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An Option Super Hedging Strategy with Shortfall Possibility in Discrete Time

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Abstract

In this article, a new strategy for super hedging European options with a given success probability, is presented. In this strategy, the portfolio could be readjusted in just a finite number of times, which is a very realistic assumption, given that not only continuously readjusting is impossible in practice, but also the existence of transaction costs constrains the number of portfolio rebalancing times. In this strategy, a discrete-time stock price process is defined by discretizing the continuous stock price process in its stopping times, and then by using American options hedge ratios on the binomial tree, the initial European option is super hedged. The performance of our strategy is also compared with that of a benchmark strategy by using Monte Carlo simulations, and it has been shown that it has a better functioning.

1 Introduction

In this article, we propose a strategy for super hedging European option with a given success probability when it is possible to readjust the hedging portfolio just in discrete times. In complete markets, any contingent claim could be perfectly hedged in continuous time. However, hedging in continuous time in practical terms is not possible, and also the existence of transaction costs further constrain the number of rebalancing times. A general contingent claim could not be perfectly hedged in discrete time even in the complete markets, like that of Black Scholes. So, here comes the notion of super hedging, that is when the portfolio makes at least the same payoff as that of the contingent claim in all states of the world. However, perfect super hedging requires a high amount of initial capital, so it would be desirable for many investors to have super hedging strategies that succeed with a high probability since in this sense the cost of the strategy would be reduced ([3]). Therefore, the purpose is to make a super hedging strategy which succeed to make such a portfolio with the least possible hedging error or shortfalls, from both number of occurrences and size point of views.

Here in our strategy, the rebalancing times are neither equidistant nor deterministic, and in fact they are a sequence of stopping times. The intuition is the same as what traders do in financial markets that is to rebalance the portfolio when the price has moved more than some extents. The price process is then discretized at the rebalancing dates, and the continuous time model is embedded into a Binomial tree. The American option hedging rations on the Binomial model is then used as the hedging ratios for the initial European option. So, the initial value of the hedging portfolio would be the same as the price of the American option, and at the rebalancing times, it is readjusted according to the American option hedging ratios.

In the literature, among the earliest works which address option hedging strategies in discrete time is [10], in which a hedging strategy is proposed, given both zero and positive transaction costs. The rebalancing times are equidistant, and the hedging error approaches zero as frequency of rebalancing increases. [7]

studies discrete time option hedging errors, and shows that the rate of convergence of this error term depends on the regularity properties of the payoff function. In [8], the asymptotic distribution of the hedging error has been derived, and in [13], the results are extended for option with discontinuous payoffs in jump-diffusion models. [5] constructs explicit robust hedging strategies under a continuous semi martingale model with small transaction costs. The rebalancing dates are a sequence of stopping times, the same as in our strategy, and the main result is the stable convergence of the discrete time hedging strategy as transaction costs converge to zero. Some other recent researches in this area include [4], [1], [11], and [6].

This paper is organized as follows. In section 2, our strategy for super hedging European options is introduced. Section 3 elaborates the numerical framework for analyzing the performance of the strategy, and section 4 compares the performance of the strategy with that of the Black Scholes discrete time delta hedging strategy and shows that our strategy has a better performance.

2 The Super Hedging Strategy

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. We consider a financial market consisting of a risk-free asset and a risky asset with price process $(S_t)_{t\geq 0}$ such that the log stock price $X_t = \log S_t$ follows a one-dimensional Markov process:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x, \tag{1}$$

where W is a standard $\mathbb{F}$-Brownian motion and μ and σ are Lipschitz continuous. We are interested in finding discrete-time strategies which allow to super hedge, with a given probability, a European option with pay-off $G_T = f(e^{X_T})$ at date T. Since the terminal date is fixed, we can assume without loss of generality (by using the risk-free asset as a numeraire) that the interest rate is zero.

2.1 Discretizing the price process

Let $B_n = \{-1, 1\}^n$ and define a sequence of mappings $a_n : B_n \to (0, \infty)$, $n = 0, \dots, \infty$ (by convention, a_0 is a constant). We then introduce a sequence of $\mathbb{F}$-stopping times $(\tau_i)_{i\geq 0}$ (readjustment dates of our discrete-time strategy) by

$$\tau_{n+1} = \inf\{t > \tau_n : |X_t - X_{\tau_n}| = a_n(I_1, \dots, I_n)\}, \quad \tau_0 = 0 \tag{2}$$

$$I_{n+1} = \begin{cases} 1 & \text{if } X_{\tau_{n+1}} - X_{\tau_n} = a_n(I_1, \dots, I_n), \\ -1 & \text{if } X_{\tau_{n+1}} - X_{\tau_n} = -a_n(I_1, \dots, I_n). \end{cases} \tag{3}$$

Further, let the discrete filtration $\hat{\mathbb{F}}$ be defined by $\hat{\mathcal{F}}_n = \sigma(I_1, \dots, I_n)$. Note that the discrete-time process $(\hat{S}_n)_{n\geq 0}$ defined by $\hat{S}_n := S_{\tau_n}$ is $\hat{\mathbb{F}}$-adapted.

2.2 The auxiliary American option

Let θ_1 and θ_2 be $\hat{\mathbb{F}}$-stopping times with $\theta_1 \leq \theta_2$. We consider a path-dependent American-style option on the discrete-time process $\hat{S}_n$, which can be exercised at the earliest at date θ_1 and at the latest at date θ_2, and has pay-off $f(S_n)$ if exercised at date n. By the classical theory of American option pricing in the binomial model, the super hedging price of this option at date n is a $\hat{\mathcal{F}}_n$-measurable random variable $P_n := p_n(I_1, \ldots, I_n)$, which can be computed via

$$P_n = \operatorname*{ess\,sup}_{\theta \in \hat{\mathcal{T}}_n,\, \theta_1 \leq \theta \leq \theta_2} \hat{\mathbb{E}}[f(e^{x+\sum_{k=1}^{\theta} I_k a_{k-1}(I_1,\ldots,I_{k-1})})|\hat{\mathcal{F}}_n], \tag{4}$$

where $\hat{\mathcal{T}}_n$ is the set of $\hat{\mathbb{F}}$-stopping times θ with $\theta \geq n$ and $\hat{\mathbb{E}}$ denotes the expectation with respect to the *risk-neutral probability of the binomial tree*, that is, the probability $\hat{\mathbb{P}}$ such that

$$e^{a_n(I_1,\ldots,I_n)}\hat{\mathbb{P}}[I_{n+1} = 1|\hat{\mathcal{F}}_n] + e^{-a_n(I_1,\ldots,I_n)}\hat{\mathbb{P}}[I_{n+1} = -1|\hat{\mathcal{F}}_n] = 1, \quad n \geq 0. \tag{5}$$

