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Multi-Armed Bandit Problem and Its Applications in Intelligent Tutoring Systems.

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Abstract

Master of Complex System

Multi-Armed Bandit Problem and Its Applications in Intelligent Tutoring Systems.

by Minh-Quan NGUYEN

In this project, we propose solutions to exploration vs exploitation problems in Intelligent Tutoring Systems (ITS) using multi-armed bandit (MAB) algorithms. ITSs on one side want to select the best learning objects available to recommends to learners in the systems but they simultaneously want to recommend learners to try new objects so that it can learn the characteristics of new objects for better recommendation in the future. This is the exploration vs exploitation problem in ITSs. We model these problems as MAB problems. We consider the optimal strategy: the Gittins Index strategy and two other MAB strategies: Upper Confidence Bound (UCB) and Thompson Sampling. We apply these strategies in two problems: recommender courses to learners and exercises scheduling. We evaluate these strategies using simulation.

Abbreviations

MAB	Multi-Armed Bandit
POMDP	Partially Observed Markov Decision Proccess
ITS	Intelligent Tutoring System
UCB	Upper Confidence Bound
EUCB	Expected Upper Confident Bound
TS	Thompson Sampling
ETS	Expected Thompson Sampling
PTS	Parametric Thompson Sampling
GI	Gittins Index
EGI	Expected Gittins Index
PGI	Parametric Gittins Index

1 | Introduction

One of the main tasks of an Intelligent Tutoring System (ITS) in education is to recommend suitable learning objects to learners. Traditional recommender systems, including collaborative, content-based and hybrid approaches [1], are widely used in ITSs and effective at providing recommendations at an individual level to learners in the system [2]. Content-based recommendation recommends learning objects that are similar to what the learners has preferred in the past. While collaborative recommendation, by assuming the similarity between learners, recommends learning objects that are useful to other learners in the past. The hybrid approach is developed to combine these two recommendation types or other types (utility-based, knowledge-based [3, 4]) in oder to gain better performance or address the shortcoming of each type.

However, with the rapid development of online education, Massive Open Online Courses (MOOCs) and other learning systems such as French Paraschool, the learning objects in many ITSs undergoes frequent changes, new courses, new learning materials added and removed. And a significant number of registered learners in these systems are new with few or no historical data. It is thus important that an ITS can still makes an useful recommend to learners even when both the learning objects and learners are new. The ITS should try to learn the characteristic of both the learners and the learning objects. But, the cost of acquiring these information can be large and this can really be harmful to learners. Education is a high-stakes domain. This raise the question of optimally balancing the two conflict purposes: maximizing the learners gain in the short term and gathering the information about the utility of learning objects to learners for better recommendation in the future. This is one of the classic problem between exploration and exploitation appearing in all level and time-scale of decision [5–8].

In this article, we will formulate exploration vs exploitation problems in ITSs as multi-armed bandit (MAB) problems. We first consider the Gittins index strategy [9, 10], which is an optimal solution to MAB problems. We also consider two other MAB strategies: UCB1 [11] and Thompson sampling [12, 13]. We will define the bandit model for these two tasks of ITSs: recommending new courses to learners and exercise scheduling, and then propose the strategies to solve each problem. We will test the strategies using simulation.

The structure of the article is organized as follows. The next chapter discussed the MAB problem, Gittins index strategy and other approximate strategies. Chapter 3 presents the application of MAB in ITS with simulation results. A closing chapter is for conclusion and future works.

2 Multi-armed bandit problem

2.1 Exploration vs exploitation problem and multi-armed bandit.

The need to balance exploration and exploitation is faced at all level of behavior and decision making from animal to human. It is faced by pharmaceutical companies to decide what drug to continue develop [14] or by computer scientists to find the best articles to recommend to web users [15]. This problem is not limited only to human. It is also faced by fungi in deciding to grow at local site or send out hyphae to explore distant sites [16], or by ant in finding the site for the nest [17].

In general, there are no optimal policy for the trade of between exploration and exploitation, even when the goals is well defined. The only optimal solution for the exploration vs exploitation problem is proposed by Gittins [9] for a class of problem when the decision is made from a finite number of stationary bandit processes in which the reward of each process is unknown but fixed and is discounted exponentially over time.

Bandit process is a popular framework to study the problem of exploitation vs exploration. In the traditional bandit problem, a player has to choose between the arm that give the best reward now (exploitation) or trying other arms with the hope of finding better arm (exploration). For an multi-armed bandit problem, there is N arms and each arm has an unknown but fixed probability of success p_i. A player has the option to play one arm at one time. The state of the arm at state x that is played changes to a new state y with a transition probability P_{xy} and gives a reward r. The states of other arms do not change. The purpose is to find the maximum expected reward, when the reward is discounted by a parameter β exponentially over time ($0 < \beta < 1$).

$$E\left[\sum_{t=0}^{\infty}\left(\beta^t r_{it}(x_{it})\right)\right] \tag{2.1}$$

2.1.1 Gittins index

For a particular form of MAB with stationary arms and no transition cost, Gittins [9] proposed this strategy and proved that it is optimal:

- Assign to each arm an index called the Gittins index.
- Play the arm with the highest index.

Where the Gittins index of an arm i is:

$$\nu_i = \max_{\tau>0} \frac{E\left[\sum_{t=0}^{\infty}\left(\beta^t r_{it}(x_{it})\right)\right]}{E\left[\sum_{t=0}^{\infty}\beta^t\right]} \tag{2.2}$$

which is a normalized sum of discounted reward over time. τ is the stopping time when selecting the process i is terminated.

This Gittins index of an arm is independent of all other arms. Therefore, one dynamical programing problem in state space of size k^N is reduced to N problem on state space of size k with k is the state space of each arm and N is the number of arms.

Since Gittins propose this strategy and prove its optimality in 1979 [9], researchers have reproved and restated the index theorem. They include Whittle's multi-armed bandit with retirement option [18], Varaiya et al.'s extension of the index from Markovian to non Markovian dynamics [19], Weber's Interleaving of Prevailing Charges [20] and Bertsimas and Nino-Mora's conservation law and achievable region approach [21]. Second edition of Gittins' book [10] has many information about the Gittins index and its development since 1979. For a review focus on the calculation of Gittins index, see the new survey of Chakravorty and Mahajan [22].

For the Gittins index of Bernoulli process with large number of trials n and discount parameter β, Brezzi and Lai [23], using a diffusion approximation for Wiener process with driff, showed that the Gittins index can be approximated by a closed form function:

$$\nu(p,q) = \mu + \sigma\psi\left(\frac{\sigma^2}{\rho^2 ln\beta^{-1}}\right) = \mu + \sigma\psi\left(\frac{1}{(n+1)ln\beta^{-1}}\right) \tag{2.3}$$

With $n = p+q$; $\mu = \frac{p}{p+q}$ and $\sigma^2 = \frac{pq}{(p+q)^2(p+q+1)}$ is the mean and variance of $Beta(p,q)$ distribution; and $\rho^2 = \frac{pq}{(p+q)^2} = \mu(1-\mu)$ is the variance of Bernoulli distribution. $\psi(s)$ is a piecewise nondecreasing function:

$$\psi(s) = \begin{cases} \sqrt{s/2} & \text{if } s \leq 0.2 \\ 0.49 - 0.11s^{-1/2} & \text{if } 0.2 < s \leq 1 \\ 0.63 - 0.26s^{-1/2} & \text{if } 1 < s \leq 5 \\ 0.77 - 0.58s^{-1/2} & \text{if } 5 < s \leq 15 \\ 2ln(s) - ln(ln(s)) - ln(16\pi)^{1/2} & \text{if } s > 15 \end{cases} \tag{2.4}$$

This approximation is good for $\beta > 0.8$ and $min(p,q) > 4$ [23].

In this article, we focus on the Bernoulli process, where the result is counted as correct (1) or failure (0). But our arguments can be applied for normal processes. For the calculation of Gittins index of normal processes, see Gittins' book [10] and the review paper of Yao [24].

Gittins index is the optimal solution to this particular form of MAB problem. The condition is that the arms are stationary, which means that the state of an arm does not change if it is not play, and there is no transition cost when players change arm. If the stationary condition is not satisfied, the problem is called restless bandit [25]. No index policy is optimal for this problem. Papadimitriou and Tsitsiklis [26] proved that the restless bandit problem is PSPACE-hard, even with deterministic transitions. Whittle [25] proposes an approximate solution to the problem using LP relaxation, in which the condition that exactly one arm is played per time step is replaced by the condition that one arm on average is played per time step. He generalizes the Gittins index and people call it Whittle index. This index is widely used in practice [27–31] and it has good empirical performance even though the theoretical basis is weak [27, 32]. When there is transition cost when player changes arm, Banks and Sundaram [33] showed that there is no optimal index strategy even in the case that the switching cost is a given constant. Jun [34] gives a survey of approximate algorithms that are available for this problem.

There are some practical difficulties for the application of Gittins index. The first one is that it is hard to compute the Gittins index in general. Computing the Gittins index is intractable for many problem that it is known to be optimal. Another issue is that the arms must be independent. The optimality and performance of Gittins index with dependent arms or contextual bandit problem [35] is unknown. Finally, Gittins' proof requires that the discount scheme is geometric [36]. In practice, arms usually are not played at equal time intervals which is a requirement for geometric discount.

