Zero-range process with random interaction

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The zero-range process is a stochastic interacting particle system that under certain conditions exhibits a phase separation. The homogeneous model (i.e. when the interaction is homogeneous) is well understood and there are analytical expressions for most thermodynamical variables. However, if we introduce a small perturbation in the interaction the problem becomes much more complex and an analytical approach is harder. In this work we first provide numerical results that allow us to understand in an intuitive manner the behaviour of the system. Later we prove analytically the existence of expected values of the free energy and critical density. We also provide an expansion that allows us to predict the system's behaviour when the perturbation is small.

PACS numbers:

I. INTRODUCTION

The zero-range process is a stochastic lattice gas where the particles hop randomly with an on-site interaction that makes the jump rate dependent only on the local particle number. It was introduced in [1] as a mathematical model for interacting diffusing particles, and since then has been applied in a large variety of contexts, often under different names, (see e.g. [2] and references therein). The model is simple enough for the steady state to factorize, on the other hand it exhibits an interesting condensation transition under certain conditions. Viz. when the particle density exceeds a critical value ρ_c the system phase separates into a homogeneous background with density ρ_c (i.e. the fluid phase) and all the excess mass concentrates on a single lattice site, thus representing a classical real-space analogue of Bose-Einstein condensation. This has been observed and studied in some detail in experiments on shaken granular media [3, 4]. It is also relevant as a generic mechanism for phase separation in single-file diffusion [5] and condensation phenomena in many complex systems such as network rewiring [6] or traffic flow [7], for a review see [2].

In the homogeneous system, if the jump rates $g(n)$ have a decreasing tail for increasing number n of particles, the transition can be caused by the growth of large clusters at the expense of small clusters. Such a model with a generic power law decay

$$g(n) = 1 + b/n^{\gamma} \tag{1}$$

with positive interaction parameters b, γ has been introduced in [9]. Condensation occurs if $0 < \gamma < 1$ and $b > 0$ or if $\gamma = 1$ and $b > 2$.

The condensation transition is hence well understood on a mathematically rigorous level in the context of the equivalence of ensembles [10, 11], and many variants of (1) have been studied [2, 12–16]. However the assumption of strict spatial homogeneity is not desirable for applications to real complex systems in which there may be some disorder due to local imperfections.

It has been shown recently that a small perturbation of the jump rates has a drastic effect on the critical behaviour. It is now clear that the condensation occurs if $0 < \gamma < \frac{1}{2}$ and $b > 0$[23] but little is known about the distributions of the thermodynamical variables of the system. In the present work we try to shed some light upon that last point.

The paper is organized as follows: first we define the zero-range process with random interaction and the quantities needed to characterize it. In section III we cover some numerical issues about the computational results and their validity. In section IV we present and confirm, by numerical means, some previous analytical results by Stefan Grosskinsky regarding the existence of the expected values of both the critical free energy and critical density. In section V we provide an expansion to compute the value of thermodynamical variables and their distributions for small noise supported with numerical results.

II. DEFINITIONS

The ZRP with disordered interaction

We consider a lattice Λ_L, which we take to be periodic and of finite size $|\Lambda_L| = L$. A configuration is denoted by $(\eta_x)_{x\in\Lambda}$ where $\eta_x \in \{0, 1, \ldots\}$ is the occupation number at site x. The dynamics is defined in continuous time, such that with rate $g_x(\eta_x)$ site $x \in \Lambda_L$ loses a particle, which moves to a randomly chosen target site y according to some probability distribution $p(y - x)$. For example in one dimension with nearest neighbour hopping, the particle moves to the right with probability p and to the left with $1-p$. The jump rate g_x depends only on the configuration of the starting site and we take it to be

$$g_x(n) = e^{\sigma\xi_x(n)+b/n^\gamma} \quad \text{for } n \geq 1 \ , \quad g(0) = 0 \tag{2}$$

where σ, b and γ are positive constants and $\xi_x(n)$, $x \in \Lambda$, $n \in \mathbb{N}$ are iidrv's with

$$\mathbb{E}\xi_x(n) = 0 \ , \quad \mathbb{E}\xi_x(n)^2 = 1 \ , \quad \text{and} \quad \delta := \log \mathbb{E}e^{-\sigma\xi_x(n)} < \infty \ . \tag{3}$$

By Jensen's inequality and strict convexity of the logarithm we have $\delta > 0$. Note that for $\sigma = 0$, the asymptotic behaviour of the jump rates is given by (1), so the present model can be interpreted as a perturbation of the generic homogeneous model. The same model has been introduced in [23].

For negative γ the rates are increasing in n for positive b and hence there is no condensation. For negative b the rates tend to zero, which means that there is condensation with critical density $\rho_c = 0$. This is an essentially trivial feature of the model which is robust against perturbation by disorder. We therefore focus on positive interaction exponent γ and $b > 0$.

Thermodynamical variables

It is well known (see e.g. [2, 17]) that the fluid phase has a grand-canonical factorized steady state $\nu_\mu^L = \prod_{x\in\Lambda_L} \nu_\mu^x$ with single-site marginal

$$\nu_\mu^x(n) \ = \ \frac{e^{n\,\mu}}{z_x(\mu)} w_x(n) \tag{4}$$

where the chemical potential $\mu \in \mathbb{R}$ fixes the particle density and the stationary weights are given by the jump rates via

$$w_x(n) = \prod_{k=1}^{n} g_x(k)^{-1} = \exp\Big(-\sigma S_x(n) - \beta(n)\Big) \ . \tag{5}$$

Here $S_x(n) = \sum_{k=1}^n \xi_x(n)$ can be interpreted as the position of a random walk on $\mathbb{R}$ after n steps with independent increments. This holds independently of the distribution of target sites $p(y - x)$ and for each realization of the $\xi_x(k)$, i.e. ν_μ^L is a *quenched* distribution. The other term in the exponent is $\beta(n) = \sum_{k=1}^n b/k^\gamma$ and acts as a drift. When the parameters are in the ranges for which there is condensation the drift eventually sends the exponent to $-\infty$. Finally, σ determines the *size* of the perturbation. Just to provide some intuition we could say that the perturbation is *big* if $\sigma \gg b$ and *small* if $\sigma \ll b$.

