.

Hull, J.C. 2011. *Options, Futures, and Other Derivatives*. Prentice Hall PTR.

Landsman, Zinoviy, & Sherris, Michael. 2001. Risk measures and insurance premium principles. *Insurance: Mathematics and Economics*, **29**(1), 103 – 115.

Liu, Chi Sang, & Yang, Hailiang. 2004. Optimal investment for an insurer to minimize its probability of ruin. *North American Actuarial Journal*.

Øksendal, B.K. 2003. *Stochastic differential equations: an introduction with applications*. Universitext (1979). Springer.

Ramasubramanian, S. 2006. An Insurance Network : Nash Equilibrium. *Insurance : Mathematics and Economics*, **38**, 374=390.

Ramasubramanian, S. 2007. A d-person differential game with state space constraints. *Applied Mathematics And Optimization*, **56**, 312–342.

Ramasubramanian, S. 2009. *Lectures on Insurance Models*. Texts and Readings in Mathematics. Hindustan Book Agency (india).

Rolski, T., Schmidli, H., Schmidt, V., & Teugels, J. 2009. *Stochastic Processes for Insurance and Finance*. Wiley Series in Probability and Statistics. John Wiley & Sons.

Schmidli, H. 2008. *Stochastic control in insurance*. Probability and its applications. Springer.

Schmidli, Hanspeter. 2002. On minimizing the ruin probability by investment and reinsurance. *Ann. Appl. Probab.*, **12**(3), 890–907.

Shreve, S.E. 2004. *Stochastic calculus for finance: Continuous-time models*. Springer finance. Springer.

Shreve, S.E. 2005. *Stochastic Calculus for Finance: The binomial asset pricing model*. Springer finance. Springer.

Wang, Shaun, & Dhaene, Jan. 1998. Comonotonicity, correlation order and premium principles. *Insurance: Mathematics and Economics*, **22**(3), 235 – 242.

Wang, Shaun S., Young, Virginia R., & Panjer, Harry H. 1997. Axiomatic characterization of insurance prices. *Insurance: Mathematics and Economics*, **21**(2), 173 – 183. ¡ce:title¿in Honor of Prof. J.A. Beekman¡/ce:title¿.

Yaari, Menahem E. 1987. The Dual Theory of Choice under Risk. *Econometrica*, **55**(1), 95–115.

Young, Virginia R. 2006. *Premium Principles*. John Wiley and Sons, Ltd.

Young, VR. 2004. Optimal investment strategy to minimize the probability of lifetime ruin. *North American Actuarial Journal*.

A Appendix

A.1 Regularly Varying Distributions

Definition A.1.1. *Let $L(.)$ be a function on $(0, \infty)$ such that*

$$\lim_{x \to \infty} \frac{L(cx)}{L(x)} = 1 \qquad for\ all\ c \geq 0$$

Then $L(.)$ is called a slowly varying at ∞ function.

The following theorem is useful in dealing with the slowly varying functions.

Theorem A.1.1. *Let $L(.)$ be a slowly varying function. Then for any $a > 0$*

$$\lim_{x \to \infty} x^{-a} L(x) = 0 \qquad \lim_{x \to \infty} x^{a} L(x) = \infty$$

That is L is small compared to any power-function x^a.

This theorem allows us to give bounds on the moments of the claim-size distributions which are slowly varying. The proof the above theorem follows from a result of *Karmata* known as *Karmata Representation Theorem.*

Theorem A.1.2 (Karmata Representation Theorem)**.** *Let L be a slowly varying function at ∞ and is positive measurable. Then it is of the form*

$$L(x) = c(x) e^{\left\{\int_{x_0}^{x} \frac{1}{t} \epsilon(t) dt\right\}} \quad \forall x > x_0 \tag{12}$$

where

- $x_0 > 0$
- *$c(x)$ is a function such that* $\lim\limits_{x \to \infty} c(x) = c > 0$
- $\lim\limits_{t \to \infty} \epsilon(t) = 0$
- *$c(.)$ and $\epsilon(.)$ are measurable.*

It is easy to check that (12) gives a slowly varying function. The proof of the other side uses another result about slowly varying functions, which states

$$\lim_{x \to \infty} \frac{L(cx)}{L(x)} = 1 \qquad uniformly\ over\ c \in K (:= \textit{compact set})$$

The complete proof can be found in (Ramasubramanian, 2009).
We now define what are called the *Regularly Varying distributions*, and provide the asymptotic results on the moments of such distributions. Such results would give us an insight into how bad the heavy tailed claims would effect the company.

Definition A.1.2. *A measurable function on $(0, \infty)$ defined as $f(x) = x^{\delta} L(x)$,($L(.)$ is a slowly varying function) is called a regularly varying function with index δ. A non-negative random variable,X, with its tail distribution*

$$P(X > x) = x^{-\alpha} L(x)$$

is called a regularly varying distribution with tail index α. That is, the tail distribution is a regularly varying function with index $-\alpha$. The class of all distributions with tail index α are denoted by $\mathcal{R}_{\alpha}$.

