

# Non-equilibrium steady state in Lorentz channel

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## Abstract

*We study the transport properties of a driven Lorentz channel. Despite the breaking of the energy shell, we show that the system reaches a non-trivial non-equilibrium steady state. We study in detail the steady state that arises in the configuration space by means of the Fokker-Planck equation. Our analytical results are in good agreement with the numerical on the condition that the velocity of the injected particles into the system is at most comparable with the maximum velocity of the driven scatterers. We also show that as the kinetic energy of the injected particles increases, the steady state approaches the one exhibited by the static Lorentz channel.*

## 1 Introduction

In dynamical systems, a billiard is a model in which a point-like particle moves along straight line within a region that has a piecewise smooth boundary [16]. The specular reflection happens when particle hits the boundary and the collision is elastic. Billiards give many natural models in optics, statistical mechanics and kinetic theory. Billiards also demonstrate all the possible behaviors of Hamiltonian systems from integrability to chaotic motions.

A particular billiard with smooth and everywhere dispersing boundary is called Sinai billiard, also known as Lorentz gas [Fig.1]. Lorentz gas was first

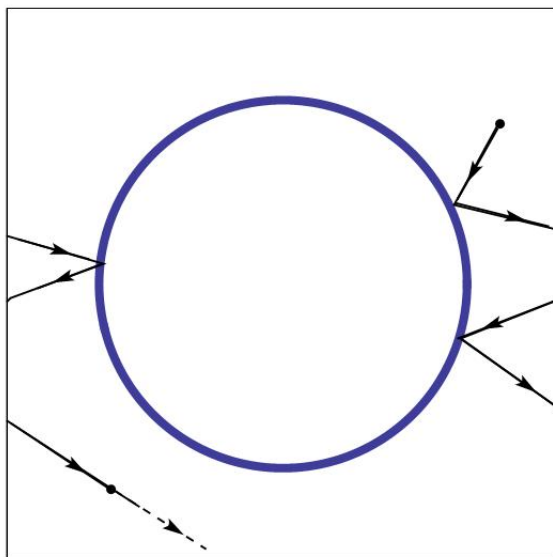


Fig. 1: Lorentz gas with periodic boundary and one trajectory of particle

introduced by H.A.Lorentz to explain the self-diffusion of electrons in metals in metals [1]. This model was no longer in use to study electric motion since physicist use quantum theory to learn it. However it was still very important in statistical mechanics and kinetic theory.

In Lorentz gas, as the mass of particle is far smaller than that of scatter, the interaction between particles was neglected. This model can be constructed in different ways depending on different placements and shapes of scatters.

In the system of  $2D$  periodic configuration with strictly convex scatters, i.e. [Fig.2], it has been proved that such system has strong mixing properties, positive Lyapunov exponent and the autocorrelation of velocity decays exponentially with the number of collisions [2]. However, the general properties of ergodicity in higher dimensional and finite systems are still unclear.

Due to these properties of infinite  $2D$  model, there are many analytical results explaining the diffusion behavior. For instance, Machta and Zwanzig have obtained the analytical expression of diffusion coefficient in the  $2D$  Lorentz gas with periodic placed scatters [3]. When the gap between cells [Fig. 2,3] is much smaller than the radius of scatters that allows the average trapping time of particle to be much longer than the average collision time, the jump between cells is then assumed uncorrelated. In this case they have

obtained good agreement between numerical and analytical results. This method also has been used to study  $3D$  Lorentz gas [9].

Recently, some different geometric shapes of scatters such as oval scatters and polygonal scatters have been studied [10][15]. It has been found that these systems have different statical properties. For instance, the mixing rate of system depends on the shapes of scatters.

In this paper, we focus on one particular direction. All scatters are circular and non-overlapping. The scatters are allowed to oscillate instead of being static which is also known as time dependent Lorentz gas. The discussion is restricted in  $2D$  and  $1D$ .