The single-site normalization is given by the partition function

$$z_x(\mu) = \sum_{n=0}^{\infty} \exp\Big(n\,\mu - \sigma S_x(n) - \beta(n)\Big) \tag{6}$$

which is strictly increasing and convex in μ [10]. Note that the exponent is a random walk with a time dependent drift $n\mu - \beta(n)$. For the existence of $z_x(\mu)$ it is necessary that the drift dominates over the stochastic part $\sigma S_x(n)$.

We denote

$$f_x(\mu) = \log z_x(\mu) \ . \tag{7}$$

However, global averaged quantities are more relevant for the characterization of the system. The way in which the average is done is of major importance since it leads to two quantities with different meanings, the annealed an quenched free energies. The annealed free energy is

$$f_A(\mu) = \log \mathbb{E} z_x(\mu) = \log \sum_{n=0}^{\infty} \exp\left(n\mu + n\delta - \beta(n)\right) \ , \tag{8}$$

where $\mathbb{E}[\cdot]$ means expected value with respect to all possible realizations of the noise $\sigma S_x(n)$. f_A can be interpreted as the free energy of a site with an *average* perturbation and it can be rewritten as a translation of the homogeneous free energy

$$f_A(\mu) = f(\mu + \delta) \ , \tag{9}$$

where $f(\mu)$ and $z(\mu)$ (and in general all quantities with no subscript) refer to the homogeneous system's variables, i.e. with $\sigma = 0$. Given that $z(\mu)$ is monotone increasing in μ, $f(\mu)$ also is and hence $f_A(\mu) > f(\mu)$ for all μ.

On the other hand, we have the quenched free energy

$$f_Q(\mu) = \mathbb{E} \log z_x(\mu) \ , \tag{10}$$

which is the average of f_x and is the physically relevant quantity as it is the systems' free energy in the thermodynamical sense. By Jensen's inequality we always have $f_A(\mu) \geq f_Q(\mu) \geq f(\mu)$ so f_A and f are upper and lower bounds for f_Q.

The local density can be calculated as usual as a derivative of the free energy

$$\rho_x(\mu) = \frac{\partial f_x(\mu)}{\partial \mu} \tag{11}$$

and it is also a strictly increasing function of μ.

As it is shown later, the grand-canonical partition function $z_x(\mu)$ does not exist for $\mu > 0$. Evaluating the quantities defined above in $\mu = 0$ one obtains the critical values. In particular, the average critical density is

$$\rho_c = \mathbb{E}\left[\rho_x(0)\right] \tag{12}$$

which is finite almost surely (*a.s.*) (see proposition 1).

III. NUMERICS

The main issue generating numerical results for this model is to compute infinite sums. The convergence of $z_x(\mu)$ (and hence of $f_x(\mu)$) is granted by the analytical results shown in section IV, however in the critical point the effect of the noise is maximal and the convergence is very slow. Because of this reason, it is useful to define and analyze *truncated* quantities

$$z_x^N(\mu) \ = \ \sum_{n=0}^{N} \exp\Big(n\,\mu + \sigma S_x(n) - \beta(n)\Big) \ , \tag{13}$$

$$f_x^N(\mu) \ = \ \log z_x^N(\mu) \tag{14}$$

and so on. By definition

$$\lim_{N\to\infty} f_x^N(\mu) = f_x(\mu) \ , \tag{15}$$

and we expect $f_x^{N+1}(\mu) - f_x^N(\mu)$ to decrease, at least in average, for increasing N. We want to find a $\bar{N}$ such that $f_x(\mu) - f_x^{\bar{N}}(\mu)$ is irrelevant compared to the numerical precision. As $f_x^N(\mu)$ are random variables, different

realizations of $S_x(n)$ will make the sum converge at different rates. To have a good estimate for $\bar{N}$ we have to average over several samples

$$\lim_{N\to\infty} \langle f_x^{N+1}(\mu)\rangle_L - \langle f_x^N(\mu)\rangle_L = 0 \tag{16}$$

where $\langle\ \cdot\ \rangle_L$ is the quenched average over L realizations of S_x and $\langle f_x^N(\mu)\rangle_L$ is a strictly increasing function of N. As $f_x^N(\mu)$ is a selfaveraging quantity, we can use $\langle\ \cdot\ \rangle_L$ as an estimator for $\mathbb{E}[\cdot]$. For the quenched free energy we have

$$\lim_{\substack{N\to\infty\\ L\to\infty}} \langle f_x^N(\mu)\rangle_L = f_Q(\mu)\ . \tag{17}$$

Note that f_Q is finite (see Theorem 3) and both f_x^N and $\mathbb{E}[f_x^N]$ are monotone increasing in N therefore the limits for N and L commute.

The optimal truncation point $\bar{N}$ has to be chosen such that $\langle f_x^N\rangle_L$ is, up to numerical precision, equal to $\langle f_x^{\bar{N}}\rangle_L$ for all $N > \bar{N}$. To find $\bar{N}$ we sample $\langle f_x^N(\mu)\rangle_L$ for increasing values of N until the previous condition is satisfied as shown in figure 1. A similar process is done for L (see figure 2).