The following result extends the uniform convergence of slowly-varying functions to regularly varying functions.

Theorem A.1.3. *Let f be a regularly varying function with index $-\alpha$, $\alpha > 0$. Then for any $A > 0$*

$$\lim_{x \to \infty} \frac{f(\lambda x)}{f(x)} = \frac{1}{\lambda^{\alpha}} \qquad uniformly\ over\ \lambda \in [A, \infty)$$

The following theorem gives the asymptotic results on the truncated moments of regularly varying distributions.

Theorem A.1.4. *Let f be a regularly varying function, and locally bounded on $[K, \infty)$, with index δ. Let*

$$\overline{I}_p(x) = \int_K^x t^p f(t) dt \qquad \underline{I}_p(x) = \int_x^{\infty} t^p f(t) dt$$

Then the following are true.

- *If $p > -(\delta+1)$ then $\overline{I}_p(x)$ diverges and*

$$\lim_{x\to\infty} \frac{x^{p+1}f(x)}{\overline{I}_p(x)} = (p+\delta+1)$$

That is, $\overline{I}_p(x)$ is a regularly varying function.

- *If $p = -(\delta+1)$ then $\overline{I}_p(x)$ may converge or diverge, and*

$$\lim_{x\to\infty} \frac{\overline{I}_p(x)}{x^{p+1}f(x)} = \infty$$

and $\overline{I}_p(x)$ is a slowly varying function.

- *If $p < -(\delta+1)$ then $\overline{I}_p(x)$ converges, and*

$$\lim_{x\to\infty} \frac{x^{p+1}f(x)}{\underline{I}_p(x)} = -(p+\delta+1)$$

and $\underline{I}_p(x)$ is a regularly varying function with index $(p+\delta+1)$.

- *If $p = -(\delta+1)$ and $\overline{I}_p(x)$ converges, then*

$$\lim_{x\to\infty} \frac{\underline{I}_p(x)}{x^{p+1}f(x)} = \infty$$

and $\underline{I}_p(x)$ is slowly varying.

The proof of this theorem uses a lemma known as the *potter's bound.*

Theorem A.1.5 (Potter's bound)**.** *If L is a slowly varying function at ∞, then for any $A > 1$ and $\eta > 0$ there exists a $K(A,\eta)$ such that*

$$\frac{L(y)}{L(x)} \leq Amax\left\{\left(\frac{y}{x}\right)^{\eta}, \left(\frac{x}{y}\right)^{\eta}\right\} \qquad \forall x, y \geq K.$$

Our main aim for above theorems, is to understand the moments of the heavy-tailed distribution. The following theorem interprets this result for claim-size distributions.

Theorem A.1.6. *Let X denote a claim-size distribution with regularly varying tails, with index α, that is $X \in \mathcal{R}_{-\alpha}$. Then for all $\alpha \geq 0$*

$$\begin{aligned} E(X^p) &< \infty \qquad for\ all\ 0 \leq p < \alpha \\ &= \infty \qquad for\ all\ p > \alpha \end{aligned}$$

and for all $\alpha > 1$,and F denoted the distribution, then , F_I , the integrated belongs to regularly varying distributions with index $(\alpha - 1)$, that is $F_I \in \mathcal{R}_{-(\alpha-1)}$.

One of the most important distributions, which is found to fit the data in actuarial context is the $Pareto(\alpha,\kappa)$ distribution. Its distribution is given by

$$F(x) = 1 - \frac{\kappa^{\alpha}}{(\kappa+x)^{\alpha}} \qquad for\ x \geq 0$$

α is the tail parameter, and κ is the scale parameter. Also $Pareto(\alpha,\kappa) \in \mathcal{R}_{-\alpha}$. Also, it is important to note that $\mathcal{R}_{-\alpha}$ is closed under convolutions, for $\alpha \geq 0$. One of the other most important properties of the regularly varying distribution is

$$\lim_{t\to\infty} \frac{P(S_n > t)}{P(M_n > t)} = 1$$

Where, S_n is the partial sum to n terms, and M_n is the partial max of the first n terms. This means that The tail behaviour of partial sum is essentially the same as tail behaviour of partial maximum. In actuarial context, this means that a few larger claims in the set would make up for most of the claim amount paid by the company. This property explains the notion of " heavy-tailed" in the distributions.

A.2 Interacting Intensities

The first of the multi-dimensional models we discuss is as given in (Bauerle & Grubel, 2008) . It supposes a insurance company with d lines of business. A business line could be a sub-company or more appropriately a type of policy the insurance company offers. If each line is independent, then ofcourse the above analysis holds. Instead, if the lines of companies are dependent on the other lines, then we have to use a multi-dimensional model.

