CHALMERS

Measures for Complexity
and the Two-dimensional Ising model

M1 project — Erasmus Mundus programme in Complex Systems Science

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1 Introduction

Complex systems science becomes one of important area in both the natural and social sciences. However, there is no concise definition of complex systems. There are various attempts to characterize a complex system. In order to define complex systems, the concept of complexity is necessary. Many scientists have tried to find proper measures of complexity with mathematical rigor to the issue. In this paper, I discuss one effective measure of complexity, based on information theory, which is suggested by Grassberger and examine this measure of complexity on the two-dimensional Ising model, the most simplest spin model. I obtain the equilibrium configurations of the two-dimensional Ising model numerically by using the Monte Carlo Simulation(MCS). From the configurations for several temperature conditions, I obtain the measure of complexity of the model through the phase transition.

2 The two-dimensional Ising model

I use the two-dimensional Ising model with nearest neighbor interaction, the Hamiltonian of which is given by

$$\mathcal{H} = -\sum_{<i,j>} J_{ij} S_i S_j - \sum_i h_i S_i, \tag{2.1}$$

where $S_i = \pm 1$ is the Ising spin at site i, J_{ij} is the coupling strength, and h_i is magnetic fields as a bias to the entire system. $< \cdots >$ denotes the nearest neighbor summation. In this work, I set J_{ij} to be uniform (i.e. ferromagnetic system with coupling strength J), and assume that there is no bias in the system, $h_i = 0$. Then, the Hamiltonian becomes

$$\mathcal{H} = -J \sum_{<i,j>} S_i S_j. \tag{2.2}$$

The magnetization of the system is defined as

$$m = \frac{1}{N} \sum_{i=1}^{N} S_i \tag{2.3}$$

where N is the total number of spins in the system. In the calculation, I use a square lattice with size L, thus $N = L \times L$. The solution of the 2d-square lattice Ising model was derived by Onsager[1]. In the thermodynamic limit,

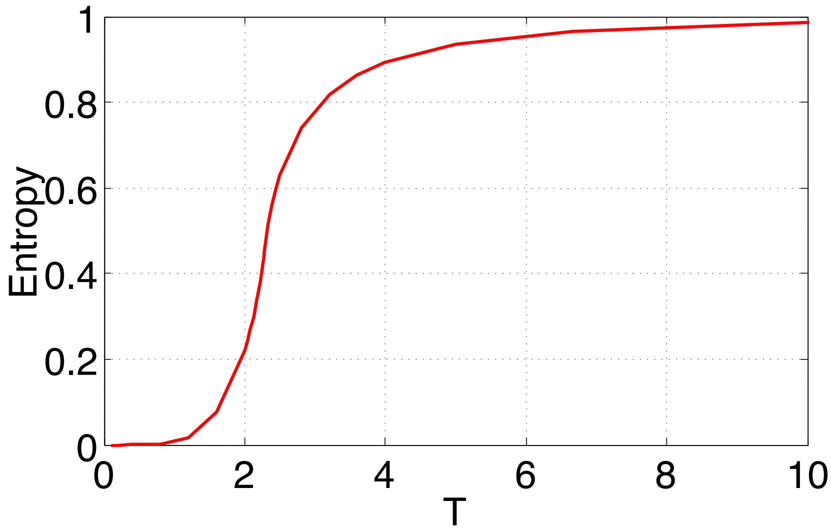

Figure 2.1: Exact entropy of the two-dimensional Ising model.

Onsager's exact free energy $F(T)$ and energy $E(T)$ at zero field are given as following:

$$\begin{aligned} F(T) &= -Nk_BT\log[2\cosh(2\beta J)] \\ &\quad -\frac{Nk_BT}{2\pi}\int_0^\pi d\phi \log\frac{1}{2}\left(1+\sqrt{1-K^2 s\sin^2\phi}\right) \qquad (2.4) \\ E(T) &= -2NJ\tanh(2\beta J)+\frac{NK}{2\pi}\frac{dK}{d\beta}\int_0^\pi d\phi\frac{\sin^2\phi}{\Delta(1+\Delta)} \qquad (2.5) \end{aligned}$$

where $\beta = \frac{1}{k_BT}$, $K = \frac{2}{\cosh(2\beta J)\coth(2\beta J)}$ and $\Delta = \sqrt{1-K^2\sin^2\phi}$. k_B is the Boltzmann constant. From Maxwell's relation,

$$s(T) = \frac{S(T)}{N} = \frac{F(T)-E(T)}{NT}, \qquad (2.6)$$

the exact entropy of the system can be obtained. In Fig. 2.1, I plot the analytic exact entropy. In this plot, I use the base of the logarithm as 2 in the entropy for comparison with information theory. Thus, maximum entropy per lattice site has to be 1 because the Ising model is a two-state model. In the figure, the entropy goes to unity as temperature increases. In