2.1.2 Upper Confidence Bound algorithms

Lai&Robbin [37] introduced a class of algorithms called Upper Confidence Bound (UCB) that guaranty that the number of time that inferior arm i is played is bounded

$$E[N_i] \leq \left(\frac{1}{K(i,i^*)+o(1))}\right) ln(T) \tag{2.5}$$

With $K(i,i^*)$ is the Kullback–Leibler divergence between the reward distributions for any arm i and the best arm i^*. T is the total number of play on all arms. This strategy guaranty that the best arm is played exponentially more often than other arm. Auer et al. [11] proposed a version of UCB which has uniform logarithmic order of regret bound. This strategy is called UCB1. The strategy is to play the arm that has the highest value of

$$\mu_i + \sqrt{\frac{2ln(T)}{n_i}} \tag{2.6}$$

with μ_i is the mean success of the arm, n_i is the number of times that arm i is played and T is the total number of plays. The algorithm for Bernoulli proccess is the algorithm 3 in Appendix A.

Define the regret of a strategy after T plays as

$$\mathcal{R}[T] = \mu^* T - \sum_{i=1}^{N} \mu_i n_i(T) \tag{2.7}$$

with $\mu^* = \max_{1 \le i \le N}(\mu_i)$. Define $\Delta_i = \mu^* - \mu_i$. The expected regret of this strategy is bounded by [11]

$$\mathcal{R}[T] \le \sum_{i,\mu_i<\mu^*} \left(\frac{8ln(T)}{\Delta_i}\right) + \left(1 + \frac{\pi^2}{3}\right) \sum_{i=1}^{N} (\Delta_i) \tag{2.8}$$

UCB algorithms is an active research field in machine learning, especially for contextual bandit problem [38–44].

2.1.3 Thompson sampling

The main idea of Thompson samping (also called posterior sampling [45] or probability matching [46]) is to randomly select an arm according to the probability that it is optimal. Thompson sampling strategy was first proposed in 1933 by Thompson [12] but it attract little attention in literature on MAB until recently when researchers started to realize its effectiveness in simulations and real-world applications [13, 47, 48]. Contrasting to promising empirical results, the theoretical results are very limited. May et al. [48] prove the asymptotic convergence of Thompson sampling but the finite time guaranty is limited. Recent works showed the regret optimally of Thompson sampling for basic MAB and contextual bandit with linear reward [49–51]. For Bernoulli process with Beta prior, Agrawal&Goyal [50] proved that for any $0 < \epsilon \le\ll 1$ the regret of Thompson sampling will satisfy:

$$E[\mathcal{R}[T]] \le (1+\epsilon) \sum_{i,\mu_i<\mu^*} \frac{lnT}{K(\mu_i, \mu^*)} \Delta_i + O\left(\frac{N}{\epsilon^2}\right) \tag{2.9}$$

where $K(\mu_i, \mu^*) = \mu_i log(\mu_i/\mu^*) + (1-\mu_i) log\left(\frac{1-\mu_i}{1-\mu^*}\right)$ is the Kullback-Leibler divergence between μ_i and μ^*. N is the number of arms. For any two functions f(x), g(x), $f(x) = O(g(x))$ if there exist two constants x_0 and c such that for all $x \ge x_0, f(x) \le cg(x)$. These regret bound is scaled logarithmically in T like UCB strategy. Other regret analysis of Thompson sampling for more general or complicated situations are also proposed [45, 52–54]. Regret analysis of Thompson sampling is an active research field.

The advantage of Thompson sampling approach compared to other MAB algorithms such as UCB or Gittins index is that it can be applied to a wide range of applications which is not limited to models that observed individual rewards alone [53]. It is easier to combine Thompson sampling with other Bayesian approaches and complicated parametric models [13, 47]. Furthermore, Thompson sampling appears to be more robust to observation delays of payoffs [13].

The algorithm of Thompson sampling for Bernoulli process [13] is algorithm 2 in the Appendix A.

2.2 Partially Observed Markov Decision Proccess Multi-armed Bandit

When the underlining Markov chain is not fully observed but the observations of a Markov chain are probabilistic, the problem is called Partially Observed Markov Decision Process (POMDP) or Hidden Markov Chain (HMC). Krishnamurthy and Michova [55] showed that POMDP for multi-armed bandit can be optimally solved by an index strategy. The state space now is $2(N_p + 1)$, in which N_p is the number of states of Markov chain p. The calculation is costly for large N_p. Follow Krishnamurthy [56] and Hauser et al. [57], we will use an sub-optimal policy called Expected Gittins Index (EGI) to calculate the index of POMDP MAB.

$$\nu_E(ik) = \sum_r p_{kr} \nu_{ir}(a_{ir}, b_{ir}) \tag{2.10}$$

with p_{kr} is the probability that learner k is in type r. ν_{ir} is the Gittins index of the course i to learners in type r with a_{ir} successes and b_{ir} failure. $\nu_E(ik)$ is the EGI of course i to student k. This policy is showed to be 99% of the optimal solution [56]. We also expand the Thompson sampling and UCB1 strategy for this problem and call them Expected Thompson Sampling (ETS) and Expected UCB1 (EUCB1). The EGI, ETS and EUCB1 algorithm for Bernoulli process are in the Appendix A (algorithm 4, 5 and 6).

3 Application of multi-armed bandit algorithms in Intelligent Tutoring Systems

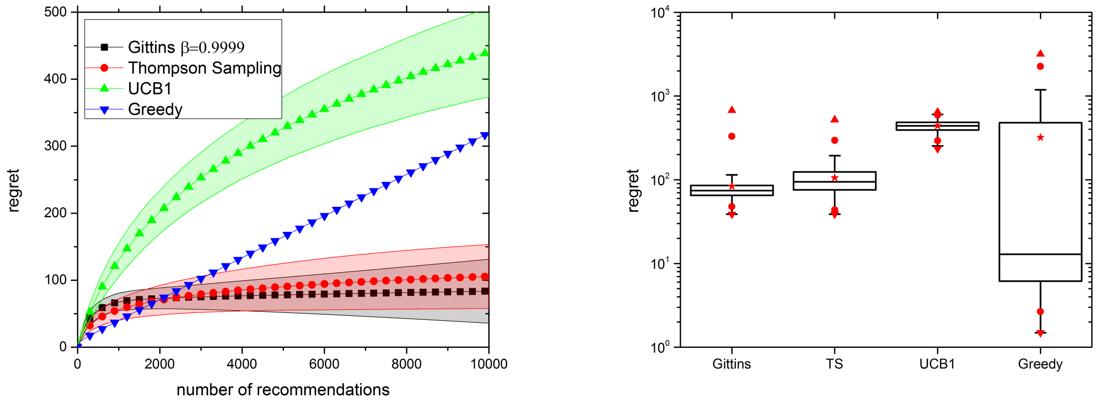
3.1 Recommend courses to learners.

One of the most important task of an ITS is to to recommend new learning objects to learners in e-learning systems. In this problem, we will focus on recommending courses to learners in e-learning systems but the method can be apply to other types of learning objects such as videos, lectures or reading materials ... with a proper definition of success and failure. Assuming that the task is to recommend courses to learners with the purpose of giving the learners courses with highest rate of finishing. There are N courses that the ITS can recommend to learner. Each course has an unknown rate of success p_i. a_i and b_i is the historical number of success (number of learners that finished the course) and failure (number of learners that dropped the course). Note that the definition of success and failure is depended on the purpose of the recommendation to learners. For example, in MOOCs, learners are classified in to different subpopulations with different learning purposes: completing, auditing, disengaging, sampling [58]. For each subpopulation, the definition of success and failure should be different. For our purpose, assuming that we are considering learners in completing group and the definition of success and failure is as above.

We will model this problem as a MAB problem. Each course i is considered to be an arm of a multi-armed bandit with a_i success and b_i failure. This is a MAB with Bernoulli process and we know that the Gittins index is the optimal solution [10]. We compare Gittins index strategy with three other strategies: UCB1, Thompson sampling and greedy. The simulation results are in Figure 3.1 and Figure 3.2. The Gittins index is calculated using Brezzi and Lai closed form approximation [23] (equation 2.3). The algorithms for these strategies for Bernoulli process are in Appendix A.