In time dependent Lorentz gas model, a particle can gain or lose energy when it hits the oscillating scatters. One can find the result from [5], the mean velocity of Lorentz gas increases in the model of  $2D$  triangular placed scatters. Such a model is connected with Fermi acceleration. Fermi acceleration consists in the increase of the mean energy of an ensemble of charged particles due to random collisions with moving magnetic inhomogeneities. It has been used to explain the acceleration of cosmic rays by Fermi [4].

The paper is organized as follows. To study the diffusion behavior in configuration space, we start from Karlis's result [5] that is the velocity density function  $\rho(V, n)$  follows a Maxwell-Boltzmann-like distribution, where  $n$  is the number of collisions of particle. The velocity density function  $\rho(V, t)$  with respect to time  $t$  is calculated. After that, under the approximation from Machta and Zawanzig that is the motions of particles can be replaced symmetric random walk [3], a super-diffusion process is established in configuration space and the steady state is obtained. In the last section, we firstly present the numerical result of static model that is a linear profile in the channel. Then, we compare the analytical and numerical results based on different parameters. Finally, some conclusions and further discussion of this model are presented.

## 2 Lorentz Gas

There are many ways to construct Lorentz gas models based on different placements of scatters. The scatters can be placed either randomly [Fig.2] or periodically [Fig.3] in the region. The scatters are either static or oscillating. We study the model with triangle placed and oscillating scatters.

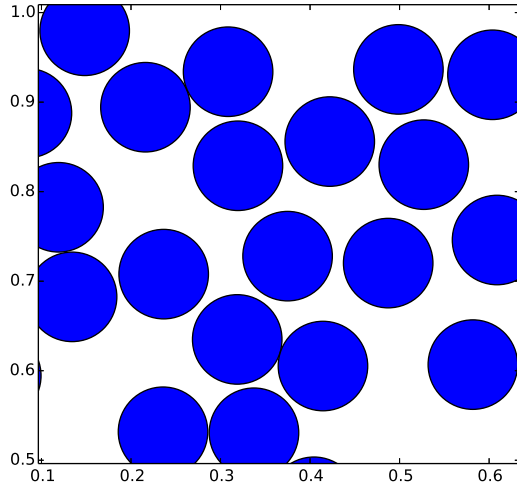


Fig. 2: Randomly placed scatters

In our model, since this dynamic depends on oscillation amplitude  $A$ , frequency  $\omega$  and the distance of centers of two neighboring scatters  $\alpha$ . In order to eliminate non-relevant parameters, we introduce a dimensionless quantity  $\epsilon = \frac{A}{\alpha}$ . Using  $\alpha$  as unit length and  $1/\omega$  as unit time. The velocity of oscillation scatter can be described by

$$u = \epsilon(\hat{\mathbf{i}}\sin\theta + \hat{\mathbf{j}}\cos\theta)\cos(t + \eta)$$

where  $u$  is the velocity of the scatters,  $\epsilon$  is the amplitude of the oscillation of scatter,  $\omega$  is the oscillation frequency,  $\theta$  is the angle between vector  $u$  and x-axis.  $\eta$  is a random number uniformly distributed in  $[0, 2\pi)$ .

The probability density function(PDF) of the magnitude of velocity  $V$  and collision times  $n$  are proposed as a Maxwell-Boltzmann-like distribution [5]. The following discussion is based on this result. The evolution of PDF in momentum space can be described by Fokker-Planck equation.

$$\frac{\partial\rho(V, n)}{\partial n} = -\frac{\partial}{\partial V}(A(V, n)\rho(V, n)) + \frac{1}{2}\frac{\partial^2}{\partial V^2}(B(V, n)\rho(V, n))$$

where the drift and diffusion coefficients are

$$A(V, n) = \frac{1}{\Delta n} \int \Delta V d(\Delta V), B(V, n) = \frac{1}{\Delta n} \int (\Delta V)^2 P d(\Delta V)$$