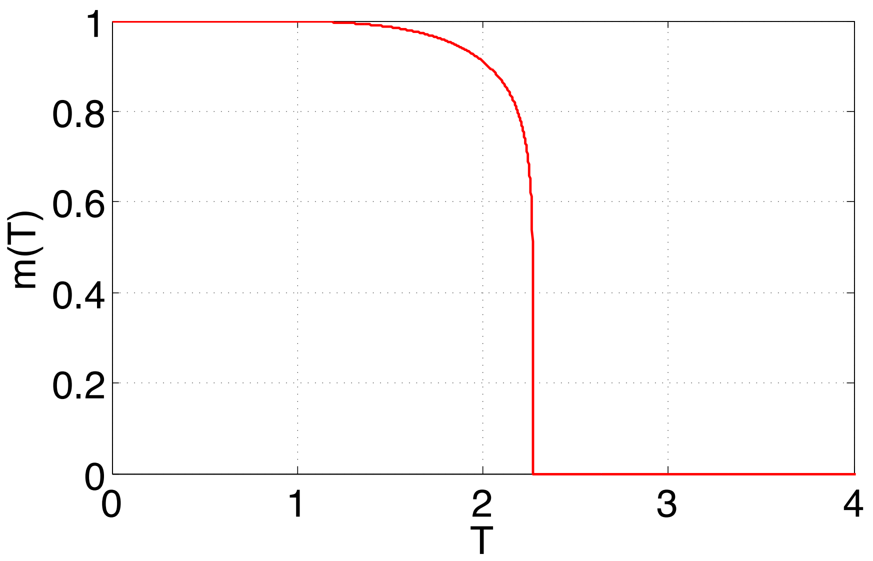

Figure 2.2: Exact magnetization of the two-dimensional Ising model.

addition, the exact solution of magnetization is

$$m = \left(1 - \left[\sinh\left(\frac{2J}{k_BT}\right)\right]^{-4}\right)^{\frac{1}{8}}. \tag{2.7}$$

The magnetization is given in Fig. 2.2. In this figure, the magnetization as an order parameter exhibits the phase transition. From some point, the order parameter becomes zero. This point is called a critical point or critical temperature and this temperature is given by

$$\tanh\left(\frac{2J}{k_BT_c}\right) = \frac{1}{\sqrt{2}} \tag{2.8}$$

$$\frac{k_BT_c}{J} = \frac{2}{\log(1+\sqrt{2})} = 2.269185... \tag{2.9}$$

Eq.(2.8) is obtained from self-duality which is the property of the square lattice Ising model. T_c denotes the critical temperature. In this report, I set k_B and the coupling strength J to be unity in order to silmplify calculation. Then, the critical temperature T_c is 2.269....

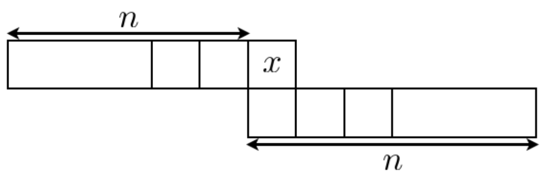

Figure 3.1: The configuration of cells B_n

3 A complexity measure of the spin system from two-dimensional information theory

In this section, I introduce the complexity measure of the spin system from the two-dimensional information theory[2].

3.1 The block entropy S

I consider an infinite two-dimensional lattice with Ising spins and let $A_{M\times N}$ be a specific $M \times N-$block occurring with probability $p(A_{M\times N})$. Then the entropy s of the system is

$$s = \lim_{M,N\to\infty} \frac{1}{MN} S_{M\times N} \tag{3.1}$$

where the block entropy $S_{M\times N}$ is given by

$$S_{M\times N} = \sum_{A_{M\times N}} p(A_{M\times N}) \log \frac{1}{p(A_{M\times N})}. \tag{3.2}$$

In this work, I define new block B_n for convenient calculations, as a certain spin-block-configuration of spins. B_n is arranged as Fig. 3.1 with two rows of symbols, each of length n. Then, the new block entropy S_{B_n} is defined by

$$S_{B_n} = \sum_{B_n} p(B_n) \log \frac{1}{(B_n)}. \tag{3.3}$$

I introduce the notation $B_n x$ for the configuration that adds the symbol x to B_n after nth spin in the top row. The conditional probability for a certain character x given that we have already observed the characters in the configuration B_n is given as

$$p(x|B_n) = \frac{p(B_n x)}{p(B_n)}. \tag{3.4}$$

This equation can be interpreted as the conditional probability for the "next" character given that characters in block B_n. From the conditional probability, the average entropy for the block configuration H_n is obtained as

$$H_n = \sum_{B_n} p(B_n) \sum_x p(x|B_n) \log \frac{1}{p(x|B_n)}. \tag{3.5}$$

For $n = 0$, I define H_0 as the entropy of the single character distribution,

$$H_0 = \sum_x p(x) \log \frac{1}{p(x)}, \tag{3.6}$$

where $x = \pm 1$. In addition, from the configurational symmetry of the system, the relation between H_n and s is given as

$$s = \lim_{n \to \infty} H_n = H_\infty. \tag{3.7}$$

Eq.(3.7) tells us that the average entropy for the block configuration converges to the real entropy if the block is large enough.