In figure 3.1, we plot the regret (2.7) of different MAB strategies. There are 10 courses that the ITS can recommends to learners and the probability of each course is randomly sampled from Beta(4,4) distribution for each run. The regret is averaged over 1000 runs. On the left is the mean and confidence interval of the regret of different strategies. On the right is the distribution of the regret at 10000^{th} recommendation. The reason why we do not plot the confidence interval of the regret of greedy strategy is that it is very large and the skewness is large too (see the right hand side). In Figure 3.2, we plot the mean and confidence interval for four other distributions in 3.1. We can see that the Gittins index strategy give the best result but the Thompson sampling work as well as the Gittins index strategy. While the UCB1 strategy guarantees that the regret is logarithmically bounded, its performance with finite number of learners is not good as good as other strategies. After 10000 recommendation, UCB1 strategy even performs worse than greedy strategy in distributions with large number of courses. The standard deviation of regret of greedy strategy is very large because this strategy exploit too much. Greedy strategy sometimes gets stuck at courses with low probability of success (incomplete learning) and this cause the linear increase of regret. In contrast UCB1 strategy has high regret because this strategy sends too much learners to inferior options for exploration which means that it explores too much. The choice of β for Gittins index strategy is depended on the number of recommend that the ITS will make. Suppose that ITS will makes about 10000 recommendations and the discount is spread out evenly then the effective discount from one learner to the next is about 1/10000 suggesting β is about 0.9999. The value of β is $\frac{T}{T+1}$ if the number of recommendation is T. $1 - \beta$ can be understand as the probability that the recommendation can stop. The distributions of probabilities we used are:

$$\begin{aligned}
&\text{Distribution 1: 2 courses with } p = 0.4, 0.6; \\
&\text{Distribution 2: 4 courses with } p = 0.3, 0.4, 0.5, 0.6; \\
&\text{Distribution 3: 10 courses with } p = 0.30, 0.35, 0.40, 0.45, 0.50, 0.55, 0.60, 0.65, 0.70, 0.75; \\
&\text{Distribution 4: 10 courses with } p = 0.30, 0.31, 0.32, 0.40, 0.41, 0.42, 0.50, 0.51, 0.52, 0.60
\end{aligned} \tag{3.1}$$

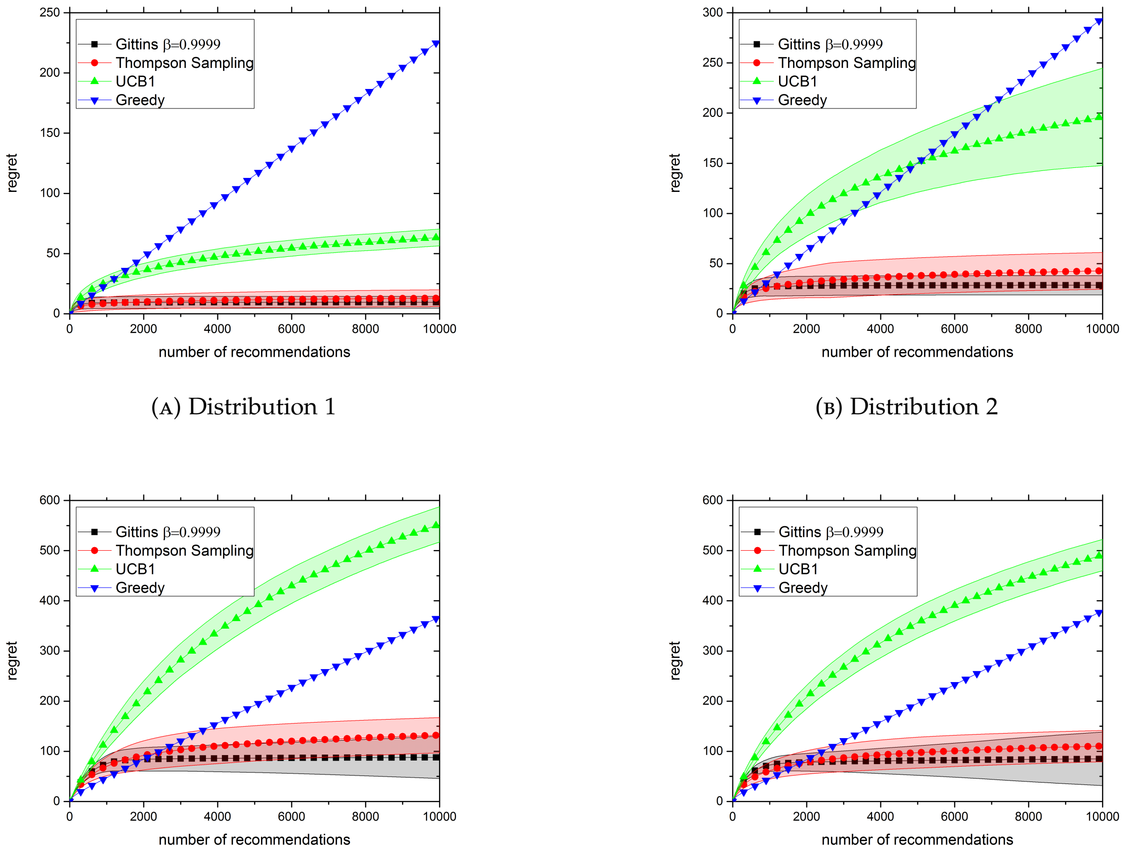


(A) Mean regret and confidence interval.

(B) Distribution of the regret at 10000^{th} recommendation.

FIGURE 3.1: The regret of different strategies: Gittins index, Thompson Sampling, UCB1 and greedy. There are 10 courses that the ITS can recommends to learners and the probability of each course is randomly sampled from Beta(4,4) distribution for each run. The regret is averaged over 1000 runs. Left: mean regret and the confidence interval as a function of number of recommendations. The confidence interval is defined as ±1 standard deviation. The confidence interval of greedy strategy is not showed because it is very large (look at the right figure). Right: distribution of the regret at 10000^{th} recommendation. The star is the mean, the circle is the 99 % and 1 % and the triangle is the maximum and minimum.

The problem with ITS is that learners are not homogeneous. In the learning context we have to consider that learners will have various individual needs, preferences and characteristics such as different levels of expertise, knowledge, cognitive abilities, learning styles, motivation, preferences, and that they want to achieve a specific competence in a certain time. Thus, we can't not treat them in a uniform way. It is of great importance to provide a personalized ITS which can give adaptive recommendation taking into account the variety of learners' learning styles and knowledge levels. To do this, ITS often classified learners into groups or types based on characteristics of learners [2, 59]. Different types of learners will have different probability of success to a course. If we know exactly what are the types of learners, we can use the data from learners of this type for calculating the Gittins index and recommend the best course to them. If a learner stay long in the systems, we can identify well the type of the learner. But the ITS has to give recommendation to learners when they are new and the ITS can only have a limited information about the learner. Assuming that student are classified into F group and we know the type f of the learner k with probability P_{fk}. Given both the



(A) Distribution 1

(B) Distribution 2

(C) Distribution 3

(D) Distribution 4

FIGURE 3.2: Regret of different recommending strategies: Gittins index, Thompson sampling, UCB1 and greedy for four different distributions of probabilities in 3.1. The regret is average over 500 runs. The confidence interval is defined as ±1 standard deviation. The confidence interval of greedy strategy is not showed because it is very large.

uncertainty of the type of learner and the usefulness of the course to the type f, we will model this problem as a POMDP MAB, where each course is an arm in multi-armed bandit but it is only partially observed with probability P_{fk}. We will solve this problem using EGI, ETS and EUCB1 strategy. The conceptual diagram for EGI is in Figure 3.3. The algorithms of EGI, ETS and EUCB1 strategies for Bernoulli process are in Appendix A.

The simulation results are in Figure 3.4 (for 2 types of learners) and 3.5 (for 3 types of learners). The probabilities are randomly sampled from Beta(4,4) distribution for each run and the regret are average over 1000 runs. We can see that in general, EGI strategy has the lowest regret compared with ETS, EUCB1 and greedy strategy. ETS strategy has excellent performance with two types of learners but the performance at three types of learner is not as good as EGI strategy. EUCB1 has the worse performance. It is even worse than greedy strategy. All strategies' regret still increase with number of recommendations because the uncertainty in the types of learners means that there is incomplete learning. All strategies' regret increase when P_f decrease. The less we know about the types of the learners, the worse the performance

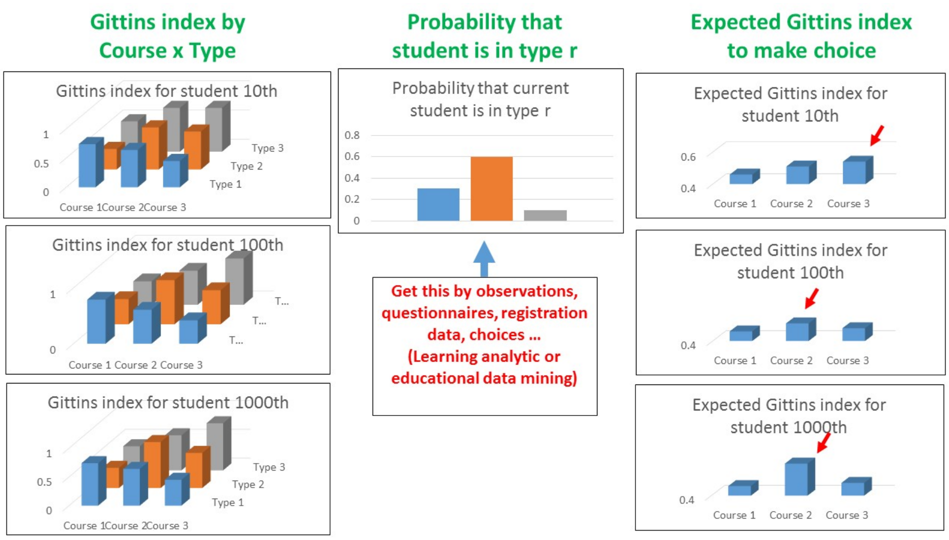


FIGURE 3.3: Conceptual diagram for recommending courses to learners with limited information about their type.

