M1 Project: Climate - Biosphere interactions using ODE models

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Abstract

There are many examples of the complex interactions of climate and vegetation through various feedback mechanisms. Climatic models have begun to take into account vegetation as an important player in the evolution of the climate. Climate models range from complicated, large scale GCMs to simple conceptual models. It is this last type of modeling that I looked at in my project. Conceptual models usually use differential equations and techniques from dynamical systems theory to investigate basic mechanisms in the climate system. They have in particular been applied to investigate glacial-interglacial cycles. These models have not often included vegetation as one of their variables, and this is what I've looked at in the project. First I investigate a simple, two equation model, and show that even in such a simple model, interesting oscillatory behaviour can be observed. Then I go on to study models with three equations, based on an existing model for temperature and ice sheet evolution. I extend this model in two ways: by adding a vegetation variable, and by adding a carbon dioxide variable. Again, oscillations are observed, but the existence depends on parameters that are linked to the vegetation. Finally I put it all together in a model with four equations. These models show that vegetation is an important factor, and can account for some specific features of glacial-interglacial cycles. They open up a lot of possibilities for further investigation and extensions.

1 Introduction

1.1 Climate - vegetation interaction

Climate has an important effect on vegetation. Plant growth is affected by temperature, carbon dioxide levels and availability of different nutrients. Also the available space is important: ice covered parts of land are not suitable for vegetation growth. It also works the other way around, though: vegetation plays an important role in the regulation of the climate. Many different effects are observed. One of the most important is the albedo effect: vegetation is darker than bare ground or ice and therefore absorbs more solar radiation and warms the planet. This phenomenon appears to be important in desert regions, where it interacts with the hydrological cycle. Charney [Charney, 1975] was the first to include this in a model, but others have followed since [Claussen et al., 1999; Zeng et al., 1999; Zeng and Neelin, 2000]. Another important region where albedo feedback is important are the high latitudes, where boreal forests mask snow in winter, causing an effective warming of the surface [Bonan, 2008; Brovkin et al., 2003].

Another effect is the uptake of carbon dioxide by plants, which in turn attenuates the greenhouse effect and cools the surface. In addition to the more obvious effects of albedo change and carbon dioxide uptake, there are many slightly more subtle mechanisms through which vegetation influences climate. Examples are the effect of plankton on cloud formation (the CLAW hypothesis [Ayers and Cainey, 2007]), evapotranspiration or more exotic feedbacks, such as the so called lightning-biota feedback as studied in [Shepon and Gildor, 2008]. A review of different mechanisms can be found in [Meir et al., 2006]. The article [Claussen, 2009] gives an introduction on the vegetation-climate interactions on long timescales.

Closely related to vegetation and climate is the global carbon cycle. The amount of carbon dioxide influences the radiative balance of the Earth through the greenhouse effect. Plants need carbon dioxide for photosynthesis, and carbon fluxes between atmosphere, ocean and the land are very important in climate modeling. It has been suggested that modeling of the carbon cycle is necessary for any model studying paleovegetation [Prentice and Harrison, 2009]. The carbon cycle is a very complex thing [Falkowski et al., 2000], with the ocean chemistry and dynamics playing an important role [Follows and Oguz, 2004].

Although many examples are known where vegetation plays an essential role in the climate system, it has only been rather recently that vegetation is included as an active player in climate models. Climatic models range from simple, conceptual ODE models up to full scale GCMs (*general circulation models* or *global climate models*). Across the whole range, vegetation can be included to better explain various climatic phenomena and trends. In some cases, predictions for models that couple atmosphere, ocean and vegetation dynamics (sometimes refered to as *Earth system models*) differ radically from models excluding vegetation [Meir et al., 2006], showing the need to include vegetation in our models to obtain better understanding of the climate.

Some studies that have included vegetation are the GCMs that were recently used for the IPCC fifth assesment report [Stocker et al., 2013; Piao et al., 2013]. GCMs are the most complex models used for climate modeling, and are meant to describe in great detail the processes in atmosphere, ocean, and land. These models are very costly computationally and are as such not often used to model climate over long time scales, which is needed when studying for example ice age cycles or transitions in past climates. More suitable for these kind of questions are models of intermediate complexity, EMICs (*Earth models of intermediate complexity*) [Claussen et al., 2002]. They simplify some of the processes and have lower resolution than the full-fledged GCMs, and are faster. They have been

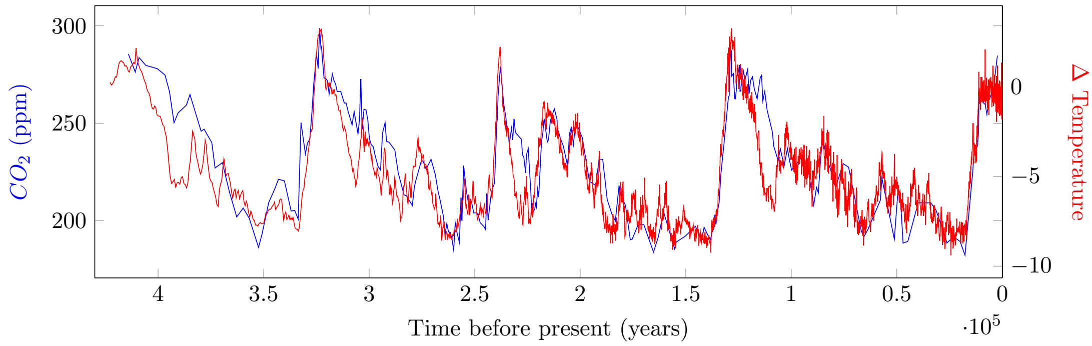

Figure 1: Data from Vostok ice core. Reproduced from [Petit et al., 1999].

succesfully used for studying paleoclimate, also coupled with vegetation models [Willeit et al., 2013; Ganopolski et al., 1998; Claussen et al., 1999].

The simplest models for climate are conceptual models, which usually use differential equations to model a simple climate system. These models are not realistic, but allow to study basic underlying mechanisms. They are also useful for exploring bifurcations, which correspond to tipping points [Lenton et al., 2008] in the climate system. Bifurcations or tipping points correspond to situations in which a small parameter change can have a large effect on the behaviour of the global system. Studying conceptual models can also provide guidance in interpreting results from larger, more complicated models [Brovkin et al., 1998; Brovkin et al., 2003]. A popular application of dynamical systems modeling for climate is the explanation of ice age cycles. We'll discuss this and review some models in the next section.