3.2 The correlation information over length n

I define the correlation information over length n as the difference between two consecutive estimates of the average entropy

$$k_{n+1} = -H_n + H_{n-1}. \tag{3.8}$$

The density information k_1 is

$$k_1 = \sum_x p(x) \log \frac{p(x)}{1/2} = 1 - H_0. \tag{3.9}$$

In the Eq.(3.9), $\frac{1}{2}$ indicates the a priori probability to get each state ± 1. This can be interpreted as an "uninformed" uniform distribution $p^{(0)}$. If the system has ν possible states, one can assign equal a priori distribution as $p^{(0)} = 1/\nu$. I combine all the correlation information of all lengths n, k_n and define the correlation information k_{corr},

$$k_{\mathrm{corr}} \equiv \sum_{n=2}^{\infty} k_n. \tag{3.10}$$

In this equation, the summation runs from $n = 2$ to infinity. Since, as I mentioned above, k_1 indicates the density information of single spin, and is not included in the correlation quatity. From expanding the Eq.(3.10) one can obtain the relation between s and k_{corr} as

$$k_{\text{corr}} = \log \nu - s - k_1 \tag{3.11}$$

$$= 1 - s - k_1 \tag{3.12}$$

Eq.(3.12) is for the Ising model which has two states. In this equation, the correlation information and density information indicates the ordered part. On the other hand, the Shannon entropy s indicates the disordered or uncertain part. If I rewrite Eq. (3.12) as

$$S_{\max} = k_{\text{corr}} + k_1 + s = \log 2 = 1, \tag{3.13}$$

the meaning of information of the system becomes more clearer. The total information of this system is composed of three parts, the correlation, the density, and the uncertainty(Shannon entropy).

3.3 The correlation complexity

From the Grassberger's correlation complexity[3], I define the measure of complexity as

$$\eta = \sum_{n=2}^{\infty}(n-1)k_n \tag{3.14}$$

$$= k_{\text{corr}} \sum_{n=2}^{\infty}(n-1)\frac{k_n}{k_{\text{corr}}} \tag{3.15}$$

$$= k_{\text{corr}}\overline{(n-1)} = k_{\text{corr}}\ l_{\text{corr}} \tag{3.16}$$

where k_{corr} is the correlation information and l_{corr} is the average correlation distance. As I mentioned in the previous section, correlation information k_{corr} indicates how ordered the system is. Then, the average correlation distance quantifies at what distance this order is to be found on average. l_{corr} is the weighted summation of the correlation information over specific lengths which implies that long distance correlation information gives high contribution on the average distance. If the observer needs information of long sequences, this means the system is more difficult to predict. Technically, unpredictability does not guarantee complexity but I can roughly assess this measure is kind of complexity quantity. From these points, the Eq.(3.16), can be a proper measure of complexity (i.e, complexity = order $\times$ unpredictability). From this measure, one can get low level of complexity in the completely ordered state and completely random state.

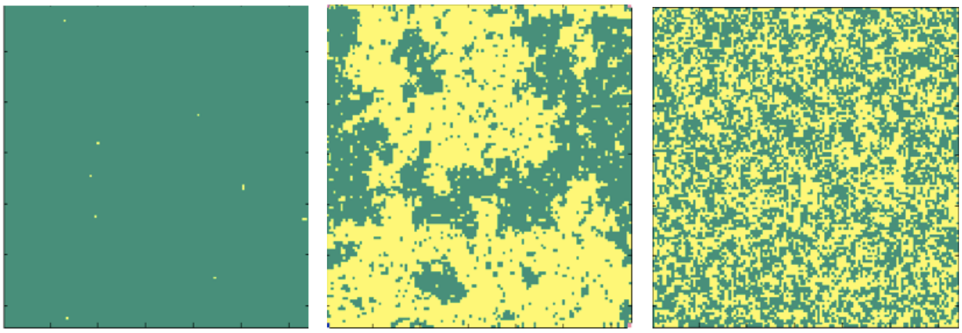

Figure 4.1: Three equilibrium configurations with different temperatures after 10000 time units. From left to right, $T = T_c/8$, $T = T_c$ and $T = 2T_c$.

4 The algorithm for the complexity measure η

In this section, I introduce an algorithm for the complexity measure of two-dimensional Ising model which is designed from two-dimensional information theory.

4.1 Equilibrium configurations

First, I obtain equilibrium configurations of Ising system at specific temperature. For this, I use the Monte Carlo Simulation(MCS)[4]. In this algorithm, I use the $L \times L$ square lattice with the periodic boundary condition.

To get a equilibrium spin-configuration,

1) Start with the random spin-configuration with ± 1. $+1$ is an up-spin and -1 is a down-spin.

2) Choose one site randomly.

3) Change the state of chosen site (i.e. If the spin is up, change to down. If the spin is down, vice versa.).

4) Compute the energy difference ΔE under the change. The energy is given from the Hamiltonian of the model, Eq.(2.2).

5) If $\Delta E \leq 0$, accept the change; otherwise accept the change with probability $e^{-\frac{\Delta E}{k_B T}}$. This probability is obtained from detailed balance condition of the Ising model.

6) Repeat above procedures till the macroscopic variable of the system converges (e.g, magnetization m). One sweep of the above procedure for all

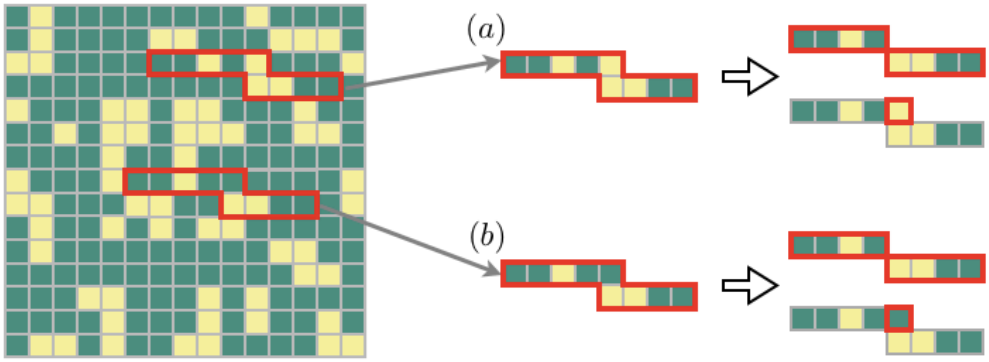

Figure 4.2: An example of different sites with the same block configuration, $B_4 = \{+1, +1, -1, +1, -1, -1, +1, +1\}$. (a) $x = -1$ (b) $x = +1$

$N = L \times L$ spins in the system corresponds to one time unit.