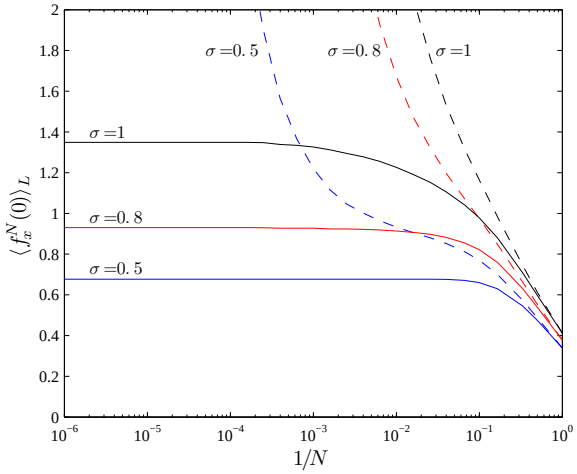


FIG. 1: Convergence of $\langle f_x^N(0)\rangle_L$ for increasing N with $\gamma = 0.4$ (solid line) and $\gamma = 0.6$ (dashed line). As we could expect, $\langle f_x^N(0)\rangle_L$ diverges for $\gamma > 1/2$. Instead, for $\gamma < 1/2$ it increases monotonically up to an upper bound which is the numerical result for $f_Q(0)$. Note that for $\gamma < 1/2$ the smaller σ is, the faster the convergence is. On the other hand, for $\gamma > 1/2$ the smaller σ is, the slower the divergence is. Here $b = 1$ and $L = 10^4$.

As the convergence is slower the larger μ is and here we have used the critical value $\mu = 0$, we expect all sums with $\mu \leq 0$, $\sigma \leq 1$, $\gamma = 0.4$ and $b = 1$ to converge for $N = 10^5$ and $L = 10^4$. Those are the parameters used in all simulations except where the contrary is explicitly stated. On the other hand, the convergence becomes slower as σ increases. A lower estimate for $\bar{N}$ with $\mu = 0$ is the point where the drift $\beta(n)$ becomes greater than the random walk $S_x(n)$ and thus successive terms of the sum start decreasing. For big n we can approximate $\beta(n) \approx \frac{b}{1-\gamma} n^{1-\gamma}$ and a random walk goes as $|\sigma S_x(n)| \sim \sigma\sqrt{n}$. This gives us an exponential relation

$$\bar{N} \gtrsim \sigma^{\frac{2}{1-2\gamma}} \tag{18}$$

that grows very fast for increasing σ. Hence obtaining numerical results for big perturbations is computationally expensive. Here we will focus on $\sigma \leq 1$ which is already big enough for our study.

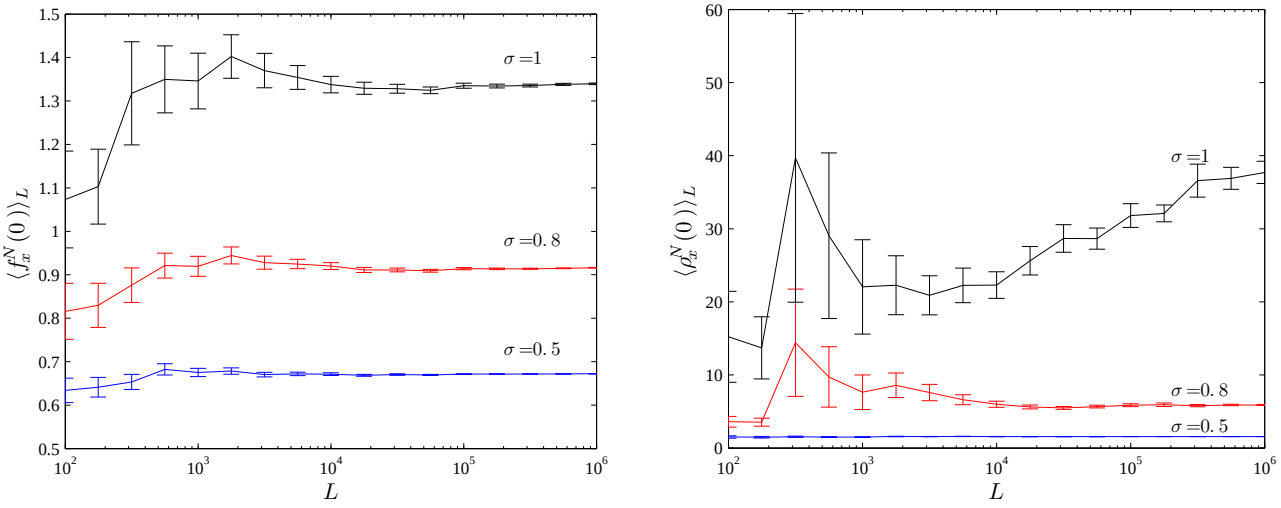


FIG. 2: Convergence of (a) $\langle f_x^N(0)\rangle_L$ and (b) $\langle \rho_x^N(0)\rangle_L$ for increasing L. The error bars represent the standard error of the mean. For small L very few rare and large values of f_x and ρ_x dominate the average but when the sampling is big enough these large contributions are compensated. Here $\gamma = 0.4$, $b = 1$ and $N = 10^5$.

IV. EXISTENCE OF EXPECTED VALUES

We now provide some analytical results regarding the existence of the thermodynamical variables defined in section II. The proofs can be found in the appendix A.

Proposition 1 *$z_x(\mu)$ and $\rho_x(\mu)$ are a.s. smooth functions of μ for $\mu < 0$. For $\gamma \in [1/2, 1)$ we have*

$$z_x(\mu) < \infty \text{ a.s.} \quad \text{iff } \mu < 0 \quad \text{and} \quad z_x(\mu) \to \infty \text{ a.s.} \quad \text{as } \mu \nearrow 0 \ . \tag{19}$$

For $\gamma \in (0, 1/2)$ we have

$$z_x(\mu) < \infty \text{ a.s.} \quad \text{iff } \mu \leq 0 \quad \text{and} \quad z_x(\mu) \to z_x(0) < \infty \text{ a.s.} \quad \text{as } \mu \nearrow 0 \ . \tag{20}$$

The same statements, respectively, hold for $\rho_x(\mu)$ and all higher moments, in particular, for the site dependent critical density

$$\rho_x(0) < \infty \text{ a.s.} \quad \text{iff} \quad \gamma \in (0, 1/2) \ . \tag{21}$$

The fact that the partition function does not exist for $\mu > 0$ means that the chemical potential can not be increased indefinitely. The chemical potential regulates the density thus the maximum density will be $\rho_x(\mu = 0)$. We have different behaviours for $\gamma \in [1/2, 1)$ and $\gamma \in (0, 1/2)$.