1.2 Dynamical systems modeling for climate

1.2.1 Ice ages

The Quaternary period (about 2.6 million years ago uptil present) is characterized by ice age cycles. During glacial periods, large ice sheets covered parts of the land, and average temperatures were about 5 degrees lower than what we have today. The main driver for these cycles are variations in the Earth's orbit, called Milankovitch cycles [Hays et al., 1976], reinforced by the internal variability and feedback mechanisms of the climate. Not only temperature and ice sheets varied during these periods, also the level of carbon dioxide changed a lot. Data from ice cores show that carbon dioxide and temperature rise and fall at more or less the same pace (Figure 1). There have also been studies that show that vegetation changes under influence of the orbital forcing [Dupont, 2011; Tzedakis, 2005], and that it plays a role in reinforcing the orbital variations [Claussen et al., 2006; Meissner et al., 2003]. Vegetation and carbon dioxide (for example through the greenhouse effect, or the influence of carbon dioxide on the growth of plants) play an important role in the glacial-interglacial cycles.

1.2.2 Climatic oscillators

Oscillations in the climatic system, and especially the ice age cycles, have been conceptually described using dynamical systems. In the book [Ghil and Childress, 1987, Part IV], the basic features of this kind of modeling are explained. The aims of such ODE models are diverse: some try to explain the typical frequencies that appear in the time series such as in Figure 1, and especially the 100ky peak that cannot be explained by orbital forcing alone and thus needs some kind of internal mechanism to be obtained [Ghil, 1994]. Other models try to explain the sawtooth shape of these curves, such as [Hogg, 2008]. The model that is explored in the long paper [Fowler et al., 2013] includes a detailed carbon cycle and a lot of physically based mechanisms, while some authors try to make their model as simple as possible [Crucifix, 2012], but still capture the basic oscillations.

Another difference among models is the extent to which their equations are based on physics. None of the ODE models are very realistic, but many are based on basic physical mechanisms. The papers [Källén et al., 1979; Le Treut and Ghil, 1983; Fowler et al., 2013] for example have an ice sheet evolution equation based on the physics of the ice. Other models, as the series by Saltzman and collaborators ([Maasch and Saltzman, 1990] and other articles) are highly parametrized, without a real physical basis for the equations.

Some models explore one specific mechanism and try to see whether it can account for some features of glacial oscillations. In [Paillard and Parrenin, 2004] the salinity of deep ocean waters plays a crucial role, while in [Gildor and Tziperman, 2001] the sea ice is a very important factor. Most models include orbital forcing, but some are able to explain parts of the spectrum with only the internal variability, such as [de la Cuesta et al., 2013], who use a Lotka-Volterra type exchange mechanism for carbon stocks as the driving mechanism behind the cycles.

Finally, these models are also studied in more theoretical context, mathematically rigourous, such as in the recent work by Widiasih [Walsh and Widiasih, 2014].

1.2.3 ODE models for climate-vegetation

The models described above do not take into account vegetation on the planet. The most recurring variables are temperature, ice mass and carbon dioxide. There are certainly models that study the interaction between climate and vegetation, but not in the ice age/paleoclimate context. The most famous of these models is probably Daisyworld [Watson and Lovelock, 1983]. The hypothetical planet with two types of daisies is used as an example of how nature regulates the planetary climate, for a wide range of external variables such as solar input. Daisyworld has had many successors and extensions. A review can be found in [Wood et al., 2008].

Daisyworld doesn't have an equation for the evolution of temperature. A series of models that incorporate such an equation was studied by von Bloh and Svirezhev [Svirezhev and von Bloh, 1996; Svirezhev and von Bloh, 1997]. In those models, multiple stable equilibria of the system are observed, with no, low or large amounts of vegetation. The appearance of multiple stable states is a recurrent feature in these models, also in slightly more complicated ones such as [Aleina et al., 2013], which models the hydrological cycle with ODEs. This feature is also sometimes observed in models of intermediate complexity [Claussen, 1998].

Apart from multiple stable states, it is interesting, also in the spirit of the climatic oscillators, to examine to possibility of internal oscillations in such models. Two extensions of Daisyworld with oscillations are described in [Nevison et al., 1999], where the oscillations arise through the addition of a temperature evolution equation to the original model, and [Gregorio et al., 1992], where a delay is introduced in the system. The appearance of oscillations will be one of the main points of interest in the models that I studied for my project.

More realistic climate-vegetation models that can be mentioned here are [Lenton and Huntingford, 2003], which examines carbon fluxes in a between soil, vegetation and atmosphere and [Lenton, 2000], which can be seen as a Earth system model using only ordinary differential equations. It is a rather extended ODE model, including a carbon cycle and vegetation, but no ice sheets. It can in principle be applied to paleoclimate but the author uses it to study present-day climate.

1.2.4 Overview of what follows

The remainder of this report is structured as follows. First I briefly described how I proceeded in studying climate-vegetation interactions, and the software I used. Then I elaborate on the different models I studied (4 in total). The first model is a very simple two equation model, which shows some interesting dynamics. In addition, it is a useful mathematical exercise when studying dynamical systems. The next two models are expansions of an existing model exhibiting oscillations, due to Källén, Crafoord and Ghil [Källén et al., 1979]. The models are explained and the main results are shown, rather qualitatively and exploratory. Then a final model is discussed which merges the two three-equation models into one four-equation model. In the end I discuss some further work that can be done, and give a conclusion. The appendices contain more figures, tables with parameter values and a draft of a paper on the two equation model.

2 Methodology

In the project I studied climate and vegetation using dynamical systems. For this I used a combination of analytical and numerical approaches. For the simple model with two equations, some analytical work was feasible. I found the book [Strogatz, 1994] a very useful companion for this. For the other models, I used the software XPPAUT [Ermentrout, 2012] for solving the systems and making the bifurcation diagrams (the program has an interface to AUTO). Python was used for small computations, and the module PyDSTool [Clewley et al., 2007] was used to double check some of the results of the numerical simulation, to make sure that some of the apparent peculiarities were not due to numerical errors. Plotting was done using the LaTeX packages Tikz/PGFPlots.

3 Models and results

3.1 A simple two equation model

Note: a draft of a paper on this model is included in the appendix. The following is a summarized version of the model and the results.