In Fig. 4.1, several configurations are given for different temperature conditions which are obtained from MSC. Dark green sites are up spins and yellow sites are down spins. The leftmost configuration shows ordered configuration in the condition, $T = T_c/8 \approx 1.13$. The middle configuration shows complex configuration which has many clusters with various sizes at critical temperature, $T = T_c \approx 2.27$. And the last figure is disordered configuration with temperature $T = 2T_c \approx 4.54$.

4.2 The statistics of block configurations B_n

From obtained configurations, I get block configurations B_n for each spin. For the convenience to compare each block configuration for statistics, I assign one configuration as one index. The index is defined with binary numbers. For example, in Fig. 4.2, there are two blocks with same configuration. In the stripe, top row has the configuration, $\{+1, +1, -1, +1\}$, and bottom row has one with $\{-1, -1, +1, +1\}$. I combine these two rows as one set and define block configuration as $B_4 = \{+1, +1, -1, +1, -1, -1, +1, +1\}$ where 4 is the distance from the chosen site. From this set, I change -1 to 0 and make binary number as $11010011_{(2)}$. This binary number corresponds to decimal number 211. From this decimal number, I assign the index i_n of this configuration as 212 which is decimal number plus one because I want to make index in the range $i_n \in [1, 2^{2n}]$. As a result, every sites corresponds to one index of specific block configuration. By using this index, I count all the indexes and make a table with all information of block configuration like below.

i_n	$x = +1$	$x = -1$	N_{i_n}
1	$N_{(1,+)}$	$N_{(1,-)}$	N_1
2	$N_{(2,+)}$	$N_{(2,-)}$	N_2
3	$N_{(3,+)}$	$N_{(3,-)}$	N_3
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$2^{2n} - 1$	$N_{(2^{2n}-1,+)}$	$N_{(2^{2n}-1,-)}$	$N_{2^{2n}-1}$
2^{2n}	$N_{(2^{2n},+)}$	$N_{(2^{2n},-)}$	$N_{2^{2n}}$
	N_+	N_-	N_{tot}

$N_{i_n}(i_n = 1, 2, \cdots, 2^{2n})$ is the counted total number of sites with same index i_n, in other words, with same block configuration. Let $N_{(i_n,+)}$ and $N_{(i_n,-)}$ denote the number of sites that spins + and − are drawn with that specific i_nth configuration of B_n. Then, the conditions

$$N_{i_n} = N_{(i_n,+)} + N_{(i_n,-)} \tag{4.1}$$

$$N_+ = \sum_{i_n=1}^{2^{2n}} N_{(i_n,+)} \tag{4.2}$$

$$N_- = \sum_{i_n=1}^{2^{2n}} N_{(i_n,-)} \tag{4.3}$$

$$N_{\text{tot}} = N_+ + N_- = \sum_{i_n=1}^{2^{2n}} N_{i_n} \tag{4.4}$$

have to be satisfied.

4.3 The average entropy of the block configuration H_n

From the Eqs.(4.1-4), I obtain the numerical probabilities,

$$\begin{aligned} p(x=+1) &= \frac{N_+}{N_{\text{tot}}} && (4.5) \\ p(x=-1) &= \frac{N_-}{N_{\text{tot}}} && (4.6) \\ p(B_n) &= \frac{N_{i_n}}{N_{\text{tot}}} && (4.7) \\ p(x|B_n) &= \frac{N_{(i_n,x)}}{N_{i_n}}. && (4.8) \end{aligned}$$

Eqs.(4.5,4.6) give the entropy of the numerical single character distribution. From Eq.(3.6),

$$\begin{aligned} H_0 &= \sum_x p(x) \log \frac{1}{p(x)} \\ &= \frac{N_+}{N_{\text{tot}}} \log \frac{N_{\text{tot}}}{N_+} + \frac{N_-}{N_{\text{tot}}} \log \frac{N_{\text{tot}}}{N_-}. && (4.9) \end{aligned}$$

The average entropy for the block configuration H_n can be obtained as

$$\begin{aligned} H_n &= \sum_{B_n} p(B_n) \sum_x p(x|B_n) \log \frac{1}{p(x|B_n)} \\ &= \sum_{i_n=1}^{2^{2n}} \frac{N_{i_n}}{N_{\text{tot}}} \sum_{x=\{+1,-1\}} \frac{N_{(i_n,x)}}{N_{i_n}} \log \frac{N_{i_n}}{N_{(i_n,x)}} && (4.10) \\ &= \frac{1}{N_{\text{tot}}} \sum_{i_n=1}^{2^{2n}} (N_{i_n} \log N_{i_n} \\ &\qquad -N_{(i_n,+)} \log N_{(i_n,+)} - N_{(i_n,-)} \log N_{(i_n,-)}). && (4.11) \end{aligned}$$

4.4 Obtain k_n and η

Finally, I calculate the correlation information k_n and the measure of complexity η, from the average entropy of the block configuration H_n, Eq.(4.11). I use Eqs.(3.8, 3.9) for k_n and Eq.(3.14) for η.