The exact model is as follows:-

- Each business line is denoted by $j = 1, 2, \cdots d$. Let $N(t) = (N_1(t), N_2(t), \cdots, N_d(t))$ be the vector containing the counting process of each of the d business lines.
- $[N(t)]_{t \geq 0}$ is a time homogeneous markov chain. At each point of time only a single claim of any type is possible. So, the possible jumps are $\{e_1, e_2, \cdots, e_d\}$, where e_k denotes the kth type of claim.
- Also, The transition probabilities are given by

  $$P(N(t+h) = k + e_j | N(t) = k) = \beta_j(k)h + o(h)$$

  Where $\beta = (\beta_1, \beta_2, \cdots, \beta_d)$ are appropriate rate functions.
- The dependence is modelled by making $\beta_j(.)$ depending on the number of claims in other lines of business.

The above assumptions completely specify the model. Note that $N(t)$ can be treated as a *pure-birth process.* However, we need further assumptions to get the results. The *first assumption* we make is

$$\eta := \sup_{k \in \mathbb{N}^d} ||\beta(k)|| = \sup_{k \in \mathbb{N}^d} (\sum_j \beta_j(k)) < \infty$$

The aove assumption avoids $\{N_j(t) \to \infty\}$ in finite time.

Also, $N(t)$ is a continuous time markov chain. Thus we have a corresponding embedded chain $X(t)$, which is discrete and consists of the sequence of the visits of $N(t)$. We can also define a chain called *Uniformised chain*, $X(t)$ which has the transition probabilities

$$p_{k,k+e_j} = \frac{\beta_j(k)}{\eta}$$

and

$$p_{k,k} = 1 - \sum_j p_{k,k+e_j}$$

This construction is useful to simulate this model. If we have a Poisson clock L, with rate $||\beta(k)||$, then we have that $N(t) = X(L(t))$. This is found useful in proving few result, and also provides a way to construct such process.

Corresponding to the above counting process, we also have the risk-reserve process. We assume that in j^{th} line of business, the claims are $\{U_{i,j}\}$, i.i.d, with distribution given by Q_j. Then if $R(t) = (R_1(t), R_2(t), \cdots, R_d(t))$ is the vector of risk reserves in each line of business, with premium rates $c = (c_1, c_2, \cdots, c_d)$, then

$$R_j(t) = R_j(0) + c_j t - \sum_{i=1}^{N_j(t)} U_{i,j}$$

and the ruin probability for individual components is given by same formulae as before, with appropriate terms.

We are interested in what are called the *fluid limits* of the above process, which gives useful results. It is defined as

$$\lim_{\gamma \to \infty} \Psi_\gamma(N(t))$$

where

$$\Psi_\gamma(f)(t) = f(0) + \frac{1}{\gamma}(f(\gamma t) - f(0))$$

The fulid limit consists of rescaling the time and space by the same factor and taking the limit. As the name suggests, these limits are commonly taken for fluids, to identify the few local properties. We write $N^\gamma = \Psi_\gamma(N(t))$ and $R^\gamma = \Psi_\gamma(R(t))$.

Define

$$W_d := \{(x_1, \cdots, x_d) \in \mathbb{R}_+^d : x_1 + x_2 + \cdots + x_d = 1\}$$

We assume that $\beta(k) \to \tilde{\beta} : W_d \to \mathbb{R}^d$, as $k \to \infty$ in the sense that, for all sequences $\{x_n\}$

$$\lim_{n\to\infty} = \infty \ and \ \lim_{n\to\infty} \frac{x_n}{||x_n||} = x \Longrightarrow \lim_{n\to\infty} \beta(x_n) = \tilde{\beta}(x)$$

where the norm $(||.||)$ is l^1 norm.

Theorem A.2.1. *Let $\beta(.)$ be as above and, Q_j has finite second moments with mean μ_j. Then every sequence $(\gamma(k))_{k\in\mathbb{N})}$ with $\gamma(k) \to \infty$, has a subsequence $(\gamma(k(l)))$ such that*

$$N^{(\gamma(k(l)))} \xrightarrow{d} \phi \qquad R^{(\gamma(k(l)))} \xrightarrow{d} r$$

Also, the paths of ϕ and r are almost surely absolutely continuous and they satisfy

$$\begin{aligned}
\phi(t) &= \phi(0) + \int_0^t \tilde{\beta}(\phi(s) - \phi(0))ds \\
r(t) &= r(0) + ct - (\mu_1\phi_1(t), \cdots, \mu_d\phi_d(t)))
\end{aligned}$$

It follows from the above theorem that, under further assumption that $\tilde{\beta}(.)$ is Lipschitz,

$$\frac{N_j(t)}{t} \xrightarrow{p} \phi_j(1) - \phi_j(0) \ \ and \ \ \frac{N_j(t)}{N_i(t)} \xrightarrow{p} \frac{\phi_j(1) - \phi_j(0)}{\phi_i(1) - \phi_i(0)}$$

Thus this gives a way to approximate the total number of claims of each type on the average.