3.1.1 Description

The first model I studied is a two equation model. Inspiration for the model were Daisyworld [Watson and Lovelock, 1983] and the models by Svirezhev and von Bloh [Svirezhev and von Bloh, 1996; Svirezhev and von Bloh, 1997]. The main difference from these models is the inclusion of ocean and sea ice. The model's equations are given in (1).

$$\begin{cases} C_T \dfrac{dT}{dt} = (1-\alpha(T,A))Q_0 - R_o(T) \\ \dfrac{dA}{dt} = \beta(T)A(1-A) - \gamma A \end{cases} \tag{1}$$

The temperature equation expresses that temperature changes as a result of the balance between incoming and outgoing energy. The variable Q_0 is the incoming solar energy, which is equal to 342.5 W/m^2. The function α denotes albedo and is given by $(1-p)\alpha_o(T) + p(\alpha_v A + \alpha_g(1-A))$. Here p is the fraction of the planet that is land, taken to be 0.3, as on Earth. The values α_o, α_v and α_g denote the albedo of the ocean, albedo of vegetation and albedo of bare ground respectively. The latter two are constant, and the essential thing is that $\alpha_v < \alpha_g$ (forests are darker and absorb more energy than bare ground). The albedo of the ocean will be taken as a function of temperature, to take into account the presence of sea ice. We will use a ramp function, as in [Sellers, 1969] and [Ghil and Childress, 1987]. The function is given by equation (2):

$$\alpha_o(T) = \begin{cases} \alpha_{ma} & \text{if} \quad T \leq T_{\alpha,\ell} \\ \alpha_{ma} + \frac{\alpha_{mi}-\alpha_{ma}}{T_{\alpha,u}-T_{\alpha,\ell}}(T - T_{\alpha,\ell}) & \text{if} \quad T_{\alpha,\ell} < T \leq T_{\alpha,u} \\ \alpha_{mi} & \text{if} \quad T_{\alpha,u} < T \end{cases} . \tag{2}$$

Here α_{ma} is the albedo of an ice-covered ocean and α_{mi} that of an ice-free ocean. The parameter values can be found in the appendix. The value of $T_{\alpha,u}$ is rather high (almost 27 degrees Celsius), which means a tiny bit of sea ice will be present even for very high global temperature.

The function $R_o(T)$ denotes the outgoing energy from the planet. Often the quartic Stefan-Boltzmann law is used, or a linearization thereof, but we opt to take into account the fact that increasing temperature entrains increasing carbon dioxide levels and thus greenhouse effect, which tends to decrease the outgoing radiation. The form for $R_o(T)$ we choose is

$$R_o(T) = B_0 + B_1(T - T_{\text{opt}}),$$

where B_0, B_1 are constants and T_{opt} is the optimal growth temperature for the vegetation. There is a lot of uncertainty on the value of these parameters, especially in B_1, the linear radiative forcing of temperature, since it all depends on which effects are taken into account and which are not. We will not attempt to obtain this values as realistic representations of reality, but for definiteness we will use $B_0 = 200$ and $B_1 = 2.5$. Despite the obvious roundness of these numbers, this parametrization is quite close to a linearization of an expression of the form $g\sigma T^4$, corresponding to Stefan-Boltzmann with a grayness factor g of about 0.55.

The second equation expresses that vegetation grows logistically, with a temperature-dependent growth rate $\beta(T)$. The shape of the function β is taken to be the same as in Daisyworld and related models (3):

$$\beta(T) = \max(0, 1 - k(T - T_{\text{opt}}))^2. \tag{3}$$

This means that growth rate is zero, except for in a certain interval, in which the dependence is parabolic with a maximum at T_{opt}, which is 283K in our model. The parameter γ is the death rate of plants. Later on γ will serve as a control parameter. Table 1 in the appendix contains an overview of the parameters and their values.

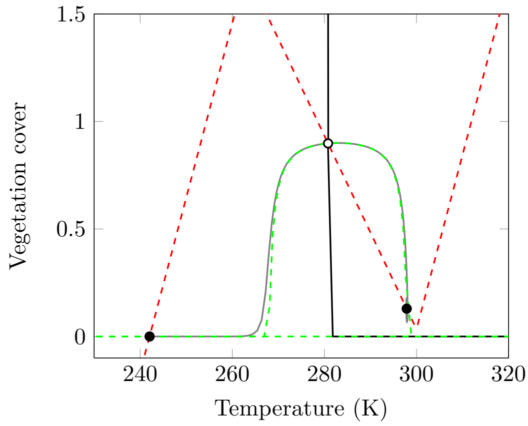

Figure 2: Phaseplane with nullclines (dashed lines) and fixed points (dots). The stable manifold of the saddle is shown in black and the unstable in gray. Coordinates and classification of the fixed points: $(242, 0)$: stable node, $(281, 0.9)$: saddle point, $(298, 0.13)$: stable focus.

3.1.2 Results

The system has three fixed points. Figure 2 shows the phaseplane (the caption includes the coordinates and classification of the fixed points). The stable state where $A = 0$ is very cold, such that no vegetation can exist ($\beta(T) = 0$), and the ocean is completely covered with ice. More interesting behaviour is observed for the fixed point with non-zero vegetation. For the standard parameter values, this point is a stable focus. Stable foci can lose their stability when the eigenvalues of the Jacobian matrix cross the imaginary axis when some parameter changes. We take γ as a control parameter, and see what happens. When γ decreases, the focus indeed loses its stability in a Hopf bifurcation and gives rise to a limit cycle.

This occurs for $\gamma = 0.02572$, so when the overturning time of vegetation ($1/\gamma$) is about 40 years. The results can be seen in the bifurcation diagram in Figure 3. Also note that when γ is higher than 0.41, no fixed point with non-zero vegetation exists. The stable fixed point, together with the saddle, are created in a saddle node bifurcation. We note that instead of changing γ, we can also obtain the Hopf bifurcation by decreasing C_T, the thermal heat capacity. It is not one of these parameters separately, but rather their product which determines the behaviour of the system. We can regard C_T as a typical timescale for temperature adjustments, and analogously $1/\gamma$ as the typical timescale for vegetation. The oscillations therefore occur when the ratio between these timescales assume a certain value. This is resemblant of other Hopf bifurcations where the time scales need to match, as for example in [Ghil and Tavantzis, 1983].

Figure 4 shows plots of temperature and vegetation cover versus time.

The oscillations have an amplitude of a few degrees for the temperature, but are very large for the vegetation. In addition, the vegetation plot show a sawtooth-like shape. It is also noteworthy that the vegetation cover stays at almost zero for long times, after which it shoots back up. The sea ice varies between 0 and about 6% cover. Even though this is not that much, the sea ice provides an essential feedback mechanism, without which the limit cycles wouldn't be there. Sea-ice has been noted to be a possible determining factor in the inception of glacial cycles [Gildor and Tziperman, 2001], which is resemblant of the importance in our present model. We can remark here that our model essentially only

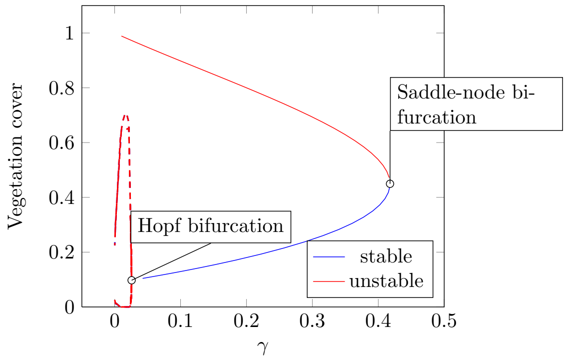

Figure 3: Bifurcation diagram. The curves were computed with XPPAUT.