4.5 Test for checker board with size 2

For testing this algorithm, I use the checker board with size 2, Fig. 4.3. First, the entropy of the single character distribution is simply obtained.

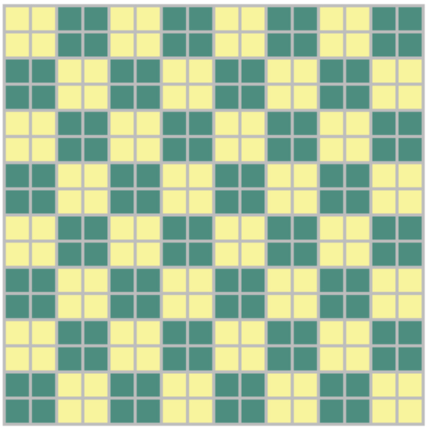

Figure 4.3: The checker board with size 2

The density of up spin and down spin are the same as $\frac{1}{2}$, thus

$$H_0 = \sum_x p(x) \log_2 p(x) = 2 \times \frac{1}{2} \log_2 2 = 1. \tag{4.12}$$

For B_1 case, there are 4 possible block configurations, $\{+1, +1\}, \{+1, -1\}, \{-1, +1\}$, and $\{-1, -1\}$. The probabilities are

$$\begin{aligned} p(B_1) &= \frac{1}{4} \\ p(x|B_1) &= \frac{1}{2} \end{aligned}$$

for every block configurations and x. From probability, H_1 is obtained as

$$\begin{aligned} H_1 &= \sum_{B_1} p(B_1) \sum_x p(x|B_1) \log \frac{1}{p(x|B_1)} \\ &= 4 \times \frac{1}{4} \log_2 2 = 1. \end{aligned} \tag{4.13}$$

For $n = 2$, theoretically there are $2^{2n} = 16$ different block configurations. However, because of the checker board condition, just 8 are possible to observe. For example, $B_2 = \{-1, -1, -1, 1\}$ is impossible to get. All possible 8 block configurations are equally probable. In addition, when I know B_2, x is always determined from B_2. Thus, the probabilities are given as

$$\begin{aligned} p(B_2) &= \frac{1}{8} \quad \text{or} \quad 0 \\ p(x|B_2) &= 0 \quad \text{or} \quad 1 \quad \text{always} \end{aligned}$$

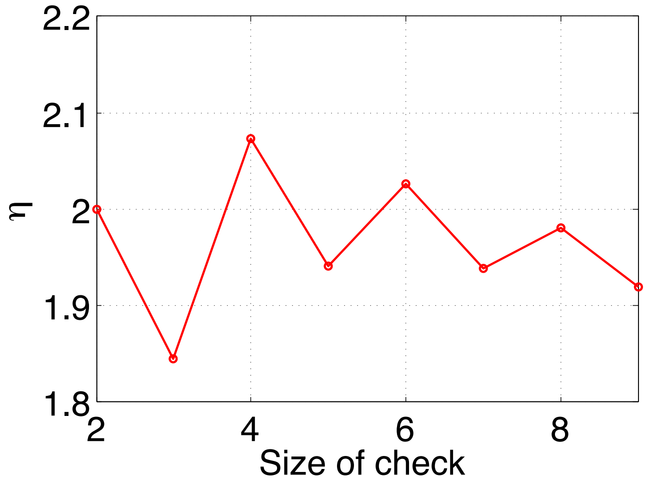

Figure 4.4: The measure of complexity of checker board with various sizes.

H_2 is

$$H_2 = \sum_{B_2} p(B_2) \sum_x p(x|B_2) \log \frac{1}{p(x|B_2)} = 8 \times \frac{1}{8} \log_2 1 = 0. \tag{4.14}$$

To sum up for checker board with size 2, $H_0 = H_1 = 1$, $H_n = 0$ for $(n \geq 2)$, $k_3 = 1$, $k_n = 0$ for $(n \neq 3)$, $\eta = 2$, $k_{\mathrm{corr}} = 1$, and $l_{\mathrm{corr}} = 2$.

I numerically test the checker board with various sizes. The result is given in Fig. 4.4. The obtained numerical result for $n = 2$ agrees with the analytical result. In the figure, when the check board size is odd number, η is smaller than even number cases. Since the length of the block is always an even number, observer can measure just correlation in even length. This implies information of relative position of the chosen site is included in the block configuration. Another thing to note is that the measure of complexity decreases as n increases for even number cases. In the large checker board case, small block can be hidden inside the check shape which makes that the short-distanced correlation information gets smaller compare to small checker board. However, the long-distanced correlation information gets larger for the complexity measure in large checker boards.