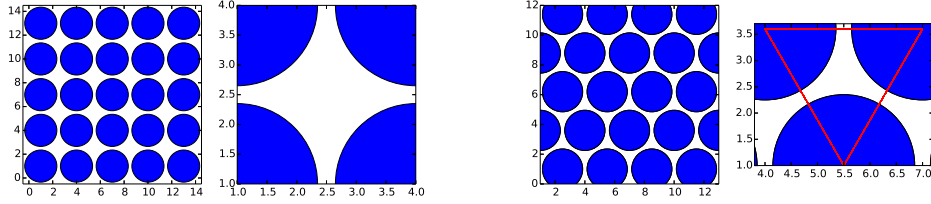


Fig. 3: Square (left) and triangle (right) placed scatters and corresponding trapping cells (white region and that of within red triangle)

$P$  is the probability of a particle being velocity  $V$  at that moment and being velocity  $V - \Delta V$  at time  $\Delta n$  collision earlier. In discrete time scale, it is convenient to use  $\Delta n = 1$ . Then  $A(V, n) = \langle \delta V \rangle$  and  $B(V, n) = \langle (\delta V)^2 \rangle$ , where  $\delta V$  is the mean increment of the magnitude of velocity. we use the results in [5]

$$A(V, n) = \left\langle \frac{\delta V}{\tau} \right\rangle = \frac{2\epsilon^2}{3\lambda}, B(V, n) = \left\langle \frac{\delta V^2}{\tau} \right\rangle = \frac{2\epsilon^2}{3\lambda} V$$

and the solution of the F-P equation under reflecting boundary at  $V = 0$  and natural boundary  $\rho(\infty, n) = 0$  is

$$\rho(V, n) = \sqrt{\frac{2}{\pi}} \frac{1}{\sigma^3} V^2 e^{-V^2/2\sigma^2}$$

$$\sigma = \sqrt{\frac{2\epsilon^2 n + V_0^2}{3}}$$

For  $n \gg 1$  numerical results show high accuracy with analytical results [5]. The mean velocity  $\langle V \rangle = 2\sigma \sqrt{\frac{2}{\pi}}$ , that is  $\langle V \rangle \sim \sqrt{n}$ . In the following section, we derive the velocity distribution with respect to time  $t$  and study the model in Lorentz channel.

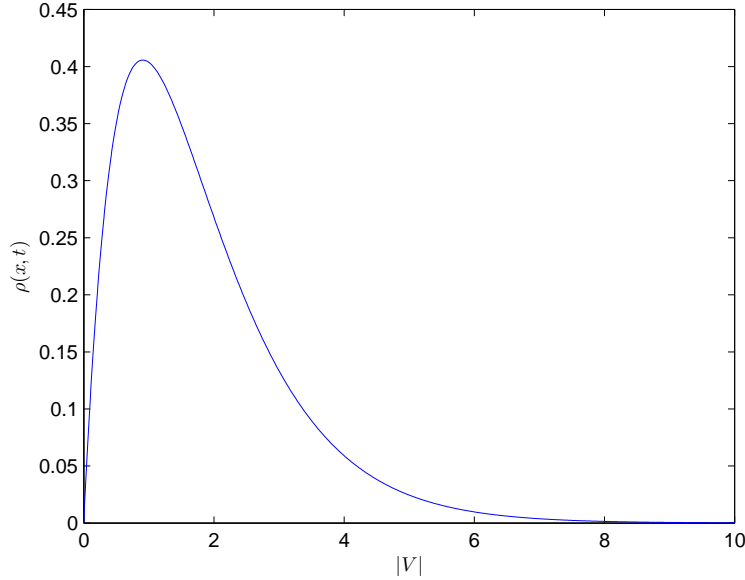


Fig. 4: Density function of absolute value of velocity for time  $n = 2 \times 10^5$  and  $\epsilon = 1/215$