Similarly, let $\Delta_n = \delta_n(I_1, \ldots, I_n)$ denote the amount of the risky asset which must be held in the portfolio between dates n and $n+1$ to super hedge this option. This amount can be computed as

$$\Delta_n = \frac{P_{n+1}(I_1, \ldots, I_n, 1) - P_{n+1}(I_1, \ldots, I_n, -1)}{S_n(e^{a_n(I_1,\ldots,I_n)} - e^{-a_n(I_1,\ldots,I_n)})}. \tag{6}$$

2.3 Embedding the continuous-time model into a binomial tree

Consider a self-financing strategy in the original (continuous-time) market, which is readjusted at dates $(\tau_i)_{i\geq 0}$, has initial cost P_0, and for which the quantity of risky assets held between dates τ_n and $\tau_{n+1} \wedge T$ is given by Δ_n. The cost process of this strategy will be denoted by V^u and is given by

$$V_T^u = P_0 + \sum_{0 \leq n \leq \theta_2 : \tau_n \leq T} \Delta_n (S_{\tau_{n+1}\wedge T} - S_{\tau_n}). \tag{7}$$

Theorem 1. *On the event that $\tau_{\theta_1} \leq T \leq \tau_{\theta_2}$,*

i. Let f be convex. Then, $V_T^u \geq f(S_T)$.

ii. Let f be Lipschitz with constant K. Then

$$V_T^u \geq f(S_T) - 2K \max_{\theta_1 \leq n \leq \theta_2} \max_{(I_1,\ldots,I_n)\in B_n} e^{x+\sum_{k=1}^{n} I_k a_{k-1}(I_1,\ldots,I_{k-1})} \sinh a_n(I_1, \ldots, I_n)$$

.

Proof. Assume that $\tau_{\theta_1} \leq T \leq \tau_{\theta_2}$ and let $\theta^* = \inf\{n : \tau_n \geq T\}$, so that $\tau_{\theta^*} \leq \tau_{\theta_2}$.

$$V^u_{\tau_{\theta^*-1}} + \Delta_{\theta^*-1} S_{\tau_{\theta^*-1}}(e^{a_{\theta^*-1}(I_1,\ldots,I_{\theta^*-1})} - 1) \geq f(S_{\tau_{\theta^*-1}} e^{a_{\theta^*-1}(I_1,\ldots,I_{\theta^*-1})})$$
$$V^u_{\tau_{\theta^*-1}} + \Delta_{\theta^*-1} S_{\tau_{\theta^*-1}}(e^{-a_{\theta^*-1}(I_1,\ldots,I_{\theta^*-1})} - 1) \geq f(S_{\tau_{\theta^*-1}} e^{-a_{\theta^*-1}(I_1,\ldots,I_{\theta^*-1})}).$$

By definition of θ^*, $\tau_{\theta^*-1} < T \leq \tau_{\theta^*}$, and therefore, by continuity of (S_t), there exists $\alpha \in [0,1]$ with

$$S_T = \alpha S_{\tau_{\theta^*-1}} e^{-a_{\theta^*-1}(I_1,\dots,I_{\theta^*-1})} + (1-\alpha) S_{\tau_{\theta^*-1}} e^{a_{\theta^*-1}(I_1,\dots,I_{\theta^*-1})}$$

In part i. we then conclude by convexity of f that

$$\begin{aligned} V^u_{\tau_{\theta^*-1}} + \Delta_{\theta^*-1}(S_T - S_{\tau_{\theta^*-1}}) &\geq \alpha f(S_{\tau_{\theta^*-1}} e^{-a_{\theta^*-1}(I_1,\dots,I_{\theta^*-1})}) \\ &\quad + (1-\alpha) f(S_{\tau_{\theta^*-1}} e^{a_{\theta^*-1}(I_1,\dots,I_{\theta^*-1})}) \geq f(S_T). \end{aligned}$$

In part ii. we get, by the Lipschitz property of f,

$$\begin{aligned} V^u_{\tau_{\theta^*-1}} + \Delta_{\theta^*-1}(S_T - S_{\tau_{\theta^*-1}}) &\geq f(S_T) + \alpha\{f(S_{\tau_{\theta^*-1}} e^{-a_{\theta^*-1}(I_1,\dots,I_{\theta^*-1})}) - f(S_T)\} \\ &\quad + (1-\alpha)\{f(S_{\tau_{\theta^*-1}} e^{a_{\theta^*-1}(I_1,\dots,I_{\theta^*-1})}) - f(S_T)\} \\ &\geq f(S_T) - \alpha K |S_{\tau_{\theta^*-1}} e^{-a_{\theta^*-1}(I_1,\dots,I_{\theta^*-1})} - S_T| - (1-\alpha) K |S_{\tau_{\theta^*-1}} e^{a_{\theta^*-1}(I_1,\dots,I_{\theta^*-1})} - S_T| \\ &\geq f(S_T) - K |S_{\tau_{\theta^*-1}} e^{a_{\theta^*-1}(I_1,\dots,I_{\theta^*-1})} - S_{\tau_{\theta^*-1}} e^{-a_{\theta^*-1}(I_1,\dots,I_{\theta^*-1})}| \\ &= f(S_T) - 2K S_{\tau_{\theta^*-1}} \sinh(a_{\theta^*-1}(I_1,\dots,I_{\theta^*-1})). \end{aligned}$$

□

In view of Theorem 1, our aim is to find $\hat{\mathbb{F}}$-stopping times θ_1 and θ_2 such that τ_{θ_1} and τ_{θ_2} approximate T. It is natural to consider stopping times based on the discrete quadratic variation of X, which can be computed from the values of the process at the readjustment dates. We define

$$Q_n := \sum_{i=1}^n \frac{(X_{\tau_i} - X_{\tau_{i-1}})^2}{\sigma^2(X_{\tau_{i-1}})} = \sum_{i=1}^n \frac{a^2_{n-1}}{\sigma^2(X_{\tau_{i-1}})} \in \hat{\mathcal{F}}_{n-1}. \tag{8}$$