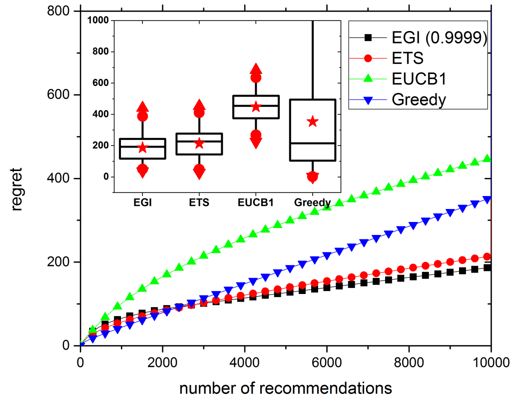
of the strategies. Simulation results for other distributions of probabilities in the Appendix B (Figure B.1, B.2 and B.3) gives the same conclusion.

3.2 Exercise scheduling.

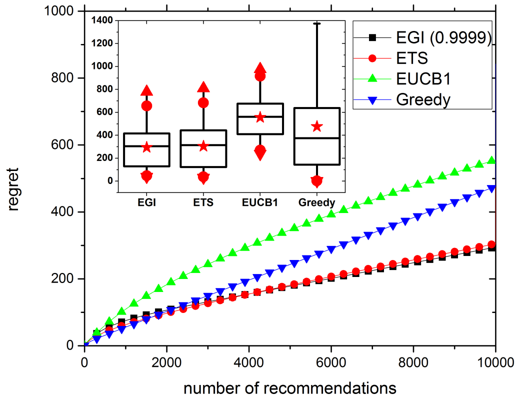
In an ITS, after a student finish learning a concept or a group of concepts, the ITS has to give him exercises to practice these concepts. The purpose is to give a student easy exercise first and hard exercise latter. Assume that we know the student type perfectly now, the problem is to schedule the exercise so that we can both achieve the goal above and learn about the difficulty of an exercise to this type of student. The main problem in this is that the ability of a student to solve an exercise is not constant but depends on many factors. One of the most important factors is how many times this student have use the concept needed to solve this exercise before. This is the theory of learning curve that the ability of a student to successfully use a concept to solve a problem depends on how much he use the concepts before. There is many model of a learning curve [60–62]. For our purpose, we will use this model of learning curve:

$$P(x) = \frac{e^{\alpha+\gamma x}}{e^{\alpha+\gamma x}+1} \tag{3.2}$$

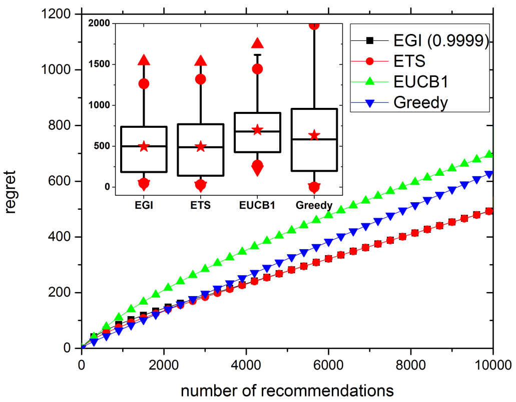
with x is the number of experience with the concept needed to solve the exercise. P(x) is the probability that the student can successfully solve the exercise. α is a parameter related to the difficulty of the exercise and γ is the parameter related to the speed of learning (learning rate). This function of P(x) has the form of logistic function. This learning curve is used in Learning Factor Analysis [63] and is related to Item Response Theory which is the theory behind standardized tests.



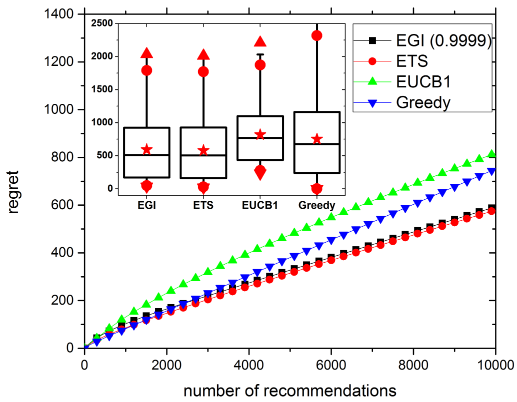
(A) $P_f = (0.95, 0.05)$



(B) $P_f = (0.9, 0.1)$



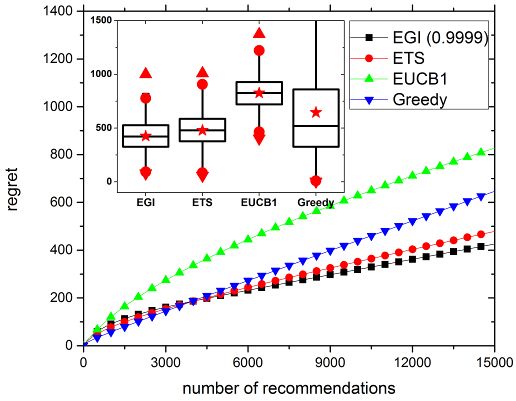
(C) $P_f = (0.8, 0.2)$



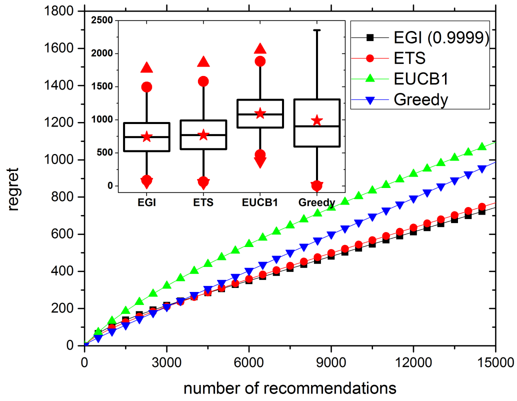
(D) $P_f = (0.7, 0.3)$

FIGURE 3.4: Regret of different recommending strategies: Expected Gittins Index (EGI), Expected Thompson Sampling (ETS), Expected UCB1 (EUCB1), and greedy. There are two types of learners and five courses. The probabilities are randomly sampled from Beta(4,4) distribution for each run. The regret is average over 1000 runs. In the large figure is mean regret as a function of number of recommends. The small figure is the distribution of the regret at 10000^{th} recommendation. The star is the mean, the circle is the 99 % and 1 % and the triangle is the maximum and minimum.

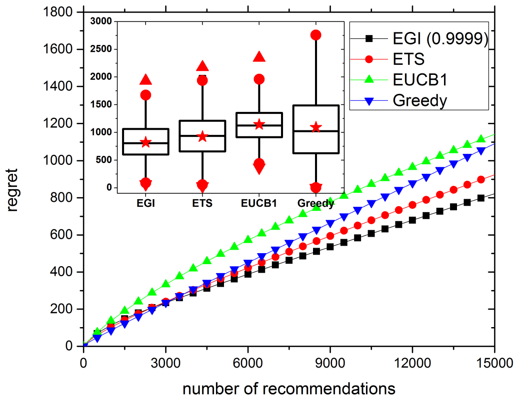
We will formulate the exercise scheduling problem as an MAB problem. The ITS will sequentially select exercises to give to student based on the history of the success and failure given the number of experience x, while it also has to adapt the selection strategy to learn about the exercise suitability to the student to maximize learning in the long run. Each exercise at each value of x is consider to be an arm of a multi-armed bandit. An naive approach is to treat each exercise at each value of x to be an independent arm and use Gittins index to calculate the index. A better approach is to use the information that the arms of one exercise (with different value of x) is related through the learning curve. To use this information from the learning curve, we propose two strategies: Parametric Gittins Index (PGI) and Parametric Thompson Sampling (PTS) with the idea of replacing the mean of success of each exercise at each value of x by the mean of success estimated from the learning curve from the history of



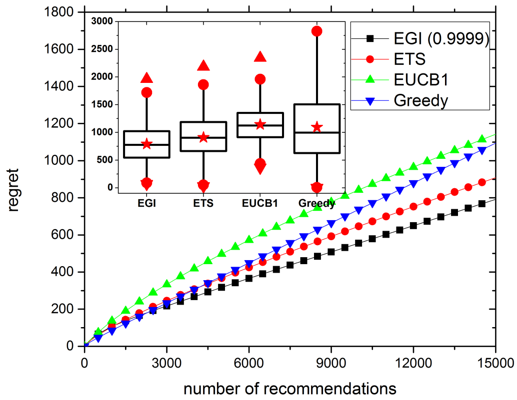
(A) $P_f = (0.9, 0.07, 0.03)$



(B) $P_f = (0.8, 0.1, 0.1)$



(C) $P_f = (0.7, 0.2, 0.1)$



(D) $P_f = (0.6, 0.3, 0.1)$

FIGURE 3.5: Regret of different recommending strategies: Expected Gittins Index (EGI), Expected Thompson Sampling (ETS), Expected UCB1 (EUCB1), and greedy. There are two types of learners and five courses. The probabilities are randomly sampled from Beta(4,4) distribution for each run. The regret is average over 1000 runs. In the large figure is mean regret as a function of number of recommends. The small figure is the distribution of the regret at 15000^{th} recommendation. The star is the mean, the circle is the 99 % and 1 % and the triangle is the maximum and minimum.

success and failure of an exercise with all x. The algorithms are in the Appendix A (Algorithm 7 and Algorithm 8). Note that the PGI and PTS are better than the independent Gittins only when the number of dependent arms (the value of x) are more than the number of parameters of the learning curve (two parameters in our model).