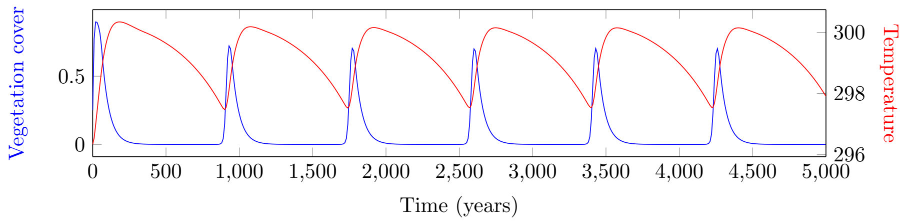

Figure 4: Trajectories for $\gamma = 0.02$, so $\gamma/C_T = 1/10$

differs from [Svirezhev and von Bloh, 1996] in the inclusion of the ocean and the sea ice, and that in that paper the authors actually prove the absence of limit cycles using Dulac's criterion.

Note that temperature jumps up quickly, but that cooling down and growth of sea ice is slower. This is reminiscent of the typical behaviour of temperature over glacial-interglacial cycles, where temperature rises quickly (so-called terminations) but the buildup of ice sheets and cooling of the planet is slower. It already indicates that inclusion of vegetation in the "ice age oscillators" might be an interesting direction to pursue, which I have done in the remainder of the project.

As further explained in the article in the appendix, it is essential for the focus to become unstable that the fixed point lies in the region where $\beta'(T)$ is negative and $\partial_T \alpha$ is negative as well. Translated to parameter values, this means that the optimal growth temperature for plants needs to be relatively low and there must be ice present for relatively high temperatures. This is one of the main questionable points of this model, since we used an optimal temperature of only 10 degrees Celsius for plant growth, and sea ice present up to 27 degrees Celsius. The model shows however how even a very simple, 2 equation model can exhibit interesting behaviour, with only few basic feedback mechanisms.

4 Three equation models

4.1 Introduction

The previous model showed how vegetation and the combination with sea ice feedback, can play a role in oscillations of the climatic system. As explained in the introduction, glacial-interglacial cycles constitute an important example of oscillatory behaviour of the climate, and many models have been proposed to explain these cycles. In the second part of my project, I extended one of this models, by Källén, Crafoord and Ghil [Källén et al., 1979] (further on called the KCG-model). The model has two equations, one for temperature evolution and one for the evolution of the ice sheet. The model can sustain internal oscillations. These arise through a Hopf bifurcation, studied in detail in [Ghil and Tavantzis, 1983]. The oscillations appear when the ratio between the typical timescales of temperature and ice sheet evolution are in certain proportion to eachother.

I extended the models by adding a third variable to the two equation model. One the one hand, I included atmospheric carbon dioxide, which is an important factor, as explained before. On the other hand, I included a vegetation cover which interacts with the climate through its albedo and the dependence of growth rate on temperature. At the very end, as briefly described in the last section, I put these two models together in a four equation model.

4.2 The KCG model

Let us very briefly describe the model we use to start with. The model, studied in [Källén et al., 1979], uses an energy balance equation for temperature and an ice sheet growth equation for the ice extent. The equations are given in the following system:

$$\begin{cases} \dfrac{d\theta}{dt} = Q_0(1 - p\alpha_{\text{land}}(\ell) - (1-p)\alpha_{\text{ocean}}(\theta)) - \kappa(T - T_\kappa) \\ \dfrac{d\ell}{dt} = \mu \dfrac{1}{\sqrt{\ell}}\big((1+\epsilon(T))\ell_T - \ell\big) \end{cases} \tag{4}$$

We will not describe the exact meaning of these equations, since this is well explained in [Källén et al., 1979] and also in the book [Ghil and Childress, 1987]. For now it suffices to know that θ is a normalized temperature, ℓ is a normalized ice sheet extent. The parameter μ denotes the ratio between time scales for temperature evolution and ice sheet evolution, and the value of this parameter determines whether the system has internal oscillations. Temperature evolution is determined by the difference between incoming radiation, partially reflected by ice, and outgoing radiation, parametrized as $\kappa(T - T_\kappa)$ in this model. The ocean albedo is of Sellers-type, as the one we used before. Land albedo is dependent on the ice sheet extent.

The equation for the ice sheet is obtained from physical principles, assuming a parabolic ice sheet profile. The function ϵ denotes the so-called *temperature - precipitation feedback*. The underlying assumption is that, for high temperature, the ratio between accumulation and ablation (melting) is higher. It is a crucial assumption of the model.

Note that we use the symbol p for the fraction of the planet that is land, whereas the original article uses γ. We do this not to confuse with γ, the vegetation death rate, used further on.

This model has already been extended to a three-equation model in [Le Treut and Ghil, 1983]. The third variable is the deflection of the Earth's mantle under the weight of the ice. It is also detailed in [Ghil and Childress, 1987]. The model has rich dynamics, especially when forced with orbital variations.

The orbital forcing in this model is done in two ways. One way is directly on the incoming radiation Q_0, which then becomes a function of time and changes due to changes in eccentricity of the Earth's orbit. The other type of forcing is through the function ϵ, which changes along with the variations in obliquity and precession of the orbit of our planet. We again refer to the original publications for the details.

4.3 Temperature, ice sheet, carbon

The first extension of the KCG model will be a model including a simplified carbon cycle. Carbon plays an essential role in the climatic system. It is one of the main greenhouse gases that warm the planet and the increase of atmospheric carbon due to burning of fossil fuels is a major cause for global warming [Stocker et al., 2013]. As mentioned in the introduction and shown in Figure 1, the amount of carbon in the atmosphere has varied along temperature over the glacial-interglacial cycles. No definitive explanation of this has been given [Gildor, 2004]. Many different mechanism play a role, the most important being probably the ocean. The book [Follows and Oguz, 2004] provides a good background on the oceanic carbon chemistry, and gives an explanation of such things as the solubility pump, organic pump and mixing, among others. We will not attempt to give an exhaustive explanation of all mechanisms involved, but let us summarize the effects that we include in the model. It is rather difficult to summarize the carbon cycle into only one equation, so only the most important factors will be taken into account. We mainly used the article [Hogg, 2008] as inspiration. The following mechanisms will be included:

Solubility of carbon dioxide The Earth's largest carbon reservoir is the ocean. The amount of carbon stored in the ocean depends on the temperature, through the solubility. Colder water is able to hold more carbon dioxide than warmer water. Thus when temperature rises, the ocean will release carbon dioxide. We model the ocean-atmosphere flux as

$$\frac{1}{\tau_c}\big(C_{oc,0}e^{b(T-288)} - C\big), \tag{5}$$

where $C_{oc,0}$ is the amount of carbon in the ocean at present day temperature, taken to be a constant (a massive simplification, we must say). This is modified with a factor $e^{b(T-288)}$, which represents the solubility effect. The exponential form and the value of the parameter b were taken from the model by Fowler [Fowler et al., 2013]. The parameter $C_{oc,0}$ was adjusted to match data more closely.