When $\gamma \in [1/2, 1)$ as μ approaches 0 the density diverges and for any real positive value of ρ we can find the corresponding chemical potential. This means that we can increase the the density of the fluid phase as much as we want and the whole system will be in the fluid phase so there is not phase separation.

On the other hand, for $\gamma \in (0, 1/2)$, the fluid phase density is bounded. If we increase the system's density above the critical value the excess mass can't belong to the fluid phase and there must be a different phase containing it.

Proposition 2 *Let $\gamma \in (0, 1/2)$. For the annealed free energy we have*

$$f_A(\mu) < \infty \quad \Leftrightarrow \quad \mu \leq -\delta \ . \tag{22}$$

For $\gamma \in [1/2, 1]$ the same holds with $\mu < -\delta$.

Proof $\mathbb{E}z_x(\mu) = \sum_n^\infty \mathbb{E}w_x(n) = \sum_n^\infty \mathbb{E}(e^{\xi_x(1)})^n e^{-\beta(n)+\mu n}$ □

Theorem 3 *Let $\gamma \in (0, 1/2)$. For the quenched free energy we have*

$$f_Q(\mu) < \infty \quad \Leftrightarrow \quad \mu \leq 0 \quad \textit{and} \quad \rho_c < \infty \; . \tag{23}$$

The last result implies that there exists a normalizable probability density function for f_x. Knowing that distribution would be extremely useful since most of the systems' features and properties could be derived from it. However, it is very hard to find analytical formulas even for its moments. In the next section we will provide some approximations that predict the probability density function of f_x and its expected value for small perturbations.

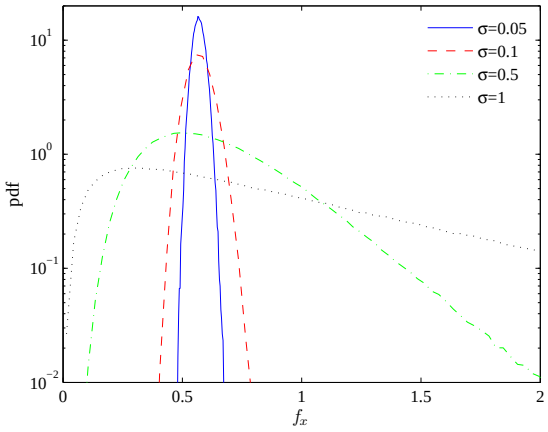
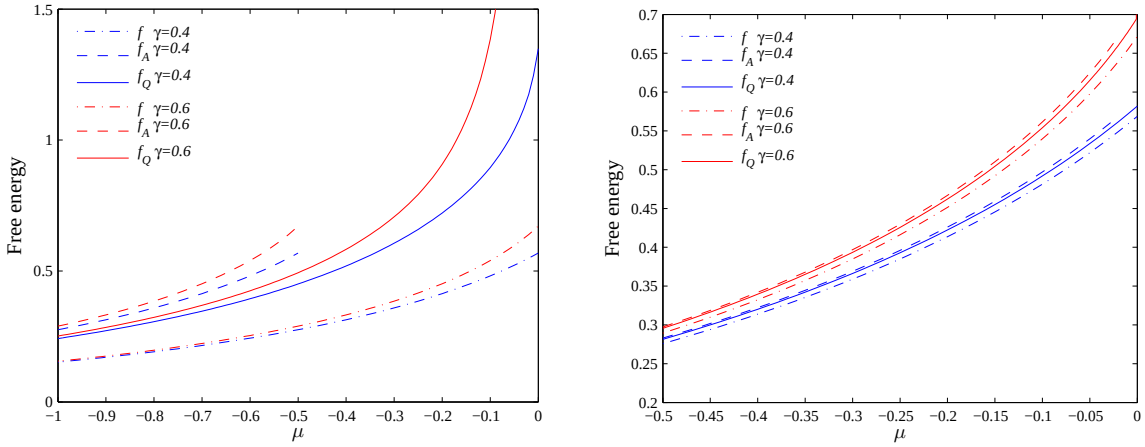


FIG. 3: Numerical results for the probability density function of f_x with Gaussian $\xi_x(n)$ and different σ. Here $\mu = 0$, $b = 1$, $\gamma = 0.4$ and $N = 10^5$. For each value of σ 10^4 samples were taken.



(a) $\sigma = 1$ (b) $\sigma = 0.2$

FIG. 4: Quenched, annealed and unperturbed free energies for $\gamma = 0.4$ and $\gamma = 0.6$. We can see that always $f_A \geq f_Q \geq f$. Also we can see that for $\gamma = 0.6$ f_Q diverges as $\mu \to 0$. For $\sigma = 0.2$ the divergence occurs so close to the origin that it is imperceptible in the plot. Each point of f_Q is the average of 10^4 samples of f_x.

V. EXPANSION FOR SMALL NOISE

The effects of noise on the thermodynamical variables are difficult to predict, however expanding the partition function for $\sigma \to 0$ one can find approximations that prove to be accurate in a range of values of σ depending on the other parameters. For $\gamma < 0.5$, $\mu \leq \mu_c$ and for all σ, as defined in (6), we have

$$z_x(\mu,\sigma) = \sum_{n=0}^{\infty} e^{n\mu - \sigma S_x(n) - \beta(n)} < \infty \quad \text{a.s.} \tag{24}$$

where we now stress the dependence on sigma. The above is continuously differentiable in σ. Recalling that $z(\mu) = \sum_{n=0}^{\infty} e^{n\mu-\beta(n)}$, we can define a probability measure $p(n)$ where the random variable X takes the value $S_x(n)$ with probability $p(n) = \frac{e^{n\mu - b_n}}{z(\mu)}$ and $E_p[\cdot]$ is the expected value with respect to the probabilities $p(n)$. Therefore

$$\frac{\partial^m z_x(\mu,\sigma)}{\partial \sigma^m}\Big|_{\sigma=0} = z(\mu) \sum_{n=0}^{\infty} S_x(n)^m p(n) = z(\mu) E_p[X^m] \ . \tag{25}$$

Now for a fixed realization of the noise $\sigma S_x(n)$ expanding around $\sigma = 0$ we have

$$z_x(\mu,\sigma) = z(\mu) \sum_{n=0}^{\infty} \frac{\sigma^n}{n!} E_p[X^n]. \tag{26}$$

According to proposition 1 all moments of $z_x(\mu)$ are finite so $E_p[X^n] < \infty$ a.s. for all n and hence $z_x(\mu)$ and $f_x(\mu)$ are analytic functions. That guarantees the convergence of the expansion.