### 3 Velocity Density Function In Time $t$

As discussed in Ref. [5], the mean increase of the magnitude of the particle velocity in the course of a single collision is

$$\langle \delta V \rangle = \frac{2\epsilon^2}{3V}, \quad \langle \delta V^2 \rangle = \frac{2\epsilon^2}{3}$$

Since the mean velocity of particles is changing, one cannot use collision number  $n$  to measure time  $t$ . To calculate the velocity density function versus time  $t$ , one has to calculate the mean free path  $\lambda$  between two sequential collisions. In the model with static scatter,  $\lambda$  only depends on the geometric properties of models [14].

$$\lambda = \frac{\pi Q}{M}$$

where  $Q$  is the accessible region in one cell and  $M$  is the total perimeter of the scatters in one cell. In triangle placed model,  $Q = \frac{\sqrt{3}}{4} - \frac{\pi R^2}{2}$  and  $M = \pi R$ , where  $R$  is the radius of scatter and  $\alpha = 1$ . As the oscillation amplitude

of scatters is small, it is reasonable to assume that the mean free path  $\lambda$  is fixed in oscillating model. We have the expressions of drift and diffusion coefficients

$$A(V, t) = \left\langle \frac{\delta V}{\tau} \right\rangle = \frac{2\epsilon^2}{3\lambda}, B(V, t) = \left\langle \frac{\delta V^2}{\tau} \right\rangle = \frac{2\epsilon^2}{3\lambda} V$$

The corresponding Fokker-Planck equation is

$$\frac{\partial \rho(V, t)}{\partial t} = -\frac{\partial}{\partial V}(A(V, t)\rho(V, t)) + \frac{1}{2} \frac{\partial^2}{\partial V^2}(B(V, t)\rho(V, t))$$

It can be simplified as

$$\frac{\partial \rho(V, t)}{\partial t} = \frac{\epsilon^2}{6\lambda} \cdot V \frac{\partial^2}{\partial V^2}(\rho(V, t))$$

we solve this equation under reflecting boundary at 0 and natural boundary in the infinity  $\rho(\infty, t) = 0$ . The solution is

$$\rho(V, t) = \left(\frac{\epsilon^2}{3\lambda}\right)^2 \frac{V}{t^2} e^{-\frac{3\lambda V}{\epsilon^2 t}}$$

Then one can find the mean velocity is linear to  $t$

$$\langle V \rangle = \frac{\epsilon^2}{3\lambda} t + V_0$$

One can also consider this in another way. Since  $\langle V \rangle \sim \sqrt{n}$  and the mean free path is fixed, then  $\Delta t_n \sim n^{-\frac{1}{2}}$ . We sum these time intervals to get  $t \sim \sum \Delta t_n \sim \sum n^{-\frac{1}{2}} \sim \sqrt{n}$ . From the relation between  $\langle V \rangle$  and  $n$ , one can obtain  $\langle V \rangle \sim t$

In configuration space, as we are interested in the case of large the radius of scatter  $R$ . This means the escape corridors between cells is narrow and particles have to collide many times before it hops to another cell. We use the approximation from Machta and Zwanzig [3]. That is the exact motion of particle can be replaced by uncorrelated random walk between cells. As it can be shown that any initial distribution of particle velocities quickly relaxes to a uniform one in respect with the direction angle [13]. It is reasonable to assume this random walk is symmetric that implies the drift coefficient in configuration space is zero. One can find that the diffusion coefficient is not constant but linear to time  $t$  which will be discussed in next section.

## 4 Lorentz Channel

In general, there are three types of periodic placed Lorentz gas system. One is that scatters are allowed to touch each other. In this case, particles are trapped in one cell forever. One is infinite trajectory system in which there exists infinite long trajectory without colliding scatters, e.g. squared placed model. The last one is finite trajectory system in which no particle can pass through system without any collision, e.g. triangle placed model with large radius. They have different statistical properties (e.g. mixing rate). We only consider finite trajectory system.