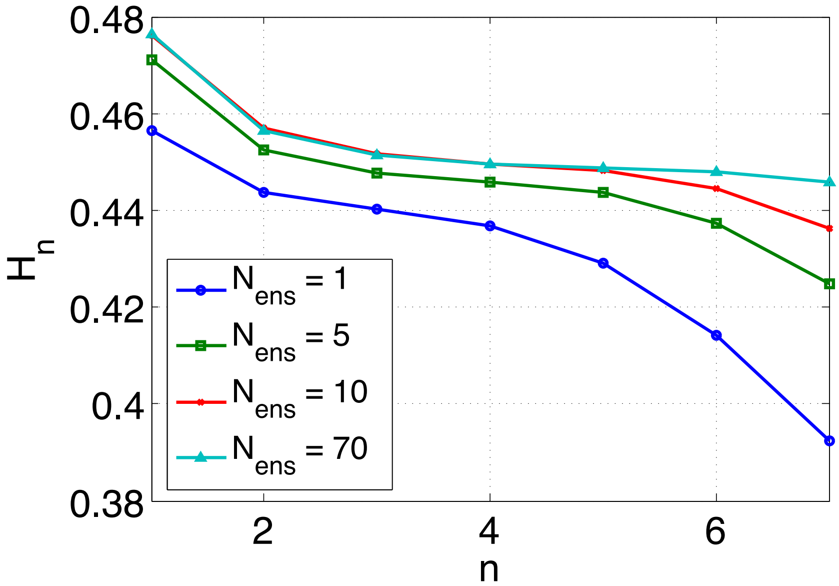

Figure 5.1: H_n vs. n for $L = 128$. I take the ensemble average for different number of ensembles at critical temperature $T = T_c$

5 Numerical results for 2-d Ising model

5.1 The average entropy of the block configuration H_n

I obtain the average entropy of the block configuration H_n from the algorithm which is described in above section. The result is given in Fig. 5.1. In this figure, I plot 4 different results. The results show that the ensemble average is needed to get proper average of entropy. When I take one sample to get H_n, the result does not converge because of statistics. In this process, statistics is the most important to get correct entropy of block configuration. For example, when $n = 6$, there are $2^{12} \approx 4096$ block configurations. However, in one equlibrium state, there are $N_{\text{tot}} = 256^2 = 2^{16}$ samples of configuration. In this case, roughly, we can get just 16 configurations for one index which means that the probability to get one specific configuration is too low (i.e. $p(B_n) \approx 2^{-12} \sim 10^{-4}$) to calculate proper probability distribution. As I increase the number of ensemble, the average entropy converges

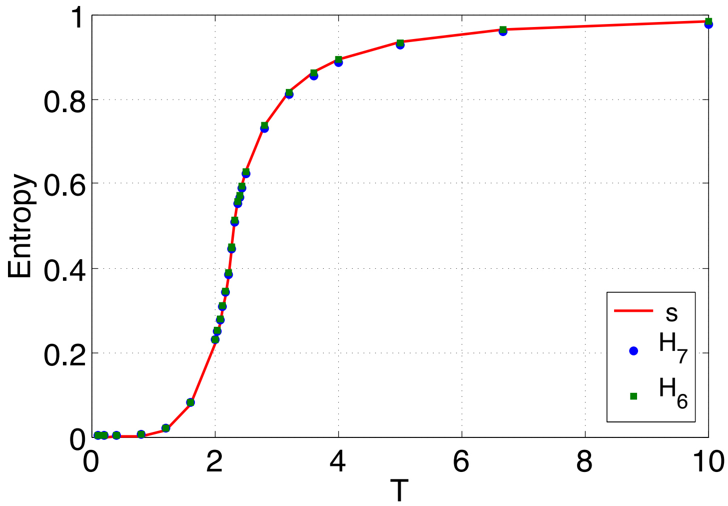

Figure 5.2: The comparison between theoretical real entropy s and numerically obtained the average entropy of the block configuration H_n. Size of system $L = 256$, the number of ensemble is $N_{\text{ens}} = 70$, the number of block configurations of $B_n \approx 4.6 \times 10^6$.

to some value. In Fig. 5.2, I compare numerical result of H_6 and H_7 to the theoretical real entropy s which is given from Onsager's solution, Eq.(2.6). According to the information theory, Eq.(3.8) shows that if the block size is large enough, H_n converges to the real entropy s. In Fig. 5.2, H_6 and H_7 almost converge to real entropy. This implies that $n = 7$ is large enough to examine this two-dimensional Ising model. Thus, the contribution of correlation information k_n for $n > 7$ are negligible to obtain the measure of complexity.

5.2 The correlation information over length n, k_n

From the results of H_n, I plot the correlation information over length n in Fig. 5.3 for different temperature conditions. In this figure, the distribution of correlation information exhibits different behavior before the phase transition and after the phase transition. In the high temperature (T = 4, 10), most of correlation information have low value of k_n. This means that the configuration is almost disordered and correlation information is not dominant. In the low temperature (T = 1.2, 2), the correlation information