Free energy distribution

Using the expansion above, the logarithm of z_x is

$$\log\big(z_x(\mu,\sigma)\big) = \log\big(z(\mu)\big) + \log\Big(1 + \sum_{n=1}^{\infty} \frac{\sigma^n}{n!} E_p[X^n]\Big) \tag{27}$$

and expanding the second logarithm in the right hand side up to second order in σ we obtain

$$f_x(\mu,\sigma) = f(\mu) + \sigma E_p[X] + \frac{\sigma^2}{2}\big(E_p[X^2] - E_p[X]^2\big) + o(\sigma^2) \ . \tag{28}$$

As f_x is analytic, we can assure that as $\sigma \to 0$ terms $o(\sigma^2)$ will go to 0 faster than the rest of the expansion. The probability density function of f_x for $\sigma \to 0$ is given by the first order term

$$\begin{aligned} E_p[X] \ &= \ \sum_{n=0}^{\infty} S_x(n)p(n) = 0 + \xi_x(1) \sum_{n=1}^{\infty} p(n) \\ &\quad + \xi_x(2) \sum_{n=2}^{\infty} p(n) + \ldots = \sum_{k=1}^{\infty} \xi_x(k) \bar{F}_\mu(k) \ , \end{aligned} \tag{29}$$

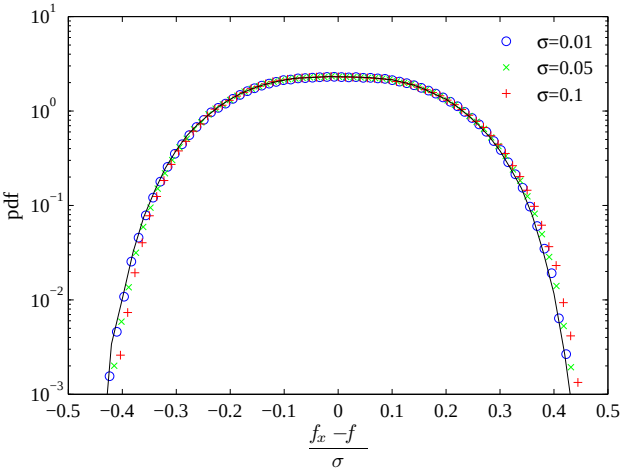
where $\bar{F}_\mu(k) = \sum_{n \geq k} p(n)$ is the tail of the distribution $p(n)$. The interpretation of this result is that for small σ the deviation of $f_x(\mu,\sigma)$ from the unperturbed systems' free energy $f(\mu)$ scales with σ. In general, as $\sigma \to 0$,

$$\frac{f_x(\mu,\sigma) - f(\mu)}{\sigma} \to \sum_{k=1}^{\infty} \xi_x(k) \bar{F}_\mu(k) \ , \tag{30}$$

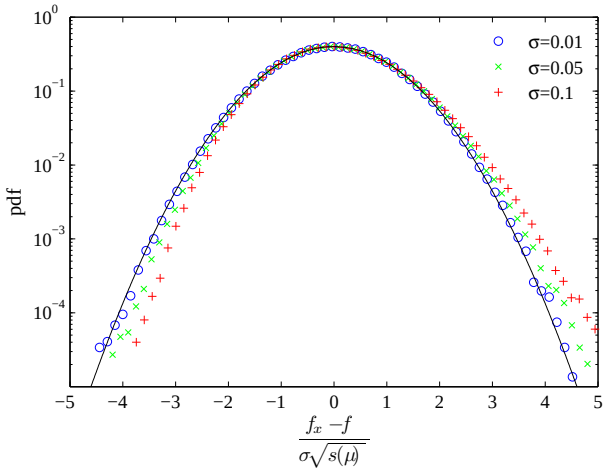
In the particular case that $\xi_x(n)$ are Gaussian this is a linear combination of independent Gaussians, and therefore,

$$\frac{f_x(\mu,\sigma) - f(\mu)}{\sigma\sqrt{s(\mu)}} \to N(0,1) \quad \text{with} \quad s(\mu) = \sum_{k\geq 1} \bar{F}_\mu(k)^2 < \infty \ , \tag{31}$$

and the fluctuations are Gaussian.



(a) Uniform noise



(b) Gaussian noise

FIG. 5: Rescaled probability density functions for uniform and Gaussian $\xi_x(n)$. In (a) the solid line corresponds to the normalized histogram of the distribution given in (30), in (b) it is a Gaussian pdf. The data for larger σ is skewed towards positive values and that is the behaviour we expect for large perturbations. Here $\mu = 0$ and the effect of the perturbation is more relevant than for $\mu < 0$ so we expect a better collapse for smaller μ. Here 10^6 samples were taken for each value of σ.