By the integration by parts formula,

$$Q_n = \sum_{i=1}^n \int_{\tau_{i-1}}^{\tau_i} \frac{\sigma^2(X_t)}{\sigma^2(X_{\tau_{i-1}})} dt + 2\sum_{i=1}^n \int_{\tau_{i-1}}^{\tau_i} \frac{X_t - X_{\tau_{i-1}}}{\sigma^2(X_{\tau_{i-1}})} dX_t = \tau_n + \varepsilon_n \tag{9}$$

with

$$\varepsilon_n = \sum_{i=1}^n \int_{\tau_{i-1}}^{\tau_i} \frac{\sigma^2(X_t) - \sigma^2(X_{\tau_{i-1}})}{\sigma^2(X_{\tau_{i-1}})} dt + 2\sum_{i=1}^n \int_{\tau_{i-1}}^{\tau_i} \frac{X_t - X_{\tau_{i-1}}}{\sigma^2(X_{\tau_{i-1}})} dX_t. \tag{10}$$

Define $\hat{\mathbb{F}}$-stopping times

$$\theta_1 = \inf\{n \geq 0 : Q_{n+1} \geq T - \delta\} \tag{11}$$

and

$$\theta_2 = \inf\{n \geq 0 : Q_n \geq T + \delta\} \tag{12}$$

Then,

$$Q_{\theta_1} \leq T - \delta \quad \text{and} \quad Q_{\theta_2} \geq T + \delta,$$

and therefore, the event $\tau_{\theta_1} \leq T \leq \tau_{\theta_2}$ is implied by the event $\{\varepsilon_{\theta_1} \geq -\delta\} \cup \{\varepsilon_{\theta_2} \leq \delta\}$.

3 Numerical Analysis

Since a great amount of machinery for the numerical analysis of the Wiener process has been developed, here it is assumed that the Markovian log-price process (X_t) is a Wiener process. In other words, its drift term $(\mu(X_t))$ is zero and its volatility term $(\sigma(X_t))$ is constant and equals to one. So, the strategy rebalancing dates $((\tau_i)_{i\geq 0}$ could be inferred from the Brownian motion hitting times to the boundaries $[-a_i(I_1,\ldots,I_i), a_i(I_1,\ldots,I_i)]_{i\geq 0}$.

The Laplace transform of the Brownian motion hitting time to $[-b,b]$, γ^b, is Equation (13). Here for notation simplicity $a_i(I_1,\ldots,I_i) = b$.

$$\mathbb{E}[\exp -\lambda\gamma^b] = \frac{1}{\cosh b\sqrt{2\lambda}} \tag{13}$$

The proof could be done by applying optional sampling theorem to the martingale $N_t^s = \cosh(sW_t)\exp(-\frac{s^2}{2}t)$, $s = \sqrt{2\lambda}$ (the complete proof could be found in Appendix 4). By taking the differentiation with respect to λ in Equation (13), and letting λ go to zero, it could be inferred that $\mathbb{E}(\gamma^b) = b^2$.

The simulation of such hitting time is a laborious work. if the hitting time of the Wiener process $x + W_t$, $-1 \leq x \leq 1$, $t > 0$ to the boundary $[-1,1]$ is called γ_x^1, according to [12], the following boundary value problem holds for $v(t,x) = \mathbb{P}(\gamma_x^1 < t) - 1$.

$$\begin{gathered}\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2},\ t > 0,\ -1 < x < 1\\ v(0,x) = -1, v(t,-1) = v(t,1) = 0\end{gathered} \tag{14}$$

If this problem is solved with the method of change of variables, Equation (15) could be concluded for the cumulative density function of the hitting time $\gamma^1 = \gamma_0^1$.

$$F_{\gamma^1}(t) = 1 - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\exp(-\frac{1}{8}\pi^2(2n+1)^2 t),\ t > 0 \tag{15}$$

If Equation (14) is solved by the Cauchy problem resulted from extending initial data in an odd way on the axis, Equation (16) would hold for the same cumulative density function.

$$\begin{gathered}F_{\gamma^1}(t) = 2\sum_{n=0}^{\infty}(-1)^k \mathrm{erf}(\frac{2n+1}{\sqrt{2t}}),\ t > 0\\ \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty}\exp(-s^2)ds,\ \mathrm{erf}(0) = 1\end{gathered} \tag{16}$$

For the calculation of large t, Equation (15) is apt, while for that of small t, Equation (16) is suitable. It is worthwhile noting that the density function derived from Equation (16) is the same as the one derived by inverting the Laplace transform in Equation (13). [2] gives some description about the shape of the distribution, such that it is unimodal, and derives some expressions for both its right and left asymptotics.

In order to generate random numbers from γ^1 distribution, it is required to solve the equation $F_{\gamma^1}(t) = U$, when U is a uniform random number in $[0,1]$ interval. A simple way to solve the equation is to use the Newton method. It is important to note that when it could be possible to generate random numbers from γ^1 distribution, using Brownian motion scaling property, generating random numbers from the desired γ^b distribution is also possible. For more information about the hitting time distribution function, please refer to [12], [9] and [2].

The initial rebalancing happens at time 0, and the difference between any two successive rebalancing dates is generated from the distribution of the Brownian motion hitting time, γ^b. So, the rebalancing dates and the stock price at those times are generated without doing any simulation of the stock price process by using time discretization, and as a result, there is not any discretization error involved.

The initial value of the hedging portfolio is the date zero price of the option calculated by Equation (4) when $n = 0$. At rebalancing date n, the amount of the risky asset which is held in the portfolio between times τ_n and $\tau_{n+1} \wedge T$, is changed to Δ_n, that is calculated according to Equation (6). In the benchmark strategy, Δ_n is the delta hedging ratio. For a given stock price process, the number of rebalancing dates would be $\theta \wedge \theta_2$ (θ_2 is the maximum possible number of rebalancing dates, for which hedging ratios exist). For the benchmark strategy, it would be θ, since hedging ratios exist till the option expiration, and θ is computed via Equation (17).

$$\theta = \inf\{n : \tau_{n+1} \geq T\} \tag{17}$$

According to part i of the Theorem 1, on the event that $\theta + 1 \leq \theta_2$, it is expected that portfolio payoff (V_T^u) be greater than or at least equal to the option payoff (G_T), or in other words, the hedging error, defined as $V_T^u - G_T$, must be non-negative. However, when this event does not hold, the hedging error could be negative.