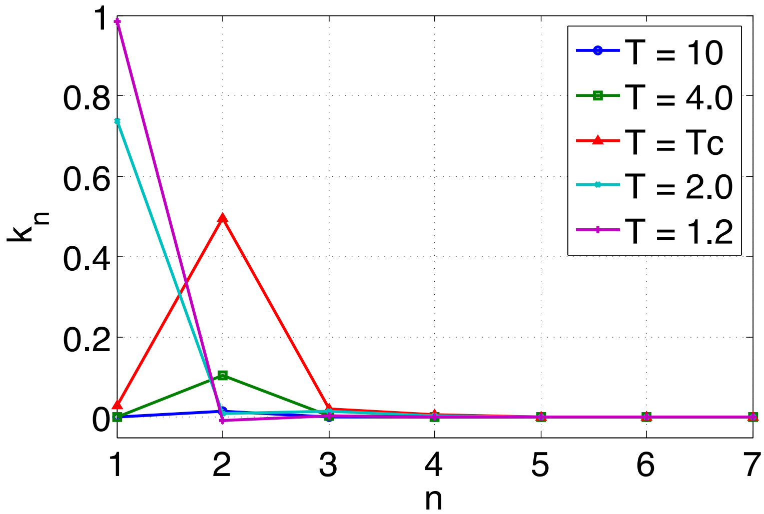

Figure 5.3: The correlation information over length n, k_n for different temperature. Size of system $L = 256$, the number of ensemble is $N_{\text{ens}} = 70$, the number of block configurations of $B_n \approx 4.6 \times 10^6$.

possesses the high value only for $n = 1$. As I mentioned in the previous section, k_1 is the density information of the system. If the temperature is lower than critical temperature T_c, density information is dominant. The interesting thing comes from the critical temperature case. The red line in Fig. 5.3 shows the characteristics of the two-dimensional Ising model. k_2 is the most high value of the correlation information. This implies that contribution of nearest neighbor is a crucial fact to get information on this system. The value of k_2 decreases as temperature increases. In addition, regardless temperature condition, k_n for $n > 5$ go to almost zero. In the Ising system, long-distanced correlation is weaker than short-distanced one. From these results, I argue that the shortest correlation information have an important role in the process of phase transition. In conclusion, the density information and the short-ranged correlation are the most important part of the two-dimensional Ising model.

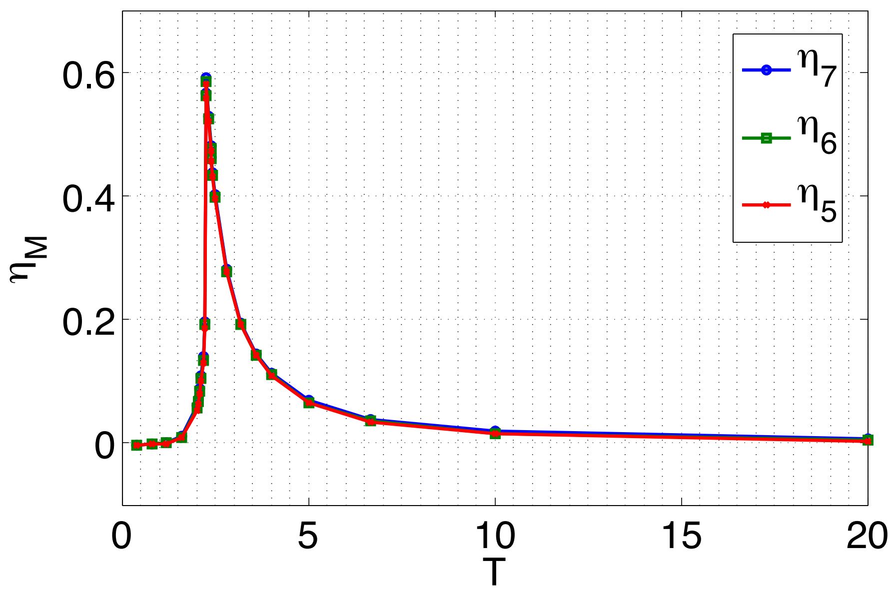

Figure 5.4: The approximated measure of complexity measure η_M with the temperature. Size of system $L = 256$, the number of ensemble is N_{ens}, the number of samples of $B_n \approx 4.6 \times 10^6$.

5.3 The measure of complexity η

From the correlation information k_n, I calculate the measure of complexity η. In the definition of the complexity measure, Eq. (3.15),

$$\eta = \sum_{n=2}^{\infty} (n-1) k_n,$$

the weighted summation over all the ranged correlation. However, because of the limitation of numerical calculation, we need to cut off the summation at reasonable point. From the results of H_n and k_n, k_n is neglected for $n > 8$. I define new approximated η as

$$\eta_M = \sum_{n=1}^{M} (n-1) k_n \tag{5.1}$$

where M is the cut-off distance which gives numerical approximated measure of complexity. In Fig. 5.4, I plot the results of η_M for different M. In this