In figure 3.6, we compare the PGI and PTS strategy with the independent Gittins index strategy (without using the information from the learning curve). There are 3 exercises with parameter $\alpha = (-3, -2, 0)$ and $\beta = (0.8, 0.5, 0.15)$ respectively. This mean that the first exercises is quite hard at first but has a high learning rate while the third exercises is easy at first but has slower learning rate. The ITS has to find the exercise with highest probability of success given the number of experience x of the learner with the concept needed to solve the exercise. The value of x for each learner is a random number between 1 and 7. Each learner is recommended one exercise only. The regret is averaged over 200 runs. From the figure, we can see that the PGI has the best performance while the PTS is a little worse. Both strategies work extremely well with small regret compared with the independent Gittins index strategy.

In figure 3.7, we plot the regret of PGI, PTS as a function of number of learners. Now, assuming that there are 11 exercises that the ITS can give learners to practice concepts. And the ITS want to give each learner 7 exercises sequentially, easy one first and hard one latter, to maximize the success. All of these exercise is related to one concept and after solving each exercise the number of experience x of the learner will increase one. The initial number of experience is 0. To speed up the simulation, we use batch update (delay) for the learning curve. The learning curve is updated after each group of three learners. Small delay like this will not affect the performance of these strategies. PGI strategy has the best performance as expected. But PTS performance is very similar to PGI strategy. After recommending exercises to 5000 learners (total 35000 recommendations), the regret is just about 100 which is very small. These strategies have very good performance. The result in figure 3.8 is the same. Now there is 8 exercises and the ITS want to give 5 exercises to each learners. The parameters of each exercises now are different and there are small different of learning rate between these exercises.

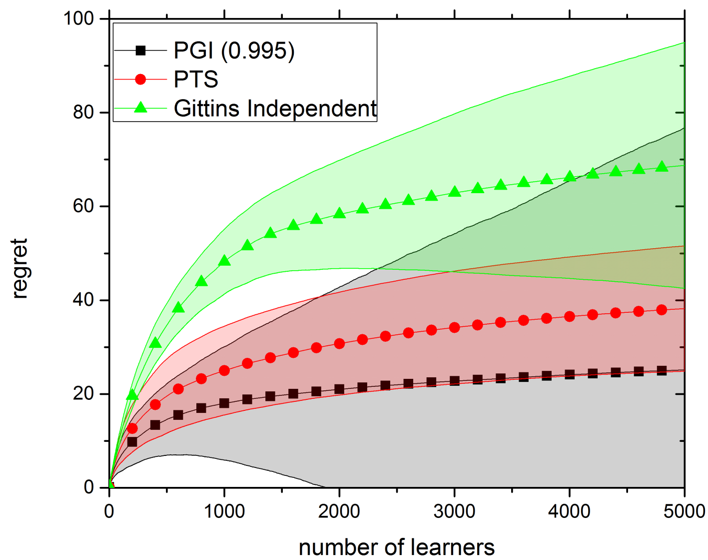


FIGURE 3.6: The mean reget and the confidence interval (±1 standard deviation) of Parametric Gittins Index (PGI), Parametric Thompson Sampling (PTS) and Independent Gittins Index. The ITS wants to give each learner one in three exercises given the number of experience x with the concept needed to solve the exercise. x is a random number between 1 and 7 for each learners. The parameters of exercises are: $\alpha = (-3, -2, 0)$ and $\beta = (0.8, 0.5, 0.15)$.

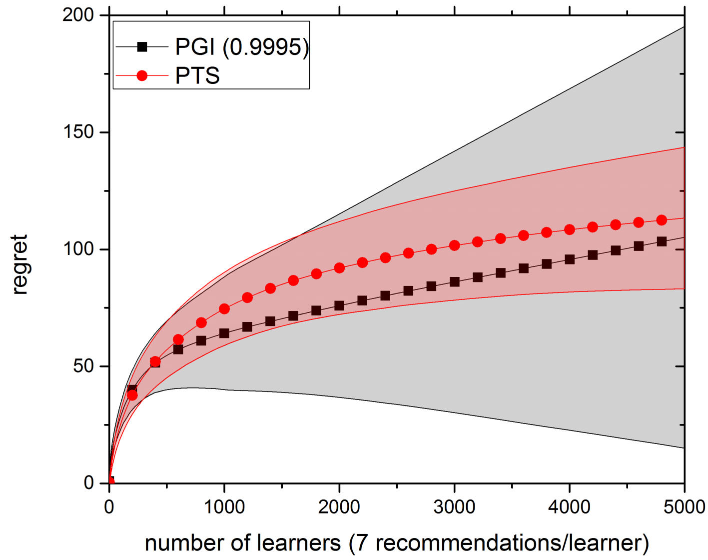


FIGURE 3.7: The mean reget and the confidence interval (±1 standard deviation) of Parametric Gittins Index (PGI) and Parametric Thompson Sampling (PTS) strategy. The ITS gives student 7 in 11 available exercises sequentially, easy one first, hard one latter, to maximize the success. The parameters of exercises are: $\alpha = (-2 : 0.3 : 1)$ and $\beta = 0.4$. Each exercise has different difficulty but similar learning rate.

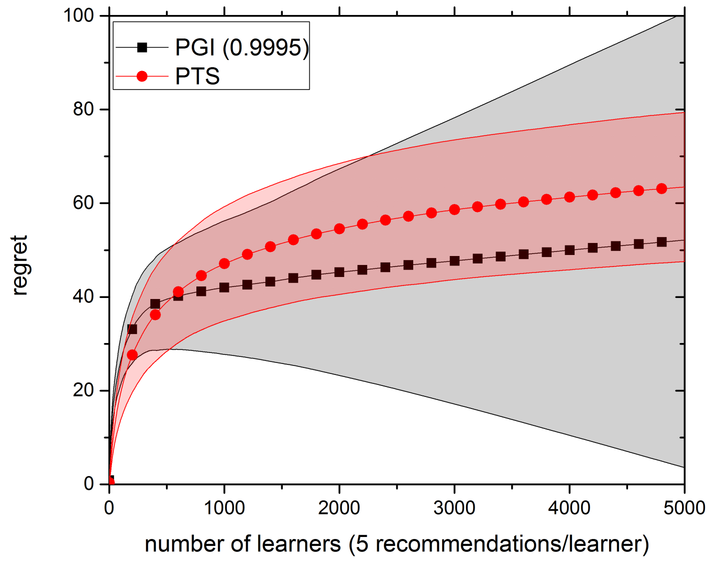


FIGURE 3.8: The mean reget and the confidence interval (±1 standard deviation) of Parametric Gittins Index (PGI), Parametric Thompson Sampling (PTS), Parametric UCB1 (PUCB1). The ITS gives student 5 in 8 available exercises sequentially, easy one first, hard one latter, to maximize the success. The parameters of exercises are: $\alpha = -2 : 0.4 : 1$ and $\beta = 0.4 : 0.02 : 0.54$.

4 | Conclusion and future work

In this project, we take an multi-armed bandit approach to ITS problems such as learning objects recommendation and exercises scheduling. We propose methods and algorithms for these problems based on MAB algorithms. We test the optimal strategies, Gittins index, together with Thompson sampling and Upper Confident Bound (UCB1) strategy. We also propose strategies to solve the learning objects recommendation with multiple types of learners: Expected Gittins Index, Expected Thompson Sampling and Expected UCB1. For the task of exercises scheduling, we modify the Gittins index strategy and the Thompson sampling to utilize the information from the learning curve and propose the Parametric Gittin Index and Parametric Thompson Sampling. We still don't know how to find the confidence bound for the UCB1 strategy in this case. We test all of these strategy using simulation. Gittins index and Gittins index based (EGI and PGI) strategies have the best performance but the Thompson sampling and Thompson sampling based (ETS and PTS) have very good performance. While Gittins index has one parameter β, Thompson sampling does not has any parameter. This is one advantage of Thompson sampling over Gittins index strategy. Furthermore, Thompson sampling has at the collective level the good property to be asymptotically convergent because all arm are tried an infinity number of time.

However, the stochastic feature of Thompson sampling can raise at the individual level some ethical criticism: it is considering learners as "cobaye". Gittins index is taking into account for each learner the parameter β, that represent the compromise between exploration and exploitation for his/her own future. There is a deep relation for a given learner and a given concept between β and the mean number of times a concept will be used in his/her future (at least until the final exam).

In the future, we want to investigate this multi-armed bandit approach to other tasks in ITSs. The recommending objects now are not only simple objects like courses or exercises but can be a collection of learning objects such as a learning paths. The result now are not Bernoulli result but we can use Gittins index, Thompson sampling or Upper Confidence Bound strategy for normal processes. The grand task we want to tackle is designing a complete ITS with there important part: a network of concepts for navigation, and a methods for classification of learners and a recommending strategies for the balance between exploration vs exploitation in the systems. This project is the first step for this task.