Weathering The carbon dioxide in the atmosphere reacts with rocks and minerals, and is as such removed from the atmosphere. Walker et al. [Walker et al., 1981] obtained an experimental formula expressing the weathering rate as a function of carbon dioxide partial pressure and temperature, but we will used a simplified version of this, as in [Hogg, 2008]. The weathering rate is

$$W_0 + W_1 C,$$

and thus only depends on the amount of carbon dioxide. It can be seen as a negative feedback that acts to counter increases in carbon dioxide in the atmosphere.

Uptake by plants Inspired by results from the four equation model (to be explained later), and evidence from the literature on the importance of the biosphere in the carbon cycle [Lenton and Huntingford, 2003; Falkowski et al., 2000], we include a partial vegetation cover in the model, and carbon uptake by vegetation. This uptake is given by $W_a C A_v$, where A_v is the fraction of land covered by vegetation and W_a is a parameter.

Sea ice The ocean-atmosphere flux we described above can of course only happen if the surface of the ocean is free. We assume that ice-covered water has no carbon dioxide exchange. Therefore we multiply equation (5) with a factor $1 - f(T)$, where f is the fraction of ocean that is covered by ice. This depends on temperature, in the same way (Sellers-type) as the albedo of the ocean depends on temperature.

Volcanism Following [Hogg, 2008], we take a constant input of carbon dioxide in the atmosphere of value V.

Let's summarize this into one equation:

$$\frac{dC}{dt} = V - (W_0 + W_1 C) - W_a A_v C + \frac{1}{\tau_c}\big(C_{oc,0}e^{b(T-288)} - C\big)(1 - f(T)) \tag{6}$$

The level of carbon dioxide doesn't influence ice sheet growth, but it does influence the temperature evolution. Because of the greenhouse effect, there is less outgoing radiation when more CO_2 is in the atmosphere, and the dependence is logarithmic [Gregory et al., 2009]. Therefore, we add a term $B \ln C/C_0$ to the temperature evolution equation. Another change to the original temperature equation lies in the land albedo. In KCG, it is given by $\alpha_{\text{land}} = \alpha_0 + \alpha_1 L$. We change this to $A_v\alpha_v + (1 - A_v)(\alpha_0 + \alpha_1 L)$, with a different value of α_0. In this model, A_v is a parameter, whereas in our other models it will be a variable. See also Remark 1 in this respect.

4.3.1 Equations

We change the system (4) into the following system:

$$\begin{cases} \dfrac{d\theta}{dt} = Q_0(1 - p\alpha_{\text{land}}(\ell) - (1-p)\alpha_{\text{ocean}}(\theta)) - \kappa(T - T_\kappa) + B\ln c \\ \dfrac{d\ell}{dt} = \mu\dfrac{1}{\sqrt{\ell}}\big((1 + \epsilon(T))\ell_T - \ell\big) \\ \dfrac{dc}{dt} = v - (w_0 + W_1 c) - W_a A_v c + \dfrac{1}{\tau_c}(c_{oc,0}e^{b(T_s\theta - 288)} - c)(1 - f(T)) \end{cases} \tag{7}$$

Here c is the *normalized* carbon dioxide level in the atmosphere, equal to C/C_0 where C is the level in ppm and C_0 is the standard preindustrial level of 280 ppm. Also, the time used in the original model was kyears, such that the constants in the carbon dioxide equation now have scaled values (this is denoted by their lowercase typeface). Values of the different parameters can be found in the appendix.

4.3.2 Results

This model exhibits, as the original KCG, internal oscillations. The oscillations do not exist for all parameter values, however. In the general spirit of the project, the parameter that we studied is A_v, the vegetation cover. Figure 5 shows plots of temperature and carbon dioxide level versus time. We plotted temperature and carbon on the same axis to see whether temperature leads carbon dioxide, as in the the data of the Vostok ice core (Figure 1). Indeed, temperature is a bit earlier than carbon, although the difference is rather small. Amplitudes of the oscillations correspond quite well to observed amplitudes. We can note here that the inclusion of sea ice in the ocean-atmosphere flux has an influence on the amplitude of the oscillations. When sea ice is included, the amplitude is higher (Figure 6).

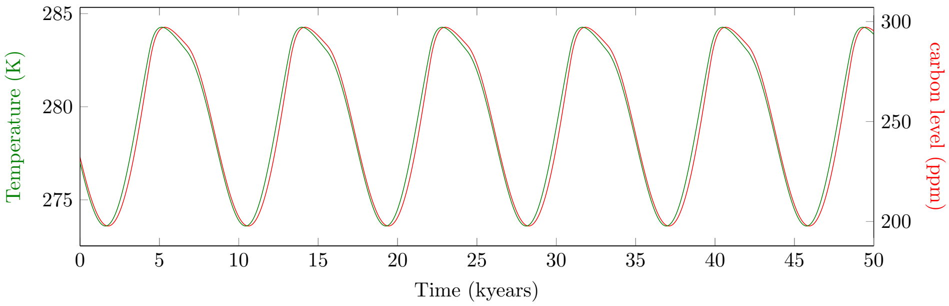

Figure 5: Plots for the non-forced model with carbon dioxide. The value of A_v is 0.4.

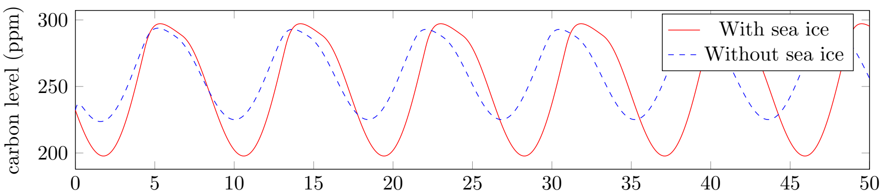

Figure 6: Comparison of the carbon dioxide for the model with the sea ice and without sea ice in the carbon evolution equation. The value of A_v is 0.4.

When the parameter A_v is lowered, the oscillations disappear and the system goes into a very cold state, with a large ice sheet.