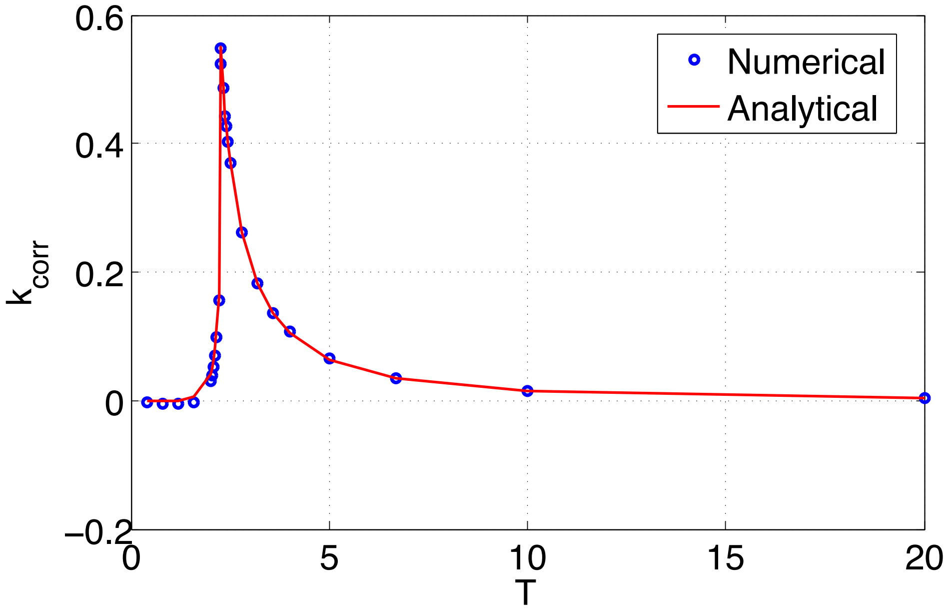

Figure 5.5: The numerical result and the analytical result of correlation information k_{corr}.

figure, the complexity measure is the highest value at critical temperature. In the low temperature, η is almost zero because most of the spins are aligned or ordered. As temperature increases through the critical point, η rapidly increases to the highest value and for higher temperature decays to zero after phase transition.

For comparison with the analytical result, I plot the correlation information in Fig.5.5. Numerical result is obtained from

$$k_{\text{corr}} = \sum_{n=2}^{7} k_n. \tag{5.2}$$

The analytical results are obtained from the Onsager's solution. From the definition of magnetization, the theoretical density information k_1 can be obtained as

$$p(x=+1) = \frac{N_+}{N} = \frac{1+m}{2}\ , \quad p(x=-1) = \frac{N_-}{N} = \frac{1-m}{2} \tag{5.3}$$

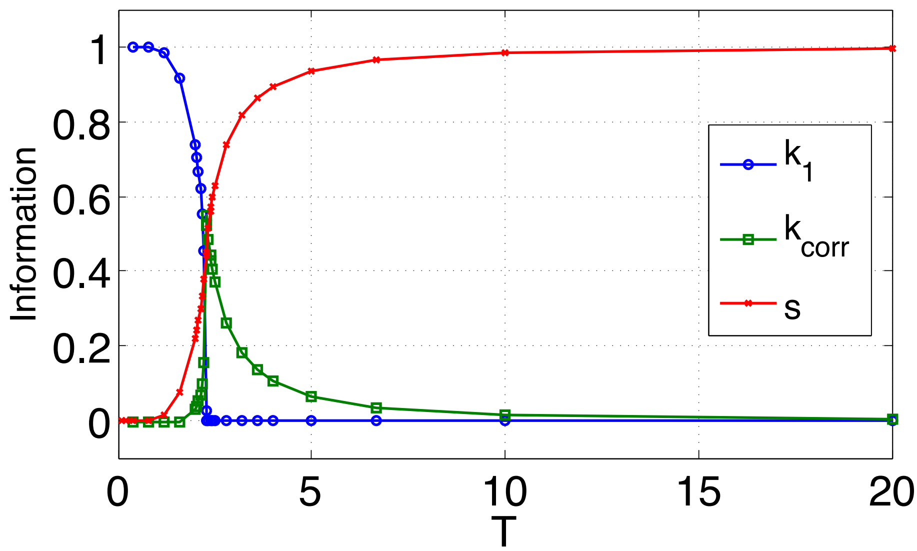

Figure 5.6: Decomposition of information for two-dimensional Ising model.

$$\begin{aligned} k_1 &= 1 - H_0 \\ &= 1 - \left(-\frac{1+m}{2} \log \frac{1+m}{2} - \frac{1-m}{2} \log \frac{1-m}{2} \right). \end{aligned} \tag{5.4}$$

The analytic result of correlation information is derived as

$$k_{\text{corr}} = 1 - s - k_1 \tag{5.5}$$

where s from Eq.(2.6), k_1 from Eq.(5.4). In the figure, numerical results are well fitted with analytic results. In Fig.5.6, I plot the decomposition of information for three parts, the density, the correlation and the uncertainty. In this figure, the correlation information is the highest at the critical point. This can be explained as that correlation information has to be increased for the qualitative change, the phase transition.

6 Conclusions

In summary, I have studied a complexity measure of two-dimensional Ising model. The Monte Carlo Simulation(MCS) has been used for obtaining equilibrium spin-configurations. From obtained configurations for different temperature condition, I calculate the effective measure of complexity η which is derived from two-dimensional information theory. Total information of the system can be divided to three parts, the density, the correlation, and the uncertainty. From this, the phase transition can be characterized in information theoretical way. When temperature is smaller than critical temperature T_c, η is small and the density information is dominant in the total information of the system. As temperature increases to T_c, the density information decreases and uncertain information, Shannon entropy increases. Near T_c, the correlation information rapidly increases for phase transition and density information gets changed to information of uncertainty. At T_c, the correlation information has the highest value and decreases as temperature increases. At high temperature, the uncertainty information becomes dominant on the system. The importance of the contribution of nearest neighbor has been pointed out in the two-dimensional Ising model. Most of the correlation information comes from nearest neighbors. Long-distanced correlation can be negligible. The measure of complexity η shows that the complexity is the highest at T_c which implies that complexity is important to make qualitative change.

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