In order to find out the payoff of the option at the expiration time, T, the stock price at T is required, which could be generated from the distribution function of $W_{T-\tau_n}$ given that $\tau_{n+1} \geq T$. This distribution follows either of Equations (18) and (19).

$$\begin{aligned} \mathbb{P}(W_s \leq \beta | \tau^b \geq s) = \frac{1}{1 - F_{\gamma^1}(s)} \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} ((-1)^k \\ + sin(\frac{\pi(2n+1)\beta}{2})) \exp{(\frac{-1}{8})} \pi^2 (2n+1)^2 s, \ \ s = T - \tau_n \end{aligned} \tag{18}$$

$$\mathbb{P}(W_s \leq \beta | \tau^b \geq s) = \frac{1}{1 - F_{\gamma^1}(s)} \sum_{n=0}^{\infty} \frac{(-1)^n}{2} (\mathrm{erf}(\frac{2n-1}{\sqrt{2s}}) - \mathrm{erf}(\frac{2n+\beta}{\sqrt{2s}}) - \mathrm{erf}(\frac{2n+2-\beta}{\sqrt{2s}}) + \mathrm{erf}(\frac{2n+3}{\sqrt{2s}})), \; s = T - \tau_n \tag{19}$$

The proof could be found in [12]. The method for generating random number from distributions (18), which is more suitable for large s, and (19), which is more appropriate for small s, is the same as the one for distributions (15) and (16).

4 Strategy Performance: General Results

In this part, the performance of the strategy is analyzed and compared with that of a benchmark strategy that is Black Scholes (BS) delta hedging. In our simulations, it is supposed that $a_i(I_1, \ldots, I_i)_{i \geq 0} = a$ is constant, and it is determined by the expected frequency of rebalancing times, such as daily, weekly, etc. So, in fact, the intuition is the same as what traders use in financial markets that is one rebalances the portfolio when the price has moved more than some certain portion of the price of the previous rebalancing date, or in other words, the new rebalancing date is the time when the difference of the log-price with that of the previous rebalancing date is greater than or equal to some threshold value, called a here.

The European option which is tried to be hedged is vanilla European call with payoff $G_T = (S_T - M)_+$ at date T when M is the strike of the option. Since here the payoff function is convex, part i of the Theorem 1 must hold.

In the model, there are two parameters, including a and δ. a is calibrated by the five expected rebalancing dates frequencies, including daily, weekly, 2-weekly, monthly, and seasonly. δ, which is introduced in equations (11) and (12), implies the shortfall probability of the super hedging strategy, that is how likely it is that the strategy fails to make a non-negative hedging error. So, the larger is the shortfall probability of the super hedging strategy, the smaller would be the initial value used to super hedge the option.

For any values of δ and a, several stock price (S_t) trajectories are generated, and for any trajectory, the hedging error for both of the strategies is computed. Here 100000 trajectories are generated from the stock price process, whose initial value (S_0) is 20 Euros. Then for all of the five expected rebalancing dates frequencies, the shortfall probability is plotted against the hedging price, that is the initial price (P_0) used to super hedge the call option (with strike $M = 22$ euros, and maturity $T = 2$ years), for both of the strategies.

As it is shown in Figure 1, for any expected rebalancing dates frequencies, our strategy performs much better than the benchmark strategy (delta hedging strategy) with the criterion of shortfall probability, since for any given hedging price, it has a much smaller shortfall probability in comparison with that of the benchmark strategy. However, it is important to note that shortfall probability, individually, could not be a perfect criterion, because although the likelihood

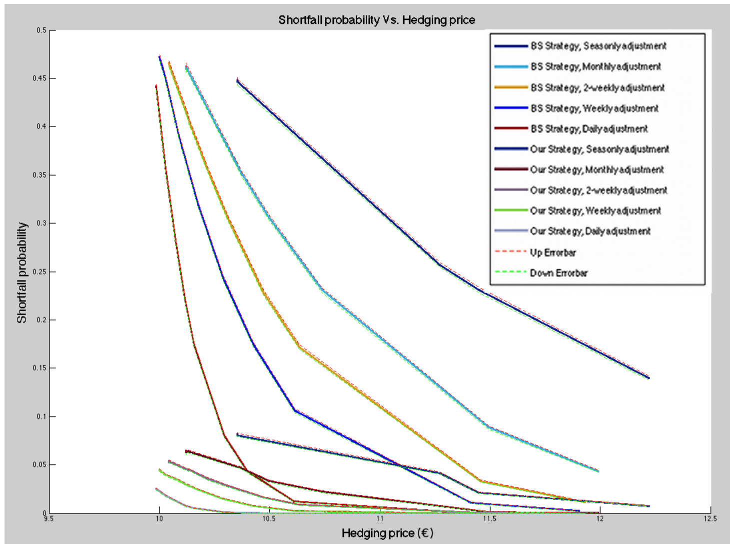

Figure 1: Shortfall probability versus hedging price, for all of the five frequencies of portfolio rebalancing, and for both the above strategy and the BS delta hedging strategy

of shortfall is small, the size of it could be big. So, it is also important to take its size into account, and as a result, Figure 2 plots expected negative hedging error, that is the average of negative hedging error times its probability (shortfall probability), versus the hedging price.

According to Figure 2, our strategy still performs better than the benchmark strategy, since for any given hedging price, it has a bit larger expected negative hedging error in comparison with that of the BS delta hedging strategy. However, the difference between the two strategies is not as significant as the one observed in Figure 1, which indicates that the shortfall size in our strategy is usually larger than that of the BS delta hedging strategy. In general, it could be concluded that our strategy performs to some small extent better than the benchmark strategy for super hedging the European option in discrete time, given the two criteria of shortfall probability and expected negative hedging error.