We also forced the model with orbital variations. We used only the precession and obliquity forcing to change the function ϵ. The exact values of the used parameters can be found in the appendix, they were taken from [Ghil and Childress, 1987, p. 432]. The result is shown in Figure 7. We only show the temperature variations here, ice sheet and carbon vary in a similar fashion. The complete plots can be found in appendix.

4.4 Temperature, ice sheet, vegetation

4.4.1 Description

The other three equation model consists of temperature, ice sheet, and vegetation. The vegetation modifies the albedo, and is itself influenced by temperature and ice sheet extent. The evolution equation for vegetation cover is

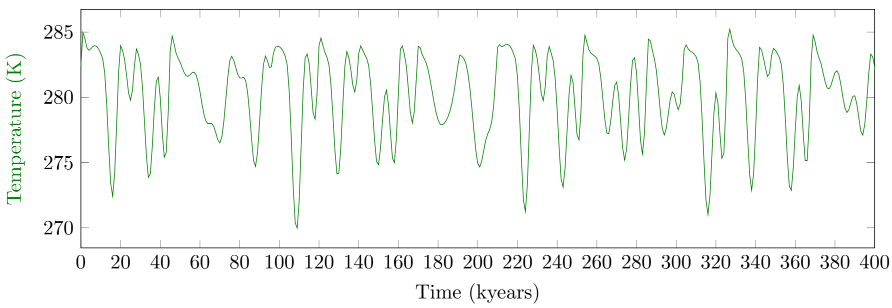

Figure 7: Plot for the forced model with carbon dioxide. The value of A_v is 0.42.

$$C_v \frac{dA}{dt} = \beta(T)A(q(L) - A) - \gamma A. \tag{8}$$

This is the same as in our two equation model, except for that the available space for plants is not 1, but $q(L)$, dependent on the ice sheet extent. If $2L$ is ice sheet extent in meters, we (rather heuristically) define q as

$$q(L) = 1 - \frac{2L}{\mathrm{TLL}}, \tag{9}$$

where TLL is *total land length*, taken to be 8000km (the model describes one hemisphere). The vegetation cover appears in the temperature evolution equation, because it modifies the albedo. Where the land albedo was $\alpha_0 + \alpha_1 L$ in the KCG model, this is now equal to $A\alpha_v + (1 - A)(\alpha_0 + \alpha_1 L)$. Note that we changed the value of α_0 from 0.25 to 0.4. Before, when no vegetation was included as variable, α_0 could be seen as some kind of average of bare ground and vegetation albedo, but when we include the vegetation in the model, α_0 only denotes the albedo of bare ground.

Remark 1. *When writing the report and going over the models, I realised that the parametrisation of the land albedo as used here, is not really realistic and coherent with the other models. Better would be to define land albedo as*

$$\alpha_{land}(L, A) = A\alpha_v + (q(L) - A)\alpha_g + (1 - q(L))\alpha_{ice\ sheet},$$

which is the weighted average over the areas covered by vegetation, land and ice respectively. I tried this, but the results of this model were much less interesting: no oscillations, and surprisingly the values of the variables in steady state solutions were less realistic. I decided to go on with the original model, since the dynamics are more interesting. In fact, using the form with $(1 - A)(\alpha_0 + \alpha_1 L)$ *introduces a nonlinear term* AL*. I have not found a physical argument to account for this, and it is a part of the model that could be reexamined. The same thing applies to the 4 equation model below, and to the carbon dioxide model above, although in the latter case it is not fundamental, since* A_v *(vegetation cover) is a parameter there and not a variable.*

The parameter values can as before be found in appendix. Note that the system exhibits interesting behaviour for many different parameter values. It is impossible to describe all possible combinations, so we fix most of the parameter values, to showcase some of the things that happen in the model.

4.4.2 Results

In analogy with the two equation model, the systems behaviour is largely determined by the death rate of the vegetation, γ. We will use this parameter as a control parameter. For all values of γ, there is a stable state with no vegetation. This state is characterized by an extremely low temperature and a large ice sheet. We are of course more interested in a state where there is vegetation present. This state exists and is stable when γ is low enough, just as in the two equation model.

When γ is not too low, nor too high, there exists a limit cycle. Plots can be seen in Figure 8. The curve for vegetation is sawtooth-shaped, as in the two equation model. Maxima of vegetation coverage coincide more or less with minima of ice sheet extent. The

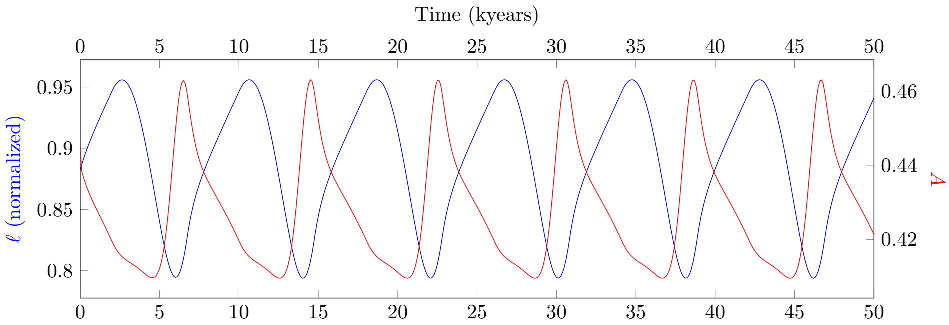

Figure 8: Plot for the non-forced model with vegetation. The value of γ is 0.265. Temperature is not shown, but it oscillates with an amplitude of about 10 K.

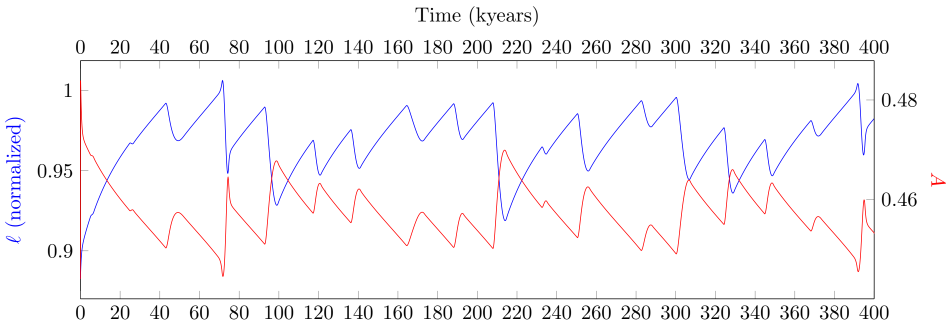

Figure 9: Plot for the forced model with vegetation. The value of γ is 0.219. The oscillations for temperature are similar. The complete plots are shown in appendix.

amplitude of the oscillations in vegetation cover is about 6%, which is much less than for the two equation model.