At all level of an Intelligent Tutorial System, there are adaptive systems with their own compromise between exploration and exploitation and with their own "life time horizon" that are equivalent to some parameter β. Thus the multi-armed strategy and its approximations will remain a must for future direction of research, especially in the context of MOOCs in education. Such big data context will allow more and more categorization for more accurate prediction at all level of each ITS.

Acknowledgements

I would like to express my gratitude to my supervisor - professor Paul Bourgine - for introducing me to the topic as well for his support during the internship. Without his guidance and persistent help this project would not have been possible. He has been a tremendous mentor for me. Furthermore, I would like to thank Nadine Peyriéras for providing me the internship. I would also like to express my gratitude to Erasmus Mundus consortium for their support during my master.

A Multi-armed bandit algorithms

A.1 Algorithms for independent Bernoulli Bandit

Algorithm 1 Gittins Index

a_i, b_i: number of success and failure of arm i until time t-1

1. Calculate Gittins index using Brezzi&Lai approximation

$$n_i = a_i + b_i$$
$$\mu_i = \frac{a_i}{n_i}$$
$$\nu_i = \mu_i + \sqrt{\frac{\mu_i(1-\mu_i)}{n_i+1}}\psi\left(\frac{1}{(n_i+1)ln(\beta^{-1})}\right)$$

2. Select arm and observe reward r

$$i^* = argmax\left\{\nu_i\right\}$$

3. Update:

$$a_{i^*} = a_{i^*} + r$$
$$b_{i^*} = b_{i^*} + (1-r)$$

Algorithm 2 Thompson Sampling

a_i, b_i: number of success and failure of arm i until time t-1

1. Sample data from Beta distribution

$\phi_i \sim Beta[a_i, b_i]$

2. Select arm and observe reward r

$i^* = argmax\ \{\phi_i\}$

3. Update:

$a_{i^*} = a_{i^*} + r$

$b_{i^*} = b_{i^*} + (1 - r)$

Algorithm 3 UCB1

a_i, b_i: number of success and failure of arm i until time t-1

1. Find the index of each arm

$$\nu_i = \frac{a_i}{a_i + b_i} + \sqrt{\frac{2ln(t-1)}{a_i + b_i}}$$

1. Select arm and observe reward r

$i^* = argmax\ \{\nu_i\}$

3. Update:

$a_{i^*} = a_{i^*} + r$

$b_{i^*} = b_{i^*} + (1 - r)$

A.2 Algorithms for POMDP Bernoulli Bandit

Algorithm 4 Expected Gittins Index

P_{fk}: probability that player k are in group f. a_{fi}, b_{fi}: number of success and failure of arm i of group f until time t-1.

1. Find the index of each arm i for each group f of learners.

$$n_{fi} = a_{fi} + b_{fi}$$
$$\mu_{fi} = \frac{a_{fi}}{n_{fi}}$$
$$\nu_{fi} = \mu_{fi} + \sqrt{\frac{\mu_{fi}(1-\mu_{fi})}{n_{fi}+1}}\psi\left(\frac{1}{(n_{fi}+1)ln(\beta^{-1})}\right)$$

2. Find the expected Gittins index of each course i to learner k

$$\nu_{ki}^{E} = \sum_{f} P_{fk}\nu_{fi}$$

2. Select arm and observe reward r

$$i^* = argmax\left\{\nu_{ki}^{E}\right\}$$

3. Update:

$$a_{i^*f} = a_{i^*f} + rP_{fk}$$
$$b_{i^*f} = b_{i^*f} + (1-r)P_{fk}$$

Algorithm 5 Expected Thompson Sampling

P_{fk}: probability that player k are in group f. a_{fi}, b_{fi}: number of success and failure of arm i of group f until time t-1.

1. Sample data for each arm i of each group f of learners.

$$\phi_{fi} \sim Beta(a_{fi}, b_{fi})$$

2. Find the expected Thompson sampling of each course i to learners k

$$\phi^E_{ki} = \sum_f P_{fk}\phi_{fi}$$

2. Select arm and observe reward r

$$i^* = argmax \left\{ \phi^E_{ki} \right\}$$

3. Update:

$$a_{i^*f} = a_{i^*f} + rP_{fk}$$
$$b_{i^*f} = b_{i^*f} + (1-r)P_{fk}$$

Algorithm 6 Expected UCB1

P_{fk}: probability that player k are in group f. a_{fi}, b_{fi}: number of success and failure of arm i of group f until time t-1.

1. Find the index of each arm i for each group f of learners.

$$n_{fi} = a_{fi} + b_{fi}$$
$$\mu_{fi} = \frac{a_{fi}}{n_{fi}}$$
$$\nu_{fi} = \mu_{fi} + \sqrt{\frac{2ln(\sum_{j=1}^{N} n_{fj})}{n_{fi}}}$$

2. Find the Expected UCB1 index of each course i to learner k

$$\nu_{ki}^{E} = \sum_{f} P_{fk} \nu_{fi}$$

2. Select arm and observe reward r

$$i^* = argmax \left\{ \nu_{ki}^{E} \right\}$$

3. Update:

$$a_{i^*f} = a_{i^*f} + rP_{fk}$$
$$b_{i^*f} = b_{i^*f} + (1-r)P_{fk}$$

A.3 Algorithms for Exercises Scheduling

Algorithm 7 Parametric Gittins Index

$a_i(x), b_i(x)$: number of success and failure of an exercise i with x is the number of experience with the concept needed to solve the exercise. x_i is the number of experience with concept needed to solve the exercise i of current learner.
Learning curve model: $P(x) = \frac{e^{\alpha+\gamma x}}{e^{\alpha+\gamma x}+1}$

1. Estimate the learning curve $P_i(x)$ from $a_i(x), b_i(x)$ using logistic regression.
2. Find the index of each exercise i given x_i

$$n_i = \sum_{x=1}^{\infty}(a_i(x) + b_i(x))/2$$
$$\mu_i = P_i(x_i)$$
$$\nu_i = \mu_i + \sqrt{\frac{\mu_i(1-\mu_i)}{n_i+1}}\psi\left(\frac{1}{(n_i+1)ln(\beta^{-1})}\right)$$

2. Select arm and observe reward r

$$i^* = argmax\,\{\nu_i\}$$

3. Update:

$$a_{i^*}(x_{i^*}) = a_{i^*}(x_{i^*}) + r$$
$$b_{i^*}(x_{i^*}) = b_{i^*}(x_{i^*}) + (1-r)$$

Algorithm 8 Parametric Thompson Sampling

$a_i(x), b_i(x)$: number of success and failure of an exercise i with x is the number of experience with the concept needed to solve the exercise. x_i is the number of experience with concept needed to solve the exercise i of current learner.
Learning curve model: $P(x) = \frac{e^{\alpha+\gamma x}}{e^{\alpha+\gamma x}+1}$

1. Estimate the learning curve $P_i(x)$ from $a_i(x), b_i(x)$ using logistic regression.
2. Find the sample of each exercise i given x_i

$$n_i = \sum_{x=1}^{\infty}(a_i(x) + b_i(x))/2$$
$$\mu_i = P_i(x_i)$$
$$\phi_i \sim Beta(\mu_i n_i, (1-\mu_i)n_i)$$

2. Select arm and observe reward r

$$i^* = argmax\,\{\phi_i\}$$

3. Update:

$$a_{i^*}(x_{i^*}) = a_{i^*}(x_{i^*}) + r$$
$$b_{i^*}(x_{i^*}) = b_{i^*}(x_{i^*}) + (1-r)$$

B | More simulation results

B.1 Simulation results for Expected Gittins index and Expected Thompson sampling.

In this section, we present some more simulation result for POMDP with different configuration of courses, types of learners and probabilities. The distributions of success probability of each course i with each types of learners P_{fi} we use are:

$$\text{Distribution 1: 4 courses with 2 types of learners} \tag{B.1}$$
$$P = \begin{pmatrix} 0.80 & 0.60 & 0.40 & 0.20 \\ 0.20 & 0.40 & 0.60 & 0.80 \end{pmatrix}$$
$$\text{Distribution 2: 3 courses with 3 types of learners}$$
$$P = \begin{pmatrix} 0.80 & 0.20 & 0.50 \\ 0.50 & 0.80 & 0.20 \\ 0.20 & 0.50 & 0.80 \end{pmatrix}$$

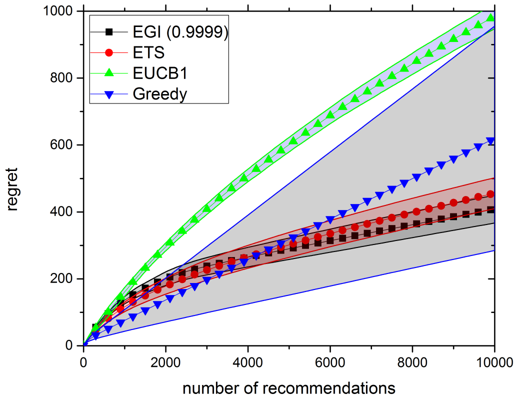
In figure B.1, we plot the mean regret and the confidence interval of Expected Gittins Index (EGI), Expected Thompson Sampling (ETS), Expected UCB1 (EUCB1), and greedy for the distribution 1 in B.1 with two types of learners. In figure B.2, we plot the mean regret and the confidence interval of Expected Gittins Index (EGI), Expected Thompson Sampling (ETS), Expected UCB1 (EUCB1), and greedy for the distribution 2 in B.1 with three types of learners. In figure B.3, we plot the mean regret and confidence interval with different randomly sampled probabilities from Beta(4,4) distribution. The probability of knowing the types of learner k is $P_{fk} = (0.6, 0.3, 0.1)$.