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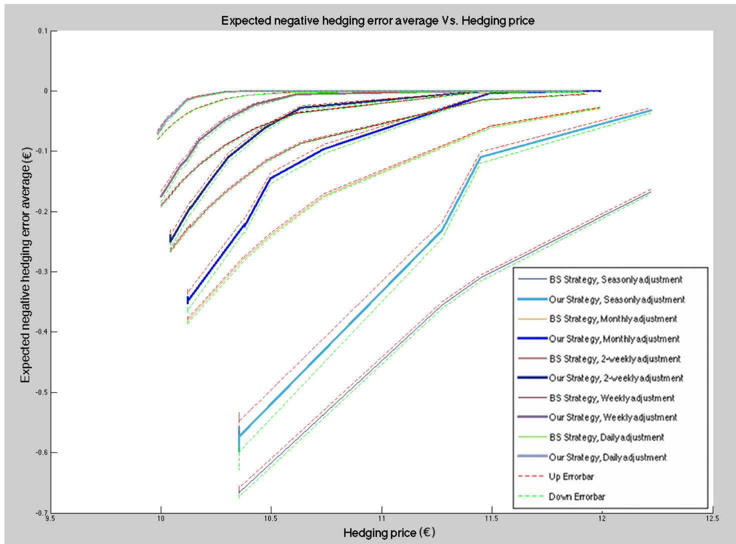

Figure 2: Expected negative hedging error versus hedging price, for all of the five frequencies of portfolio rebalancing, and for both the above strategy and the BS delta hedging strategy

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Appendix: Laplace transform of the Brownian motion hitting time

Here the Laplace transform of the Brownian notion hitting time to the boundary $[-b, b]$, shown by γ_b, has been derived. Since $M_t^s = \exp(sW_t - \frac{s^2}{2}t)$ is a martingale, so as to $N_t^s = \frac{M_t^s + M_t^{-s}}{2} = \cosh(sW_t)\exp(-\frac{s^2}{2}t)$. Then, based on the optional sampling theorem, it could be concluded that $N_{t\wedge\gamma_b}^s$ is also a martingale.

$$\mathrm{N}_{t\wedge\gamma_b}^s = \cosh(sW_{t\wedge\gamma_b})\exp(-\tfrac{s^2}{2}(t\wedge\gamma_b))$$

$$\lim_{t\to\infty} N_{t\wedge\gamma_b}^s = \lim_{t\to\infty} \mathbb{I}(\gamma_b < \infty)\tfrac{1}{2}\exp(-\tfrac{s^2}{2}(t\wedge\gamma_b))[\exp(sb) + \exp(-sb)]$$

$$\lim_{t\to\infty} \mathbb{E}[N_{t\wedge\gamma_b}^s] = \lim_{t\to\infty} \mathbb{E}[\mathbb{I}(\gamma_b < \infty)\tfrac{1}{2}\exp(-\tfrac{s^2}{2}(t\wedge\gamma_b))[\exp(sb) + \exp(-sb)]]$$

If $s = 0$, then:

$$1 = \mathbb{E}[\mathbb{I}(\gamma_b < \infty]) = \mathbb{P}(\gamma_b < \infty)$$

Since N_t^s is a martingale, so its expectation should be constant.

$$1 = \mathbb{E}[\frac{1}{2}\exp(-\frac{s^2}{2}(t\wedge\gamma_b))[\exp(sb) + \exp(-sb)]]$$

And since $\frac{s^2}{2} = \lambda$.

$$\mathbb{E}[-\lambda\gamma_b] = \frac{1}{\cosh b\sqrt{2\lambda}}$$

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Abstract

In this article, a new strategy for super hedging European options with a given success probability in stochastic volatility models, is introduced. Stochastic volatility is a more realistic assumption because the fact that volatility is not constant over time is a long observed feature in the market. In this strategy, the option is super hedged by using the hedge ratios of the Timer option. A comparison has been also established between the performance of our strategy and that of the benchmark strategy, that shows our strategy has a better performance.

1 Introduction

In this article, a new strategy for super hedging European options, with a given success probability, in stochastic volatility models, is proposed, in which the portfolio is readjusted at continuous time. Stochastic volatility is a more realistic assumption because the fact that volatility is not constant over time is a long observed feature in the market. In stochastic volatility models, the market is incomplete when it just consists of a risk-free asset, and a risky asset, because there are two sources of uncertainty, one from the stock price and the other from the variance, which could not be completely hedged with just one risky asset. In such an incomplete market, it is not possible to perfectly hedge all contingent claims, but it would be still very useful to super hedge those contingent claims that is to make a portfolio whose payoff is at least the same as that of the contingent claim in all states of the world. Perfect super hedging could be very expensive, and many investors who do not want to incur such a cost, may be interested in an strategy which could super hedge not perfectly, but with a high probability of success, to lessen the strategy initial capital requirement ([5]). In this regard, we have presented a new such a super hedging strategy and its performance from both success probability and expected shortfall size point of views, indicate that it is superior to the benchmark strategy.

In this strategy, Timer options has been used to super hedge the initial European option. Timer option is an Exotic option, being first marketed by Société Générale Corporate and Investment Banking (SG CIB) at 2007, that let the buyers fix the level of volatility in the pricing model. Based on the level of volatility, a volatility target is determined, so, the expiration of the option would be random and is the time that market accumulated volatility hits the volatility target. In fact, the option expires according to how different market volatility is from the initial constant volatility used in the pricing model. If the market volatility be higher than the initial volatility, then the option expires earlier than expected and vice versa. When the contingent claim is the Timer option, it is possible to perfectly hedge the claim with just one risky asset if the interest rate is zero by transferring the volatility uncertainty to the randomness of the expiration, using Brownian motion time change technique. For more information regarding Timer options, please refer to [1] and [8]. In our strategy, based on the maximal level of shortfall possibility, the variance budget (volatility target) is determined, and then by using the hedging ratios of the Timer option, the initial European option is super hedged, with a given success probability.

The pioneering works for super hedging with a given success probability are [5], in which the aim has been to make a hedging strategy which works well with highest probability given a capital constraint, and [6], in which the aim to construct strategies which minimize the shortfall risk. [11] and [12] investigate the same problem while using convex and coherent risk measures to quantify shortfall risk. The same problem has been also addressed by [9] and [10].

The organization of this article is as follows. Section 2 presents the super hedging strategy. Section 3 elaborates on the numerical analysis of the strategy and how it is compared with a benchmark strategy. The general results of the performance of the strategy could be found in Section 4.