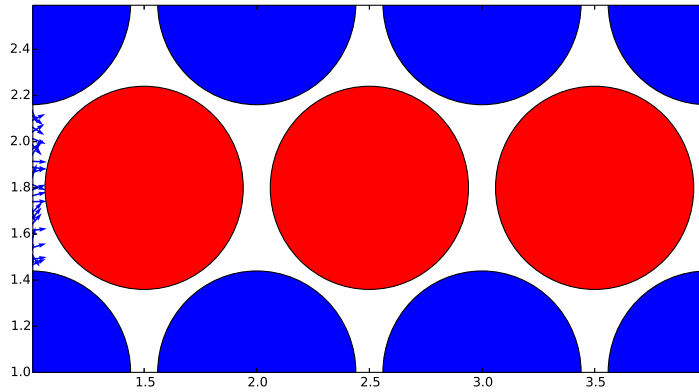


Fig. 5: Lorentz channel. The blue scatters are static and the red scatters are oscillating

There is a channel with length  $L$  in which scatters are placed in triangular way [Fig.5]. Only middle scatters are allowed to oscillate with small amplitude. The scatters on the top and bottom are static. The size of middle scatter is chose large enough so that no particle can pass through this channel without collision. This system is open. There is a steady flux that ejects from left boundary and particles can escape from both sides.

In this channel, the escape corridors in each cell is  $l_0 = 2(1 - R)$ . The billiard accessible region  $\Omega = \frac{\sqrt{3}}{4} - \frac{\pi R^2}{2}$ . The total perimeter of scatters is given by  $M = \pi R$ . Then, the diffusion coefficient with respect to  $x$ -axis



direction is given by

$$D_x = \frac{\langle x^2 \rangle}{2t} = \frac{\langle l_x^2 \rangle N}{2t}$$

$$N = \frac{A(V, t)t^2}{2H} = \frac{\epsilon^2}{3H}t^2$$

$$H = \lambda \frac{M}{l_0} = \frac{\pi\Omega}{l_0}$$

Where  $H$  is the average length of trajectory of particle staying in one cell,  $N$  is the number of passing cells of a particle,  $l_x$  is the projection on x-axis of the length of one cell.  $l_x = \frac{1}{2}$  then  $\langle l_x^2 \rangle = \frac{1}{4}$ . One find the diffusion coefficient

$$D_x = \frac{\epsilon^2 R(1 - 2R)}{3\pi(\sqrt{3} - 2\pi R^2)^2}t$$

The F-P equation in configuration space is

$$\frac{\partial \rho(x, t)}{\partial t} = D_x(t) \frac{\partial^2 \rho(x, t)}{\partial x^2}$$

Denote  $D_x = kt$ , after changing parameter by  $t' = t^2$ , the F-P equation is given by

$$\frac{\partial \rho(x, t')}{\partial t'} = \frac{k}{2} \frac{\partial^2 \rho(x, t')}{\partial x^2}$$

This is a Heat equation.

For open boundaries of finite channel, that is a source at left boundary, absorbing boundaries  $\rho(0, t') = \rho(L, t') = 0$  on the both sides and initial condition  $\rho(x, t) = \delta(x - x_0)\delta(t - t_0)$ . Solving this equation and changing back the parameter  $t'$  to  $t$ , one has the general solution

$$\rho(x, t|x_0, 0) = \sum_{n=1}^{\infty} \frac{2}{L} e^{-\frac{kn^2\pi^2}{2L^2}(t-t_0)^2} \sin\left(\frac{n\pi x_0}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