Figure 9 shows the results of this model with forcing. The sawtooth-like shape is present, but amplitudes are rather low. This is due to the fact that we decreased γ a bit. The amplitude of the oscillations decreases with γ, and if we have large oscillations, the forcing can kick the system out of the basin of attraction of the limit cycle and push it into a very cold state (this also occurs in the model with carbon, by the way). This doesn't happen if we lower γ, to have lower amplitude internal variations. We note that the range of values for γ for which this works is very narrow, which can be regarded as a limitation of the model.

4.5 Four equation model

After seeing the results for the three equation models, we might want to look at what happens when we combine all the variables together. We will briefly describe the resulting model. The equations for temperature and ice sheet remain the same. In the carbon equation we now have the term W_aAC instead of W_aA_vC. The only difference is indeed that A_v, which was a parameter in the three equation carbon model, is now replaced by the variable A. In the equation for vegetation, we change the growth rate. The growth rate of the plants, which previously only depended on temperature, now also depends on available carbon dioxide in the atmosphere. The more there is available, the better the plants grow, with a saturation effect for high values of carbon.

The evolution equation for A is

$$\frac{dA}{dt} = f_1(C)\beta(T)A(q(L) - A) - \gamma A. \tag{10}$$

The only thing that has changed from the three equation model, is the addition of the function f_1, expressing the dependence of growth rate on carbon level. The function is given by

$$f_1(C) = \frac{\mu_1(C - k_c)}{k_m + C - k_c}. \tag{11}$$

We directly took this function, together with the parameters, from [Lenton and Huntingford, 2003]. The results of this model are similar to the other two models: oscillations exists if γ is in the right range. For γ too high, vegetation cannot be sustained and the system drops to a very low temperature state. More or less the same remarks as for the three equation model with vegetation apply: we observe sawtooth shapes, and the model with forcing only works for a limited range of values for γ. The period of oscillations is larger than for the three equation models. This model is not completely developed yet, and many more can be done as described below. In the appendix plots can be found for forced and not forced versions of the model.

5 Further work

The work I've done consisted mostly in setting up the models, and doing initial exploration of their behaviour, particularly focused on limit cycles. The next step would to extend results, and more thorougly investigate these models separately. Here are some things that can be done.

For the two equation model, a spatial, one-dimensional version could be interesting, where the incoming solar energy would depend on latitude. In such a model, there could be an interplay between for example desert vegetation and sea ice, mediated by global temperature. The problem of our not so realistic T_{opt} in the zero dimensional model could be resolved in this way. Another possibility is, since the ocean is very important in the model, to include ocean vegetation: plankton. It is known that plankton has an effect on climate, so it would make sense to include this in a model. Finally, the mathematical treatment of the model can be further extended, for example with a proof of the existence of the Hopf bifurcation.

For the other models, there is a lot that can be done. I have only started with a first analysis of the influence of some of the parameters (linked to vegetation), but many others are uncertain and could be varied, such as B, the greenhouse effect, the optimal growth temperature for plants, the carbon flux parameters, ... The inconsistency in the parametrization of the land albedo (Remark 1) should be eliminated or justified. A careful analysis of which mechanism has which effect can be done, such as what I have shown for the role of the sea ice in the ocean atmosphere carbon exchange. Another possibility is to obtain bounds on the parameter values, either from the literature or by numerically exploring the behaviour and possibly fitting to data.

Following [Ghil and Childress, 1987], an analysis of the spectra of the forced models can be interesting to see what they reproduce of the typical spectra of ice age cycles, and a comparison with other models such as [Le Treut and Ghil, 1983] can be done.

Further, a more quantitative bifurcation analysis, such as done for the 2 equation model, should be done as well, together with a quantitative description of the other stable states.

6 Conclusion

In this project, I investigated the role of vegetation in dynamical systems models for climate. There have been many examples where vegetation and climate interact, but climatic modeling has not often included vegetation as a variable, especially in the case of ODE models.

First I studied a two equation model, based on Daisyworld and derivations thereof. The main feedback mechanism is through the albedo of plants and sea ice. The model exhibits oscillations, which arise through a Hopf bifurcation when the death rate of vegetation is small enough. It is one of the smallest models for climate-vegetation which shows such oscillations. This model will be the subject of a paper. It opens the way for a lot of interpretations and possible extensions.

The next part consisted of examining the role that vegetation can play in climatic oscillators, usually applied to model ice age cycles. I took an existing model for temperature and ice sheet evolution, and extended it to two three-equation models and a four-equation model. In my project I only did the setting up of the models and initial explorations, with focus on the parameters related to the vegetation. These parameters, γ, the death rate, and in the model with carbon A_v, the vegetation cover, determined the existence of oscillations or not. For the models in which vegetation was included as a variable, the shape of the oscillations showed to be sawtooth-like, which is promising for the role of vegetation in modeling of paleoclimate. All models allow oscillations, but for some of them the range of parameters for which oscillations exists is quite narrow.

There is many further work that can be done on these models, but they can already take their place in the series of climatic oscillator models, and they show once more that vegetation can be a determining factor in the behaviour of the climate system.

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Appendix

This appendix contains

- Plots for the different models, both with and without astronomical forcing, for all variables. Some variables (such as temperature and carbon dioxide) are plotted on the same axis to facilitate comparison with data from Vostok ice core (Figure 1).
- Tables with values for all parameters used in the models. Parameters that are used in different models are mentioned once, unless their value is different in other models.
- The (early) draft of an article on the two equation model. This contains a bit more information than the report.