2 The Super Hedging Strategy

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. We consider a financial market consisting of a risk-free asset and a risky asset with price process $(S_t)_{t\geq 0}$ such that it follows the Markov process:

$$dS_t = \mu_t S_t dt + \sqrt{\nu_t} S_t dW_t^1, \quad S_0 = s, \tag{1}$$

Here μ_t is the instantaneous drift of stock price returns and is Lipschitz continuous, and the variance process $(\nu_t)_{t\geq 0}$ follows:

$$d\nu_t = \alpha(S_t, \nu_t, t)dt + \eta\beta(S_t, \nu_t, t)\sqrt{\nu_t} dW_t^2, \quad \nu_0 = \nu, \tag{2}$$

Where W^1 and W^2 are standard $\mathbb{F}$-Brownian motions which have correlation ρ with each other ($< dW^1 dW^2 >= \rho dt$). Here η is called the volatility of volatility. We are interested in finding continuous-time strategies which allow to super hedge, with a given probability, a European option with pay-off $G_T = f(S_T)$ at date T when f is the payoff function. Since the terminal date is fixed, we can assume without loss of generality (by using the risk-free asset as a numeraire) that the interest rate is zero.

We will use Timer options hedge ratios in order to hedge this European option.

2.1 Finding the random expiration time of the Timer option

The integrated variance process Y_t, which is the realized variance, is defined as:

$$Y_t = \int_0^t \nu_s \, \mathrm{d}s, \quad Y_0 = 0, \tag{3}$$

Then, the $\mathbb{F}$-stopping time τ (random expiration time) is introduced by:

$$\tau = \inf\{t > 0 : Y_t \geq Q\} \tag{4}$$

When Q is the variance budget, defined such that:

$$\mathbb{P}(\tau \geq T) = 1 - \alpha \tag{5}$$

or, in other words:

$$\mathbb{P}(Y_T \leq Q) = 1 - \alpha \tag{6}$$

When α is the maximum possible shortfall probability. In the next part, pricing and hedging of the Timer option with random expiration τ, variance budget Q, and payoff function f are discussed.

2.2 Pricing and hedging the Timer option

If we assume that there exists a risk neutral probability $\acute{\mathbb{P}}$, then the price process (P_t) of the Timer option expiring at time τ with payoff $G_\tau = f(S_\tau)$ is $\mathcal{F}_t$-measurable and follows:

$$P_t = \mathbb{E}^{\acute{\mathbb{P}}}[f(S_\tau)|\mathcal{F}_t], \quad 0 \leq t \leq \tau \tag{7}$$

Where $\mathbb{E}^{\acute{\mathbb{P}}}$ denotes the expectation with respect to the risk-neutral probability $\acute{\mathbb{P}}$. It could be shown that in the case that interest rate is zero like here, Timer option price process (7) would be equal to the price process of a non-Timer European option in Black Scholes model.

To demonstrate this, X_t and A_t are defined as followings:

$$X_t = \int_0^t \sqrt{\nu_s} dW_s^1 \tag{8}$$

$$A_t =< X >_t= \int_0^t \nu_s ds \tag{9}$$

Since X_t is a continuous martingale, and $A_t \to \infty$ as $t \to \infty$, X_t could be expressed as a time-changed Brownian motion ($X_t = W_{A_t}^1$). Then, risky asset price process (1) could be written as:

$$\begin{aligned} S_t &= S_0 \exp{(X_t - \frac{1}{2}A_t + \int_0^t \mu_s ds)} \quad \text{under } \mathbb{P} \\ S_t &= S_0 \exp{(W_{A_t}^1 - \frac{1}{2}A_t + \int_0^t \mu_s ds)} \quad \text{under } \mathbb{P} \end{aligned} \tag{10}$$

So, under risk neutral probability $\acute{\mathbb{P}}$, the term $\int_0^t \mu_s ds$ would be replaced with $\int_0^t r_s ds$ term when r_t is the possible interest rate process.

$$S_t = S_0 \exp{(W_{A_t}^1 - \frac{1}{2}A_t + \int_0^t r_s ds)} \quad \text{under } \acute{\mathbb{P}} \tag{11}$$

If variance be constant ($\nu_t = \nu$) like in Black Scholes model, then the resulting price process $(\tilde{S}_t)$ would be:

$$\tilde{S}_t = S_0 \exp\,(\sqrt{\nu} W_t^1 - \frac{1}{2}\nu t + \int_0^t r_s ds) \quad \text{under } \acute{\mathbb{P}} \tag{12}$$

Based on Equations (11) and (12), S_τ equals to $\tilde{S}_{\acute{T}}$ under $\acute{\mathbb{P}}$ when $\nu = \frac{A_\tau = Q}{T}$, if and only if interest rate is zero ($r_t = 0$) which is the case in here.

When S_τ equals to $\tilde{S}_{\acute{T}}$ under $\acute{\mathbb{P}}$, then the Timer option price process (7) would be the same as the price process of an European option with payoff $f(S_{\acute{T}})$ at expiration $\acute{T}$ in Black Scholes model with constant variance $\nu = \frac{Q}{\acute{T}}$ ([1]). The expiration $\acute{T}$ could be considered as any positive number since the variance ν would adjust itself accordingly.

In [3], it has been shown that in order to hedge the Timer option, the same hedging ratios could be used as those utilized in Black Scholes model to delta hedge the European option with payoff $f(S_{\acute{T}})$ at expiration $\acute{T}$ when $\acute{T} = Q$, and consequently $\nu = 1$, and time t is conceived as $\int_0^t \nu_s ds$.

2.3 Building the self-financing strategy

Consider a self-financing strategy, which is readjusted in continuous time, and it has initial cost R_0, and for which the quantity of risky assets held at time t is given by Δ_t. The cost process of this strategy will be denoted by V^u and is given by:

$$V_T^u = R_0 + (\int_0^{\tau \wedge T} \Delta_s \, \mathrm{d}S_s) + \Delta_{\tau \wedge T}(S_T - S_{\tau \wedge T}) \tag{13}$$

Theorem 1. *Let f be convex, Then, $V_T^u \geq f(S_T)$ on the event that $\tau \geq T$.*

Proof. According to Ito lemma:

$$df(S_t) = \frac{\partial f}{\partial S} dS_t + \frac{\partial^2 f}{\partial S^2} d < S >_t$$

Since f is convex ($\dfrac{\partial^2 f}{\partial S^2} \geq 0$), and interest rate is zero, then it could be concluded that the payoff process ($G_t = f(S_t)$) is a submartingale with respect to any risk neutral probability ($\acute{\mathbb{P}}$). In other words:

$$G_t \leq \mathbb{E}^{\acute{\mathbb{P}}}[G_T | \mathcal{F}_t], \quad \forall \acute{\mathbb{P}} \in \Theta$$

Where Θ is the set of all risk-neutral probabilities.