Since there is a steady flux at left boundary, one has to integral  $\rho(x, t)$  with respect to starting time  $t_0$  and let  $t$  goes to infinity. We derive the approxi-

mate solution. The steady state is

$$\begin{aligned} \rho(x, x_0) &= \lim_{t \rightarrow \infty} \int_0^t \rho(x, t | x_0, t_0) dt_0 \\ &\approx \frac{i\pi \log\left(\frac{1 - \cos(\pi(x-x_0)/L)}{1 - \cos(\pi(x+x_0)/L)}\right)}{2\pi\left(\pi x_0 - iL \log\left(\frac{1+e^{-\frac{i\pi x_0}{L}}}{1+e^{\frac{i\pi x_0}{L}}}\right)\right) + LG} \end{aligned}$$

Where  $G = 4Li_2(e^{-\frac{i\pi x_0}{L}}) - 2Li_2(e^{\frac{i\pi x_0}{L}}) + Li_2(e^{-\frac{2i\pi x_0}{L}}) - 2Li_2(e^{-\frac{i\pi(L+x_0)}{L}})$  and  $Li_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$  is the Polylogarithm function. As  $\rho(x, x_0)$  is the density function,  $\frac{i\pi}{2\pi(\pi x_0 - iL \log(\frac{1+e^{-\frac{i\pi x_0}{L}}}{1+e^{\frac{i\pi x_0}{L}}})) + LG}$  is a normalization factor

## 5 Analytical and Numerical Results

In this section, we compare analytical and numerical results based on different parameters. The simulation was running for  $10^7$  time unites. At each time unit, 10 particles are ejecting from left side of this channel. The initial velocity of particles  $V_0$  are the same. The angle between the initial velocity and y-axis are randomly chose from  $[0, \pi)$  that means all particles towards right in x-axis.

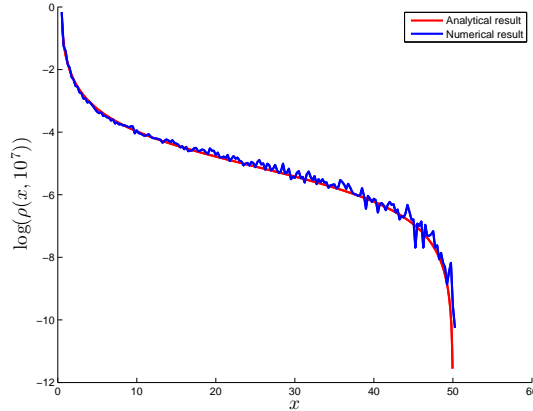


Fig. 6: Logplot of analytical and numerical density function  $\rho(x, 10^7)$  with  $x_0 = 0.5$ ,  $L = 50$ ,  $V_0 = 0.004$  and  $\epsilon = 0.004$

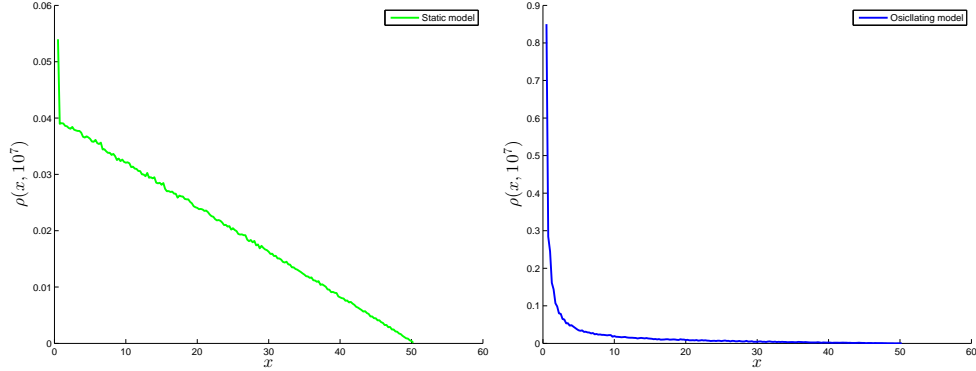


Fig. 7: Plot of  $\rho(x, 10^7)$  of static (left) and oscillating (right) models