B.2 Effect of delay

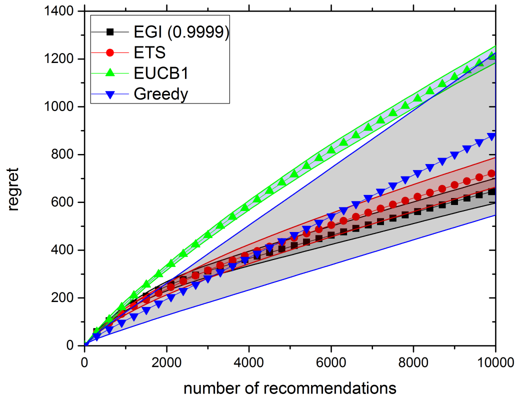
In real ITSs, the feedback of learners is usually not sequential. There is usually an delay in feedback. The reason can be various runtime constraints or learners learning at the same time. So the data of success and failure normally arrive in batches over a period of time. We now try to quantify the impact of the delay on the performance of MAB algorithms.

Table B.1 show the mean regret of each MAB algorithms: Gittins index, Thompson sampling and UCB1 after 10000 recommendations with different value of delay. We consider 10 courses with the probability of each course is drawn from beta(4,4) distribution. The first conclusion is that the regret of all MAB algorithms increase when the delay increases. Thompson sampling is quite robust to the delay. It is because Thompson sampling is a random algorithm and this alleviates the effect of the delay. On the other hand, Gittins index and UCB1 are deterministic strategies so they have larger regrets when the delay increase. In figure B.4, we plot the regret of each strategies with different delays as a function of number of recommends.

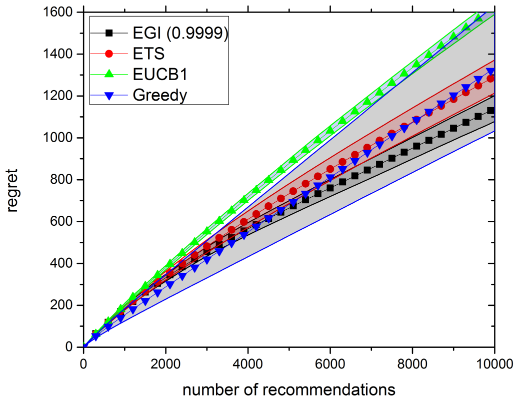
Table B.2 show the mean regret of strategies for POMDP MAB: Expected Gittins Index (EGI), Expected Thompson Sampling (ETS) and Expected UCB1 (EUCB1). As in the case with original MAB algorithms, ETS is extremely resilient with delay. Even with delay equals 500, the regret of ETS does not change much. EGI and EUCB1 are not so resilient with delay but the regret of these strategies does not increase so much as the Gittins index and UCB1. We can think of two reason for this. The first reason is because of the uncertainty in the type of



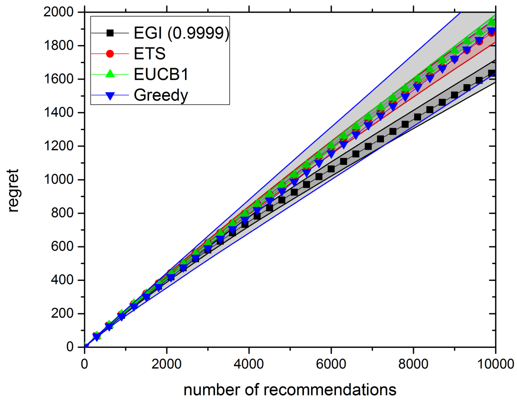
(A) $P_f = (0.95, 0.05)$



(B) $P_f = (0.9, 0.1)$



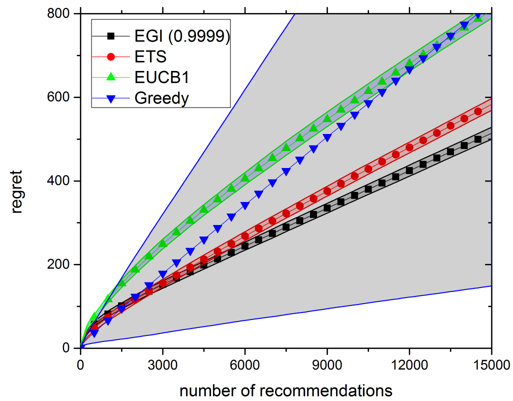
(C) $P_f = (0.8, 0.2)$



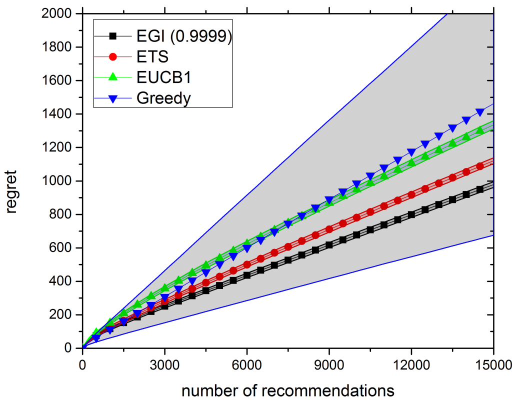
(D) $P_f = (0.7, 0.3)$

FIGURE B.1: The mean regret and the confidence interval ($\pm$ standard deviation) of different recommending strategies: Expected Gittins Index (EGI), Expected Thompson Sampling (ETS), Expected UCB1 (EUCB1), and greedy for the distribution 1 in B.1 (two types of learners). The regret is average over 200 runs. Note that with this distribution, the confidence interval of greedy strategy is very large while the confidence interval of EGI, ETS and EUCB1 is quite small.

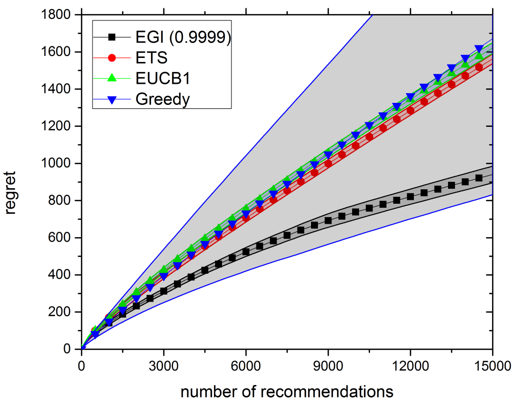
learners. This make EGI and EUCB1 less deterministic. The second reason is because of the type of learners. There are three types of learners so it means that the effective delay for each type of learners is the delay divided by three. In figure B.5, we plot the mean regret of these 3 strategies as a function of number of recommends with different delays.



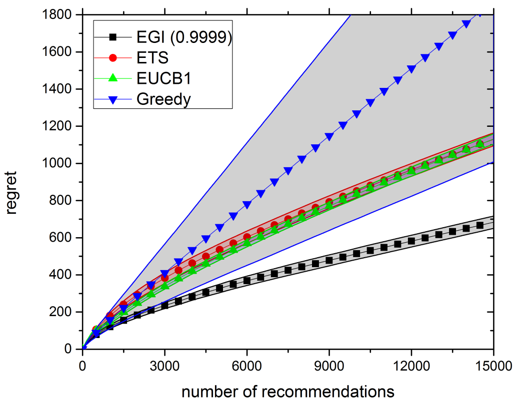
(A) $P_f = (0.9, 0.07, 0.03)$



(B) $P_f = (0.8, 0.1, 0.1)$



(C) $P_f = (0.7, 0.2, 0.1)$

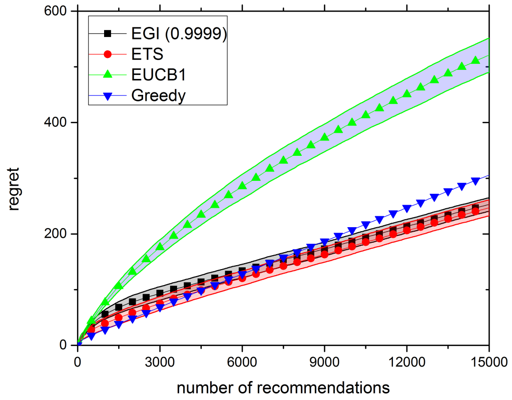


(D) $P_f = (0.6, 0.3, 0.1)$

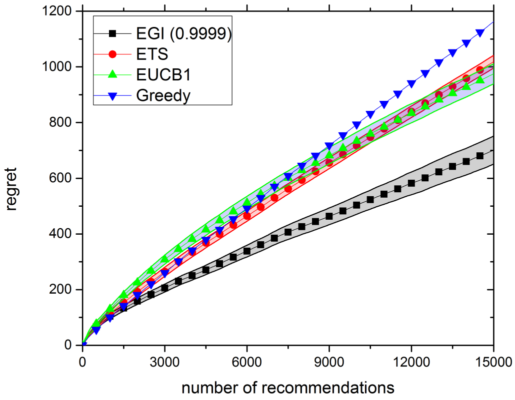
FIGURE B.2: The mean regret and the confidence interval ($\pm$ standard deviation) of different recommending strategies: Expected Gittins Index (EGI), Expected Thompson Sampling (ETS), Expected UCB1 (EUCB1), and greedy for the distribution 2 in B.1 (Three types of learners). The regret is average over 200 runs. Note that with this distribution, the confidence interval of greedy strategy is very large while the confidence interval of EGI, ETS and EUCB1 is quite small.