So, it would be never optimal to exercise the option earlier than its expiration time. Also since the interest rate is zero, the timer option can be priced and perfectly hedged by using the Black Scholes strategy. Assume $\tau \geq T$, then $V_\tau^u = f(S_\tau)$. Since it is never optimal for the option to be exercised earlier than its expiration, then:

$$V_t^u \geq f(S_t), \quad \forall t \leq \tau$$

Since $\tau \geq T$, then as a result, $V_T^u \geq f(S_T)$

□

In view of Theorem 1, our aim is to find variance budget Q such that with the probability greater than or equal to $(1-\alpha)$, the Timer option expiration τ is greater than or equal to the initial call option expiration T.

3 Numerical Analysis

For the numerical analysis, Heston model has been selected among the stochastic volatility models, so in the variance process (2), $\alpha(S_t, \nu_t, t)$ and $\beta(S_t, \nu_t, t)$ are:

$$\begin{aligned} \alpha(S_t, \nu_t, t) &= -\lambda(\nu_t - \bar{\nu}) \\ \beta(S_t, \nu_t, t) &= 1 \end{aligned} \tag{14}$$

Here λ is the speed of reversion of variance process to its mean $\bar{\nu}$.

In order to simulate the Heston process, it is required to discretize the stock price process and the variance process. Time is discretized as $t_i = (i-1)h, i \in \mathbb{N}$, when h is the time step. The stock price process is discretized by simple Euler scheme.

$$S_{t_{i+1}} = \sqrt{\nu_{t_i}} S_{t_i} \sqrt{h} Z \tag{15}$$

Where $Z \sim N(0,1)$ under the risk neutral measure. The variance process is discretized according to Milstein scheme.

$$\nu_{t_{i+1}} = \nu_{t_i} - \lambda(\nu_{t_i} - \bar{\nu})h + \eta\sqrt{\nu_{t_i}}\sqrt{h}W + \frac{\eta^2}{2}h(W^2 - 1) \tag{16}$$

Where $W \sim N(0,1)$ such that $\mathbb{E}(ZW) = \rho$. In the variance process discretization, if $\nu_{t_i} = 0$ and $\frac{4\lambda\bar{\nu}}{\eta^2} > 1$, then $\nu_{t_{i+1}} > 0$ ([7]), so the problem of the negative variance would be solved if the parameters satisfy this condition, which is the case in our simulations.

There is a closed form formula for the Laplace transform of Y_T in Heston model ([2]), and it could be used in order to find variance budget Q such that with the probability greater than or equal to $(1-\alpha)$, the Timer option expiration τ is greater than or equal to the initial call option expiration T.

$$\mathbb{E}\{e^{-uY_T}\} = \frac{\exp\left(\frac{\lambda^2\bar{\nu}T}{\eta^2}\right)}{\left(\cosh\frac{\gamma T}{2} + \frac{\lambda}{\gamma}\sinh\frac{\gamma T}{2}\right)^{\frac{2\lambda\bar{\nu}}{\eta^2}}} \exp\left(-\frac{2\nu_0 u}{\lambda + \gamma\coth\frac{\gamma T}{2}}\right) \tag{17}$$

Where $\gamma = \sqrt{\lambda^2 + 2\eta^2 u}$. This Laplace transform function could be inverted numerically in order to find out the probability distribution function of Y_T, from which it is possible to infer Q, given any value of α. For Laplace numerical inversion, the Fourier transform technique, introduced in [4], is used.

The European option which has been selected for the numerical analysis, is European call with convex payoff $G_T = (S_T - M)_+$, when M is the strike of the option. The performance of our strategy for super hedging this option is evaluated by using Monte Carlo simulations, and it is also compared with the Heston delta hedging strategy.

The Timer option with variance budget Q, expiration τ, and payoff $(S_\tau - M)_+$ could be hedged with the same ratios as those of the Black Scholes strategy, used to hedge the vanilla option with strike M, expiration equals to the variance budget Q, and when time is the integrated variance, and volatility is one. So, the price process P_t in Equation (7) for this option would be:

$$P_t = S_t N(d_1) - M N(d_2) \tag{18}$$

When $N(x)$, d_1 and d_2 are computed as follows:

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp \frac{-z^2}{2} \, \mathrm{d}z \tag{19}$$

$$d_{1,2} = \frac{\log \frac{S_t}{M} \pm 0.5(Q - \int_0^t \nu_s \, \mathrm{d}s)}{\sqrt{Q - \int_0^t \nu_s \, \mathrm{d}s}} \tag{20}$$

Then Δ_t, that is the amount of the risky asset which must be held in the portfolio at time t to super hedge this option, can be calculated as:

$$\Delta_t = \frac{\partial P_t}{\partial S_t} = N(d_1) \tag{21}$$

The initial value of the hedging portfolio is the time zero price of the option calculated by Equation (18) when $t = 0$. At time t, the amount of the risky asset held in the portfolio would be Δ_t, that is calculated according to Equation (21). In the benchmark strategy, Δ_t is the Heston delta hedging ratio, that is the derivative of the available closed form formula for the European call price in the Heston model with respect to the stock price at time t (for more information, please refer to Appendix 4 or [7]). The hedging portfolio is readjusted all the time till $\tau \wedge T$, when τ is the expiration time of the Timer option, calculated by Equation (4). However, in the benchmark strategy, it is done till the expiration of the initial European option, that is T.

According to Theorem 1, on the event that $\tau \geq T$, it is expected that portfolio payoff (V_T^u) be greater than or at least equal to the option payoff (G_T), or in other words, the hedging error, defined as $V_T^u - G_T$, must be non-negative. However, when this event does not hold, the hedging error could be negative.