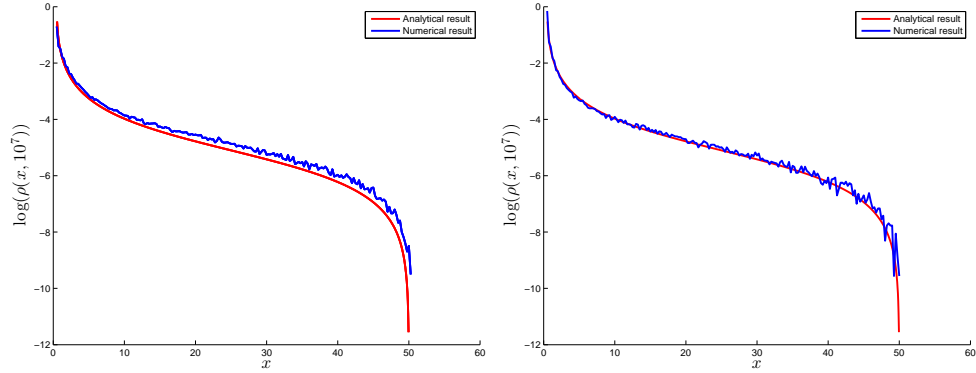


Fig. 8: Small (left)  $\epsilon = 0.002$  and large (right)  $\epsilon = 0.004$  amplitude of oscillating scatter with  $V_0 = 0.004$

The steady state is

$$\begin{aligned} \rho(x, x_0) &= \lim_{t \rightarrow \infty} \int_0^t \rho(x, t | x_0, t_0) dt_0 \\ &\approx \log\left(\frac{1 - \cos(\pi(x - x_0)/L)}{1 - \cos(\pi(x + x_0)/L)}\right) / \text{Normalization factor} \end{aligned}$$

In analytical result,  $x_0$  is the starting point of particle which cannot be zero. If  $x_0 = 0$ , there is only trivial solution. So we choose  $x_0 = 0.5$  that is less than one cell's length. The total length of channel is  $L = 50$ .

In Fig.6, one can find the agreement of analytical and numerical results is good. Then, we study the steady state based on different parameters.

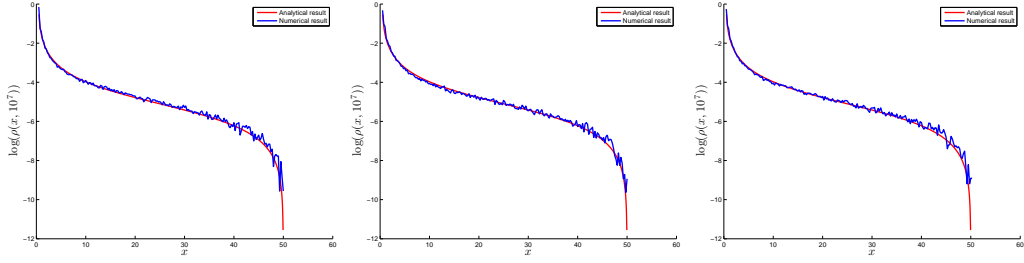


Fig. 9: The values of oscillating angle  $\theta$  are  $0.5\pi$ ,  $0$ ,  $0.25\pi$ , with  $\epsilon = 0.004$  and  $V_0 = 0.004$

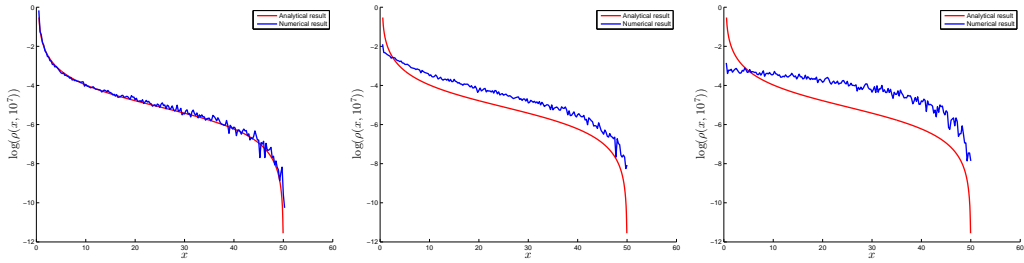


Fig. 10: The initial velocities of particles  $V_0$  are  $0.004$ ,  $0.04$ ,  $0.4$ , with amplitude  $\epsilon = 0.004$ .