TABLE B.1: The effect of delay on the regret of MAB strategies after 10000 recommendations. There are ten courses and the probability of each course is randomly sampled from Beta(4,4) distribution for each run. The regret are averaged over 500 runs.

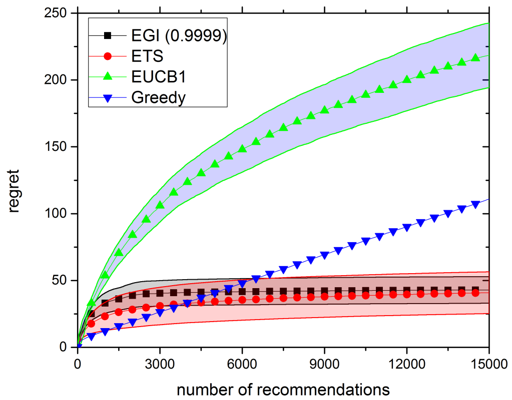
delay	1	10	30	100	500
Gittins Index	83.08	98.20	124.11	276.81	1261.8
Thompson Sampling	100.31	102.93	105.23	106.44	177.06
UCB1	445.83	457.66	484.82	560.02	1360.8



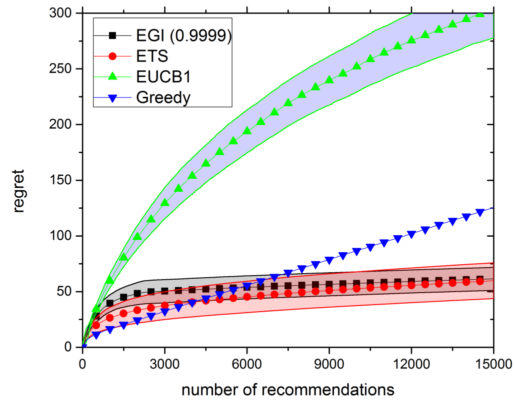
(A)



(B)



(C)

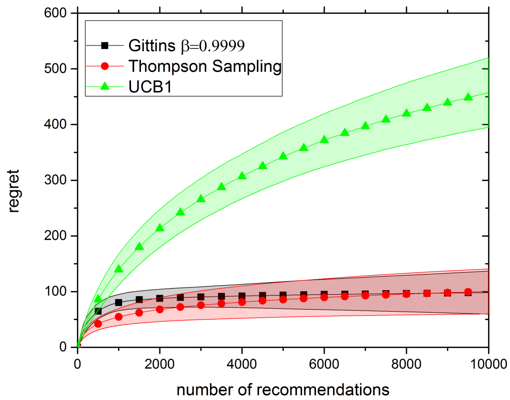


(D)

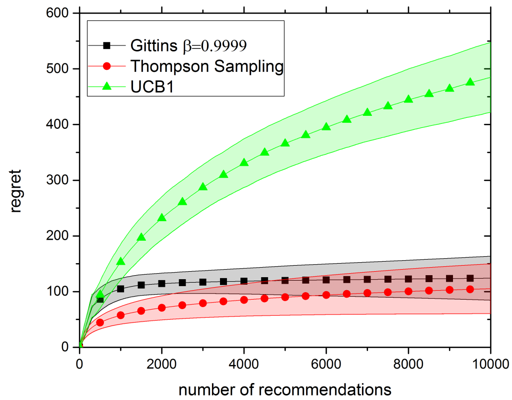
FIGURE B.3: The mean regret and the confidence interval ($\pm$ standard deviation) of different recommending strategies: Expected Gittins Index (EGI), Expected Thompson Sampling (ETS), Expected UCB1 (EUCB1), and greedy for four distributions of probabilities. There are three courses and three types of learners and the probabilities are randomly sampled from Beta(4,4) distribution. $P_f = (0.6, 0.3, 0.1)$. The regret is average over 200 runs. The confidence interval of greedy strategy is not show because it is very large.

TABLE B.2: The effect of delay on the regret of Expected Gittins Index (EGI), Expected Thompson Sampling (ETS) and Expected UCB1 strategy after 15000 recommendations. There are five courses and there types of learners and the probabilities of each run are randomly sampled from Beta(4,4) distribution. $P_f = (0.7, 0.2, 0.1)$. The regret is average over 1000 runs.

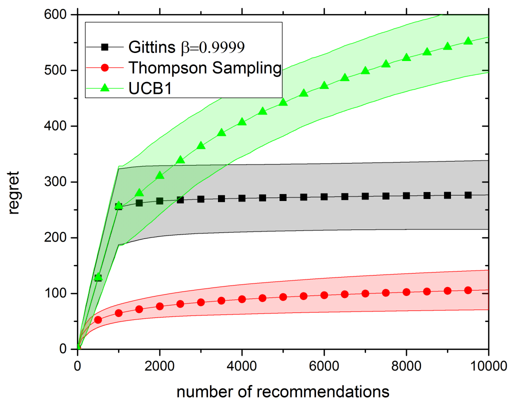
delay	1	10	30	100	200	500
EGI	822.34	815.97	828.12	828.27	919.27	1097.1
ETS	924.72	920.27	935.45	910.44	954.75	951,54
EUCB1	1142.7	1121.5	1137.4	1138.9	1191.7	1255.3



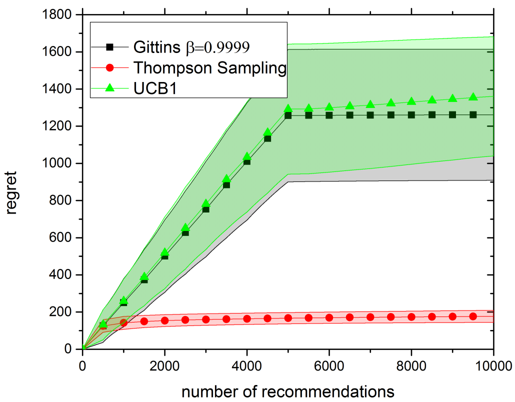
(A) Delay=10



(B) Delay=30

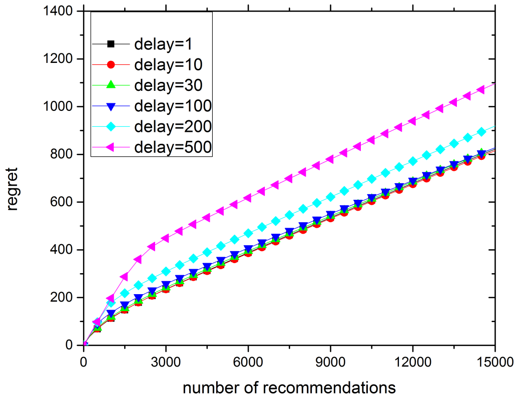


(C) Delay=100

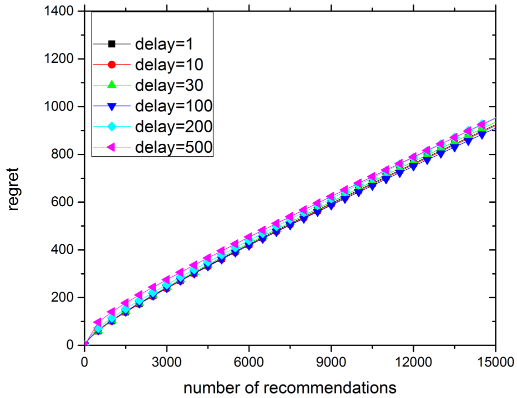


(D) Delay=500

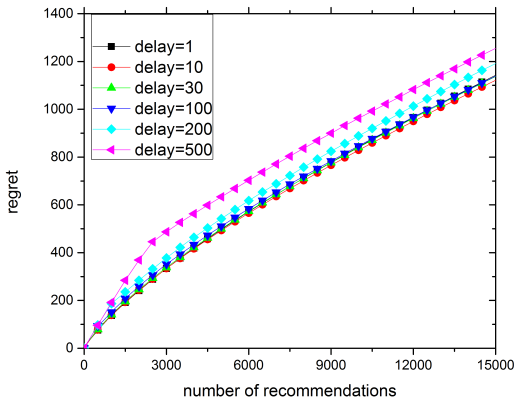
FIGURE B.4: Regret of MAB strategies with different delays. There are 10 courses and the probability of each course is randomly sampled from Beta(4,4) distribution for each run. The regret is averaged over 500 runs.



(A) EGI



(B) ETS



(C) EUCB1

FIGURE B.5: Regret of Expected Gittins Index (EGI), Expected Thompson Sampling (ETS) and Expected UCB1 (EUCB1) strategy with different delays. The regret is average over 1000 runs. There are five courses and there types of learners and the probabilities are randomly sampled from Beta(4,4) distribution for each run.

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