Expected values

Using the same expansion one can estimate $f_Q = \mathbb{E}[f_x]$ and $\rho(\mu,\sigma) = \mathbb{E}[\rho_x(\mu)]$. First, we have that $\mathbb{E}[S_x(n)] = 0$ and also $\mathbb{E}[S_x(n_1)S_x(n_2)S_x(n_3)] = 0$ and so on for all odd powers. Therefore we only have even powers and the expansion of the quenched free energy is

$$f_Q = \mathbb{E}[\log z_x(\mu,\sigma)] = f + \frac{\sigma^2}{2}\left(\mathbb{E}\left[E_p[X^2]\right] - \mathbb{E}\left[E_p[X]^2\right]\right) + o(\sigma^3) \ . \tag{32}$$

The main difficulty now is to compute the expected values of even powers of the random walks $S_x(n)$. After some calculations (see appendix B) we can rewrite the quenched free energy and the density as

$$f_Q(\mu) = f(\mu) + \sigma^2\phi(\mu) + o(\sigma^3) \tag{33}$$

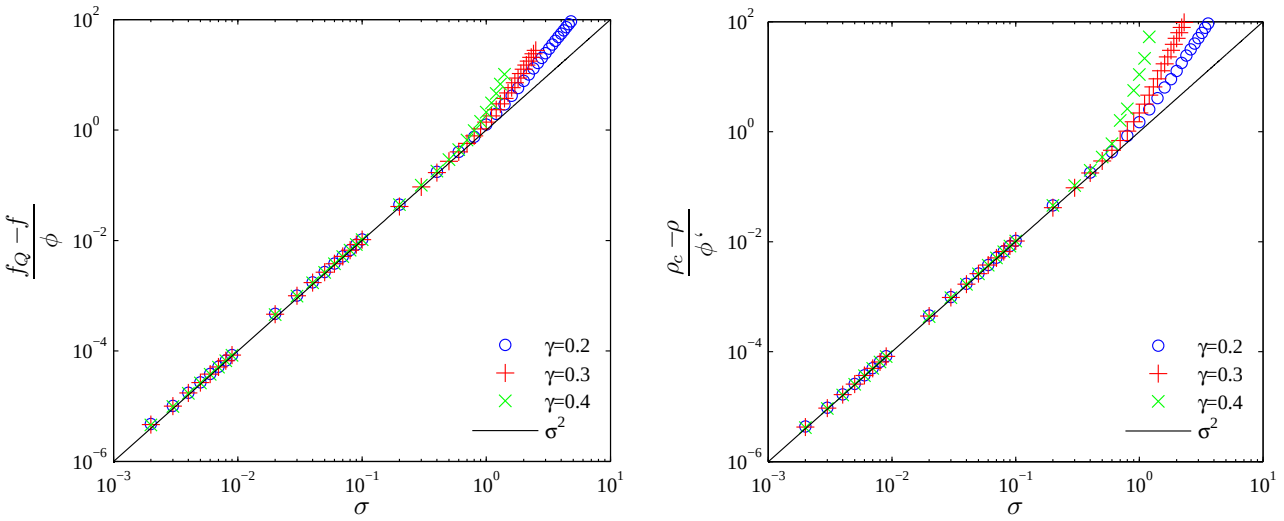
$$\mathbb{E}[\rho_x(\mu,\sigma)] = \rho(\mu) + \sigma^2\phi'(\mu) + o(\sigma^3) \tag{34}$$

where

$$\phi(\mu) = \frac{1}{2}\sum_{n=0}^{\infty} \bar{F}_\mu(n)(1 - \bar{F}_\mu(n)) \tag{35}$$

$$\phi'(\mu) = \frac{\partial\phi(\mu)}{\partial\mu} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{\partial\bar{F}_\mu(k)}{\partial\mu}(1 - 2\bar{F}_\mu(k)) \tag{36}$$

are coefficients that can be easily computed and are totally deterministic. Note that unlike the expressions for the probability density functions of $f_x(\mu)$, these results do not depend on the distribution of $\xi_x(n)$.



(a) $f_Q(0)$ (b) $\rho_c(0)$

FIG. 6: Data collapse of the expansions of $f_Q(0)$ and $\rho_c(0)$. The expansion up to second order is very accurate for $\sigma < 0.5$. For $\sigma > 1$ higher order terms become relevant. Here $N = 10^5$, $L = 10^5$.

A fourth order term can be computed. For the coefficient of the σ^4 term the cumulant expansion gives:

$$\frac{1}{24}\mathbb{E}\left[\Big(-6E_p[X]^4 + 12E_p[X]^2E_p[X^2] - 3E_p[X^2]^2 - 4E_p[X]E_p[X^3] + E_p[X^4]\Big)\right] \tag{37}$$

and we now have to compute terms of the form $\mathbb{E}[S_x(n_1)S_x(n_)S_x(n_3)S_x(n_4)]$. Using nested conditional distributions we obtain

$$\mathbb{E}[S_x(n_1)S_x(n_2)S_x(n_3)S_x(n_4)] = 3n_1^2 + n_1(n_2 - n_1) + n_1(n_3 - n_2) = 2n_1^2 + n_1n_3 \ . \tag{38}$$

It is difficult to find a closed analytical formula for the coefficient, but it can be computed numerically. However, a fourth order expansion does not predict the curves obtained numerically. This can be expected since the slopes of the curves in figure 6 are different for $\sigma \gtrsim 1$ so there are different contributions of several terms $o(\sigma^2)$.

VI. DISCUSSION

In this work we have deepened the knowledge of the zero-range process with disordered interaction. We have studied the model within the frame of the grand canonical ensemble because its partition function its well defined and provides a very good starting point. The main goal of this project was to generate good numerical results and simple analytical expressions for the main system's thermodynamical variables.

We first focused on producing reliable numerical results. This implies a deep analysis of the partition function an its numerical properties. We can now be confident about the values obtained numerically in a large range of the system's parameters.

We have also presented some results by Stefan Grosskinsky concerning the system's partition function, quenched and annealed free energies and densities (site dependent and critical density) and we have checked that the numerical values confirm the analytical predictions.

Finally we provide some expressions for approximations derived from a Taylor expansion of the partition function. We can predict very accurately the probability density function of f_x for $\sigma < 0.1$ and we have expressions for f_Q and ρ_c that predict the actual values with great precision for $\sigma < 0.5$.