First of all, we compare the numerical results between static and oscillating models [Fig.7]. There is a linear profile in static model while a quantitative different profile is in oscillating model. The linear distribution in static model has been discussed by Gaspard [17]. We represent his result numerically.

Then, we use different oscillation amplitudes of scatters [Fig.8]. When  $\epsilon$  is small, the agreement is not good. After increasing  $\epsilon$ , results show better agreement.

After that, we run simulation with different oscillation angles  $\theta$  of scatters [Fig.9]. One can find that the steady state does not depend on  $\theta$ .

Finally, we choose different initial velocities of particles [Fig.10]. When  $V_0$  is very high, the agreement breaks. However, if we compare the oscillating model with high  $V_0$  with static model. The oscillating model with high  $V_0$  gives a linear profile which shows good agreement with static model regardless of some fluctuation [Fig.11].

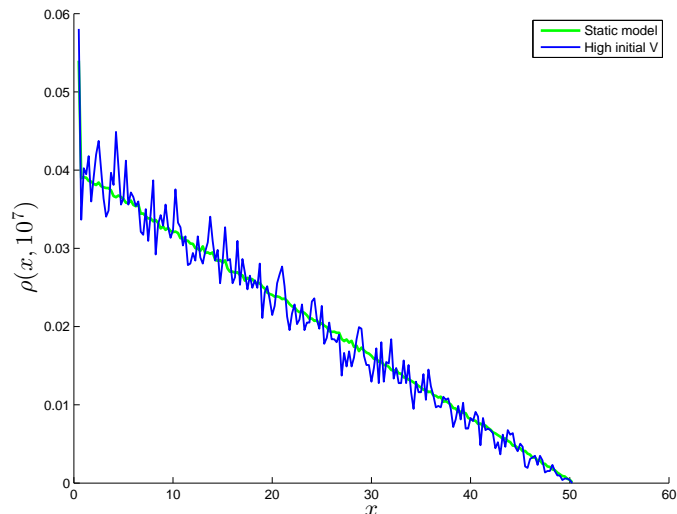


Fig. 11: Plot of  $\rho(x, 10^7)$  of static and high initial velocity ( $V_0 = 0.4$ ) models, with amplitude  $\epsilon = 0.004$ .

## 6 Conclusion and Further Work

One can find quantitative difference of the steady states between static and oscillating models. We represent the result of static model that is a linear profile in configuration space [17]. One can find different numerical result in oscillating model [Fig.7].

In oscillating model, from the comparison between analytical and numeric results, one can find that the steady state does not depend on the oscillation angles  $\theta$ . It depends on the oscillation amplitude of scatters  $\epsilon$  and the initial velocity of particle  $V_0$ . When  $\epsilon$  is comparable to  $V_0$ , the agreement between analytical and numerical results is good. If  $V_0 \gg \epsilon$ , one obtains a linear profile in configuration space. This linear profile shows good agreement with that of static model despite of some fluctuation. The reason is that when  $V_0 \gg \epsilon$ , the relative movement of scatter is very small. We obtain a linear result similar with the one exhibited by the static model [Fig.11].

In this paper, we did not consider different length of channel  $L$ . We always use  $L = 50$  units. However,  $L$  is an important parameter. It changes the scale of this system. The time of reaching steady state depends on it. The following work is to study how the time of reaching steady state varies with  $L$ .

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