4 Strategy Performance: General Results

In the model, there is one parameter, that is α, and it represents the maximum possible shortfall probability, that is how likely it is that the strategy fails to make a non-negative hedging error. So, the larger is the shortfall probability of the super hedging strategy, the smaller would be the initial price (calculated according to the equation (18) when $t = 0$) used to super hedge the option.

For any value of α, several trajectories of the stock price process (S_t) and the variance process (ν_t) are generated, and for any trajectory, the hedging error for both of the strategies is computed. Here 100000 trajectories are generated from the stock price process, whose initial value (S_0) is 20 Euros, and the variance process, whose initial value (ν_0) is 0.11. Besides, initial European option is a call with strike $M = 19.5$ Euros and expiration $T = 0.1$ year. Also, in the simulations, the Heston parameters, λ, $\bar{\nu}$, η and ρ, are taken as 2.0, 0.11, 0.9 and -0.25 respectively, and the time step (h) is fixed at 0.0001 year.

In Figure 1, shortfall probability is plotted against hedging error for both of the strategies. It is important to note that although there exists a portion related to discretization error in the values for shortfall probability, it could still serve as a good criterium for making a comparison between the strategies. As it is demonstrated, for any hedging price, our strategy has a much smaller shortfall probability than that of the benchmark strategy, so, it could be concluded that our strategy performs much better given this criterium.

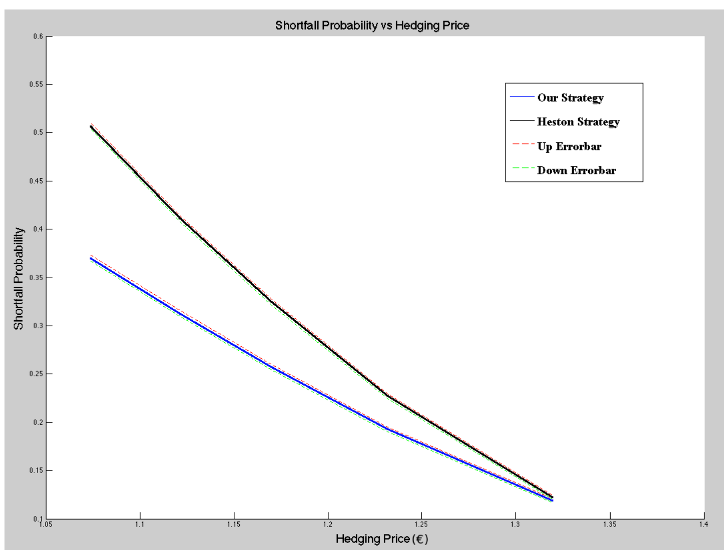

Figure 1: Shortfall probability versus hedging price, for both the above strategy and the Heston strategy

One drawback of shortfall probability is that it fails to take the shortfall size into account. So, Figure 2 plots expected negative hedging error, that is the average of negative hedging error times its probability (shortfall probability), against the hedging error.

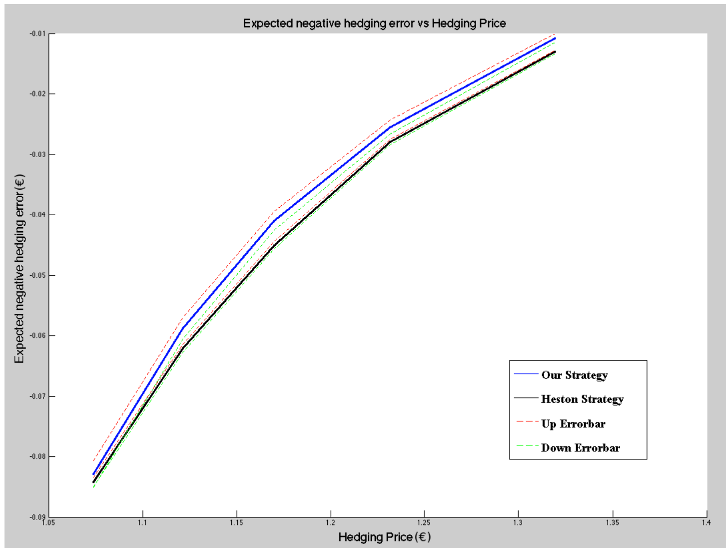

Figure 2: Expected negative hedging error versus hedging price, for both the above strategy and the Heston strategy

Based on Figure 2, it could be still concluded that our strategy has a better performance in comparison with that of the Heston delta hedging strategy, because for any given hedging price, it has a bit larger expected negative hedging error. It should be also noted that the difference between the two strategies given this criterium is not as significant as the difference observed given the shortfall probability criterium. Consequently, it can be mentioned that the shortfall size in our strategy is bigger although its likelihood is smaller. In general, it could be concluded that our strategy shows a bit better performance for super hedging the European option.

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Appendix: Heston delta hedging

The hedging ratios for European call options, in a closed form, in the Heston delta hedging strategy, follow:

$$\Delta_t = \frac{\partial P_1}{\partial \phi} + P_1 - \frac{M}{S_t}\frac{\partial P_0}{\partial \phi}$$

When $\phi = \log(\frac{S_t}{M})$, $\tau = T - t$, and $P_{1,2}$ are computed as:

$$P_j(\phi, u, \tau) = 0.5 + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\,\{\frac{\exp\{C_j(u,\tau)\bar{\nu} + D_j(u,\tau)\nu + iu\phi\}}{iu}\}\,\mathrm{d}u, \quad j = 0, 1.$$

where:

$$D(u,\tau) = r_- \frac{1 - e^{-d\tau}}{1 - ge^{-d\tau}}$$
$$C(u,\tau) = \lambda\{r_-\tau - \frac{2}{\eta^2} \log\{\frac{1 - e^{-d\tau}}{1 - g}\}\}$$
$$g = \frac{r_-}{r_+}$$
$$r_\pm = \frac{\beta \pm d}{\eta^2}$$
$$d = \sqrt{\beta^2 - 4\alpha\chi}$$
$$\alpha = -\frac{u^2}{2} - \frac{iu}{2} + iju$$
$$\beta = \lambda - \rho\eta j - \rho\eta iu$$
$$\chi = \frac{\eta^2}{2}$$

The integrals are numerically calculated by using Adaptive Quadrature rules.