It would be very interesting to shed some light upon the behaviour for large perturbations since it would give a broad understanding of the model, however it is still a difficult problem to tackle. It would also be interesting to analyze how the distribution of $\xi_x(n)$ affects the model. Finally, how the variables diverge for $\gamma \in [1/2, 1)$ is a question of major interest since in finite systems metastable condensations can occur in that range.

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APPENDICES

A. Proofs of Proposition 1 and Theorem 3 from section IV.

Proof of Proposition 1 The law of the iterated logarithm holds.

$$\limsup_{n\to\infty} \frac{|\sigma S_x(n)|}{\sqrt{n\log\log n}} = \sigma\sqrt{2} \quad a.s. \tag{39}$$

see e.g. [18]. Further, we write

$$\beta(n) = \sum_{k=1}^{n} \frac{b}{k^\gamma} \simeq \frac{b}{1-\gamma}\, n^{1-\gamma} \quad \text{as } n\to\infty\ . \tag{40}$$

We thus have $\beta(n) \gg \sigma S_x(n)$ $a.s.$ as $n\to\infty$ if and only if $\gamma\in(0,1/2)$, which implies

$$\sigma S_x(n) - c\beta(n) \to -\infty \ a.s. \quad \text{and} \quad n^q e^{\sigma S_x(n) - c\beta(n)} \to 0 \ a.s. \tag{41}$$

for all $c>0$, $q>0$. This implies in particular that $w_x(n)\to 0$ $a.s.$ as $n\to\infty$ and with $z_x(0)=\sum w_x(n)$ we get

$$\mathbb{P}(\exists\text{finitely many } n : w_x(n) > e^{-\beta(n)}) = 1 \quad \Rightarrow \quad z_x(0) < \infty \ a.s.\ . \tag{42}$$

The same holds for higher moments

$$\mathbb{P}(\exists\text{finitely many } n : n\, w_x(n) > e^{-\beta(n)}) = 1 \quad \Rightarrow \quad \rho_x(0) < \infty \ a.s.\ . \tag{43}$$

If $\gamma\in[1/2,1)$ this only works for $\mu<0$. □

Proof of Theorem 3 Write $z_x^N(0) = \sum_{n=0}^N w_x(n)$. Then $z_x^0(0)=1$ and $z_x^N(0)>1$ $a.s.$. Let $\Delta_N := \mathbb{E}[f_x^N(0) - f_x^{N-1}(0)]$. Then

$$\begin{aligned}\Delta_{N+1} &= \mathbb{E}[f_x^{N+1}(0) - f_x^N(0)] = \mathbb{E}\log(z_x^{N+1}(0)/z_x^N(0)) = \\ &= \mathbb{E}\log\Big(1+\frac{w_x(N+1)}{z_x^N(0)}\Big) \le \mathbb{E}\frac{w_x(N+1)}{z_x^N(0)}\end{aligned} \tag{44}$$

since $\log(1+x)\le x$ for $x>0$. With $w_x(N+1) = w_x(N)e^{-\sigma\xi_x(N+1) - b/(N+1)^\gamma}$ and by independence of $\xi_x(N+1)$ we get

$$\mathbb{E}\frac{w_x(N+1)}{z_x^N(0)} = e^{\delta - b/(N+1)^\gamma}\, \mathbb{E}\frac{w_x(N)}{z_x^N(0)}\ . \tag{45}$$

Now, $w_x(N)\le z_x^N(0)$ $a.s.$ and we can estimate

$$\mathbb{E}\frac{w_x(N)}{z_x^N(0)} \le 1\,\mathbb{P}\Big(\frac{w_x(N)}{z_x^N(0)} > 1/N^2\Big) + 1/N^2\ . \tag{46}$$

Since $z_x^N(0)\ge 1$ $a.s.$ we have

$$\mathbb{P}\Big(\frac{w_x(N)}{z_x^N(0)} > 1/N^2\Big) \le \mathbb{P}(w_x(N) > 1/N^2) = \mathbb{P}\Big(\sigma S_x(N) > \frac{b}{1-\gamma}\, N^{1-\gamma} - 2\log N\Big)\ , \tag{47}$$

whish is bounded above by $\exp\Big[-n^{1-2\gamma}\big(\frac{b}{1-\gamma}\big)^2\big(\frac{1}{2\sigma}\big)\Big]$, see [24]. Thus $\sum_{k=1}^\infty \Delta_k < \infty$ and thus

$$f_Q(0) = \lim_{N\to\infty} \mathbb{E} f_x^N(0) = \lim_{N\to\infty} \sum_{k=1}^N \Delta_k < \infty\ . \tag{48}$$

The second statement, $\rho_c < \infty$, can be shown very similarly. Now define

$$\rho_x^N(0) := \Big(\sum_{n=0}^{N} n w_x(n)/z_x(0) \Big) < \infty \tag{49}$$

for all $N > 0$, since $1/z_x(0) \in [0,1]$ *a.s.*. With $\Delta_N = \rho_x^N(0) - \rho_x^{N-1}(0)$ we get

$$\Delta_N = \frac{N w_x(N+1)}{z_x(0)} < N \text{ a.s. .} \tag{50}$$

Analogous to the above this leads to

$$\mathbb{E}\Delta_N \leq N\, \mathbb{P}\Big(\frac{w_x(N)}{z_x(0)} > 1/N^3 \Big) + 1/N^2 \ , \tag{51}$$

and since $z_x(0) > 1$ *a.s.* we have

$$\mathbb{P}\Big(\frac{w_x(N)}{z_x(0)} > 1/N^3 \Big) \leq \mathbb{P}(w_x(N) > 1/N^3) = \mathbb{P}\Big(\sigma S_x(N) > \frac{b}{1-\gamma}\, N^{1-\gamma} - 3\log N \Big) \ , \tag{52}$$

which is again bounded by $\exp\Big[-n^{1-2\gamma} \big(\frac{b}{1-\gamma} \big)^2 \big(\frac{1}{2\sigma} \big) \Big]$. The rest follows analogously. □

B. Calculations of section V.

As S_x has independent increments, $\mathbb{E}[S_x(n) S_x(k)] = \min\{n,k\}$. The first expected value in the bracket in (32) is $\sum_{n=0}^{\infty} n p(n) = \rho(\mu)$. For the second expected value we need to compute terms of the form

$$\sum_{n,k=0}^{\infty} \min\{n,k\}\, p(n) p(k) = -\sum_{n=0}^{\infty} n p(n)^2 + 2 \sum_{n=0}^{\infty} n p(n) \bar{F}_\mu(n) \tag{53}$$

Now use $p(n) = \bar{F}_\mu(n) - \bar{F}_\mu(n+1)$ and the trick

$$\big(\bar{F}_\mu(n) - \bar{F}_\mu(n+1) \big) \bar{F}_\mu(n) = \frac{1}{2} \big(\bar{F}_\mu(n)^2 - \bar{F}_\mu(n+1)^2 + \underbrace{(\bar{F}_\mu(n) - \bar{F}_\mu(n+1))^2}_{p(n)^2} \big) \tag{54}$$

and summation by parts

$$\sum_{n=0}^{\infty} n (\bar{F}_\mu(n+1)^2 - \bar{F}_\mu(n)^2) = -\sum_{n=0}^{\infty} \bar{F}_\mu(n)^2 \ , \tag{55}$$

which leads to

$$\sum_{n,k=0}^{\infty} \min\{n,k\}\, p(n) p(k) = \sum_{n=0}^{\infty} \bar{F}_\mu(n)^2 \ . \tag{56}$$

One finally obtains

$$\mathbb{E}\left[E_{p_n}[X^2] - E_{p_n}[X]^2 \right] = \rho(\mu, 0) - \sum_{n=0}^{\infty} \bar{F}_\mu(n)^2 = \sum_{n=0}^{\infty} \bar{F}_\mu(n)(1 - \bar{F}_\mu(n)) \tag{57}$$

To estimate the density we simply derive the free energy expansion wrt μ. It is useful to compute first

$$\frac{\partial \bar{F}_\mu(k)}{\partial \mu} = \sum_{n \geq k} p(n)\, (n - \rho(\mu)) = \sum_{n \geq k} \bar{F}_\mu(n) - \rho(\mu) \bar{F}_\mu(k) \ . \tag{58}$$

[1] F. Spitzer, Adv. Math. **5**, 246–290 (1970).
[2] M.R. Evans, T. Hanney, J. Phys. A: Math. Gen. **38**, R195–R239 (2005).
[3] J. Eggers, Phys. Rev. Lett. **83**(25), 5322-5325 (1999). D. van der Meer, J.P. van der Weele, D. Lohse, Phys. Rev. Lett. **88**, 174302 (2002). F. Coppex, M. Droz, A. Lipowski, Phys. Rev. E **66**, 011305 (2002).
[4] J. Török, Physica A **355**, 374–382 (2005). D. van der Meer, K. van der Weele, P. Reimann, D. Lohse, J. Stat. Mech.: Theor. Exp. P07021 (2007).
[5] Y. Kafri, E. Levine, D. Mukamel, G.M. Schütz, and J. Török. Phys. Rev. Lett. **89**(3), 035702 (2002).
[6] A.G. Angel, M.R. Evans, E. Levine, D. Mukamel, Phys. Rev. E **72**, 046132 (2005). A.G. Angel, T. Hanney, M.R. Evans, Phys. Rev. E **73**, 016105 (2006).
[7] J. Kaupuzs, R. Mahnke, R.J. Harris, Phys. Rev. E **72**(5), 056125 (2005).
[8] M.R. Evans, Europhys. Lett. **36**, 13-18 (1996). J. Krug, P.A. Ferrari, J. Phys. A: Math. Gen. **29**, L465–L471 (1996). I. Benjamini, P.A. Ferrari, C. Landim, Stoch. Proc. Appl. **61**, 181–204 (1996).
[9] M.R. Evans, Braz. J. Phys. **30**(1), 42–57 (2000).
[10] S. Grosskinsky, G.M. Schütz, H. Spohn, J. Stat. Phys. **113**(3/4), 389–410 (2003).
[11] M.R. Evans, S.N. Majumdar, R.K.P. Zia, J. Stat. Phys. **123**, 357–390 (2006).
[12] M.R. Evans, T. Hanney, S.N. Majumdar, Phys. Rev. Lett. **97**, 010602 (2006).
[13] J.M. Luck, C. Godreche, J. Stat. Mech.: Theor. Exp. P08005 (2007).
[14] A.G. Angel, M.R. Evans, E. Levine, D. Mukamel, J. Stat. Mech.: Theor. Exp. P08017 (2007).
[15] S. Grosskinsky, G.M. Schütz, J. Stat. Phys. **132**(1), 77-108 (2008)
[16] Y. Schwarzkopf, M.R. Evans, D. Mukamel, J. Phys. A: Math. Theor. **41**, 205001 (2008).
[17] E. Andjel, Ann. Probability **10**(3), 525–547 (1982).
[18] O. Kallenberg, *Foundations of Modern Probability, 2nd edition* (Springer, New York, 2002).
[19] P.A. Ferrari, C. Landim, V.V. Sisko, J. Stat. Phys. **128**, 1153-1158 (2007).
[20] C. Kipnis, C. Landim, *Scaling Limits of Interacting Particle Systems* (Springer, Berlin, 1999).
[21] A.G. Angel, M.R. Evans, D. Mukamel, J. Stat. Mech.: Theor. Exp. P04001 (2004).
[22] The authors are only aware of results on integrals of exponential Brownian motions with constant drift, e.g. M. Yor, J. Appl. Prob. **29**, 202-208 (1992).
[23] S. Grosskinsky, P. Chleboun, G.M. Schütz, Phys. Rev. E **78**(3), 030101(R) (2008)
[24] Amir Dembo and Ofer Zeitouni, *Large Deviations Techniques and Applications*. Springer, Application of Mathematics, vol. 38, 1998