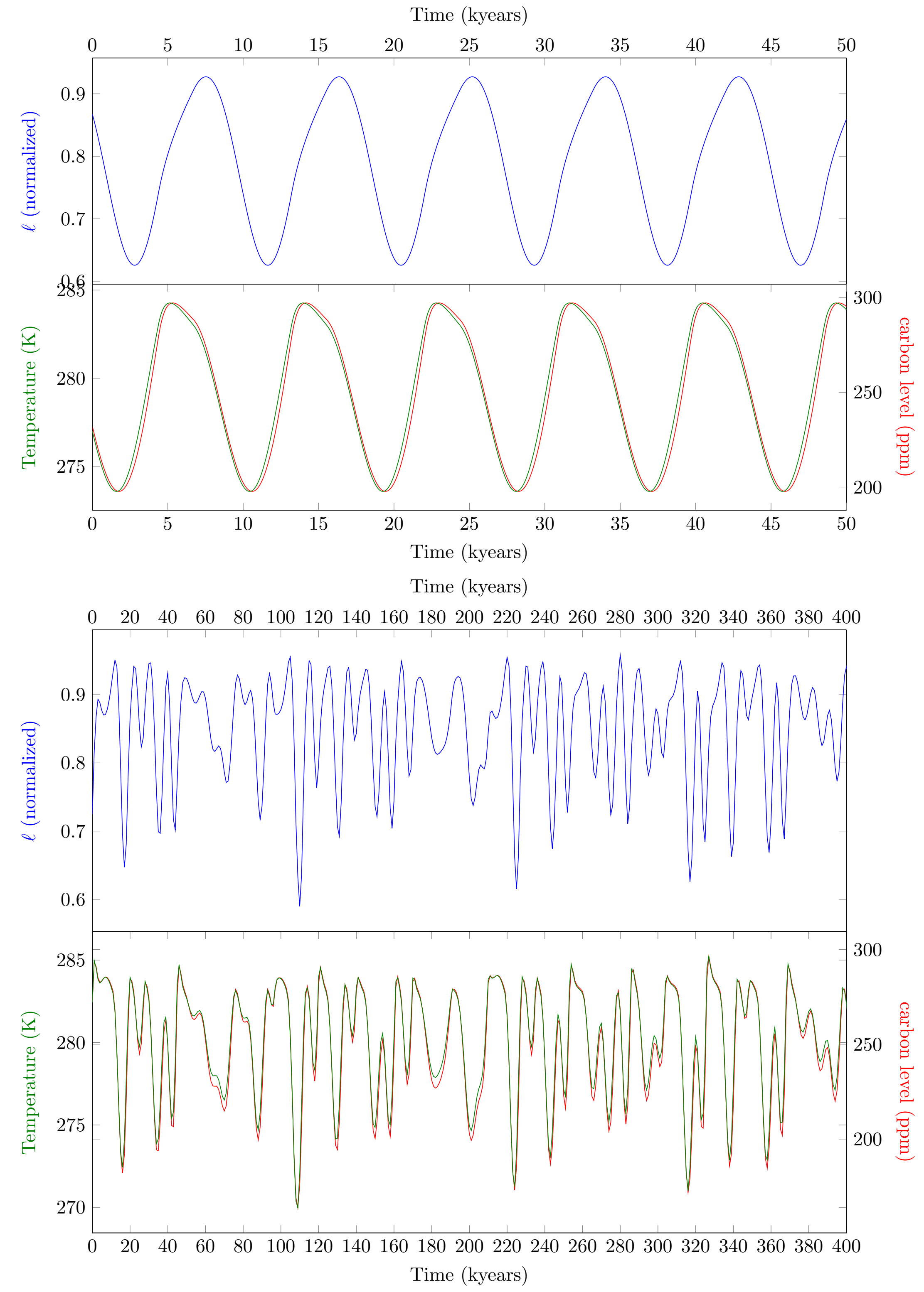

Figure 10: Plots for the three equation model with carbon dioxide. Above: not forced, $A_v = 0.4$. Below: forced, $A_v = 0.42$.

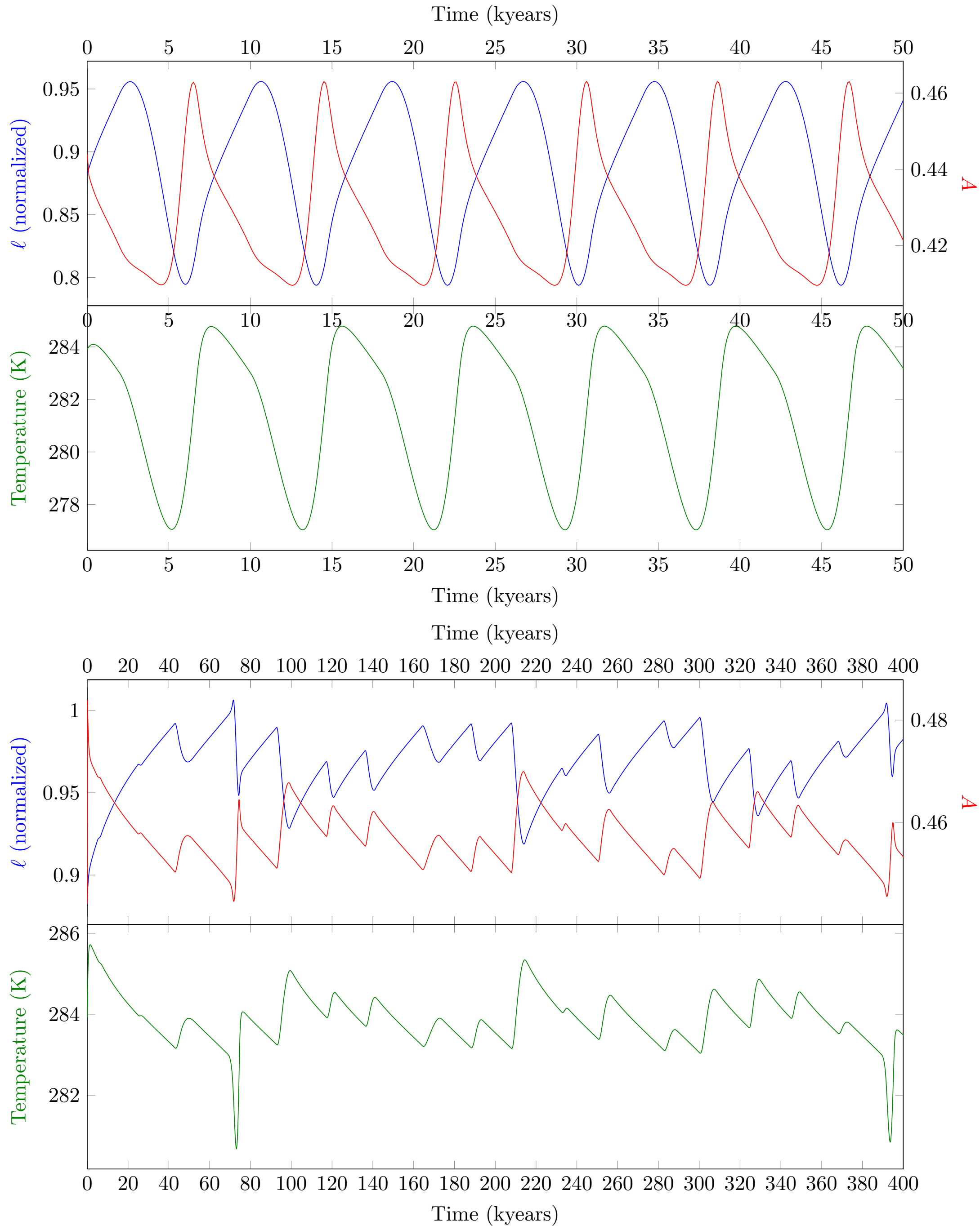

Figure 11: Plots for the three equation model with vegetation. Above: not forced, $\gamma = 0.265$. Below: forced, $\gamma = 0.219$

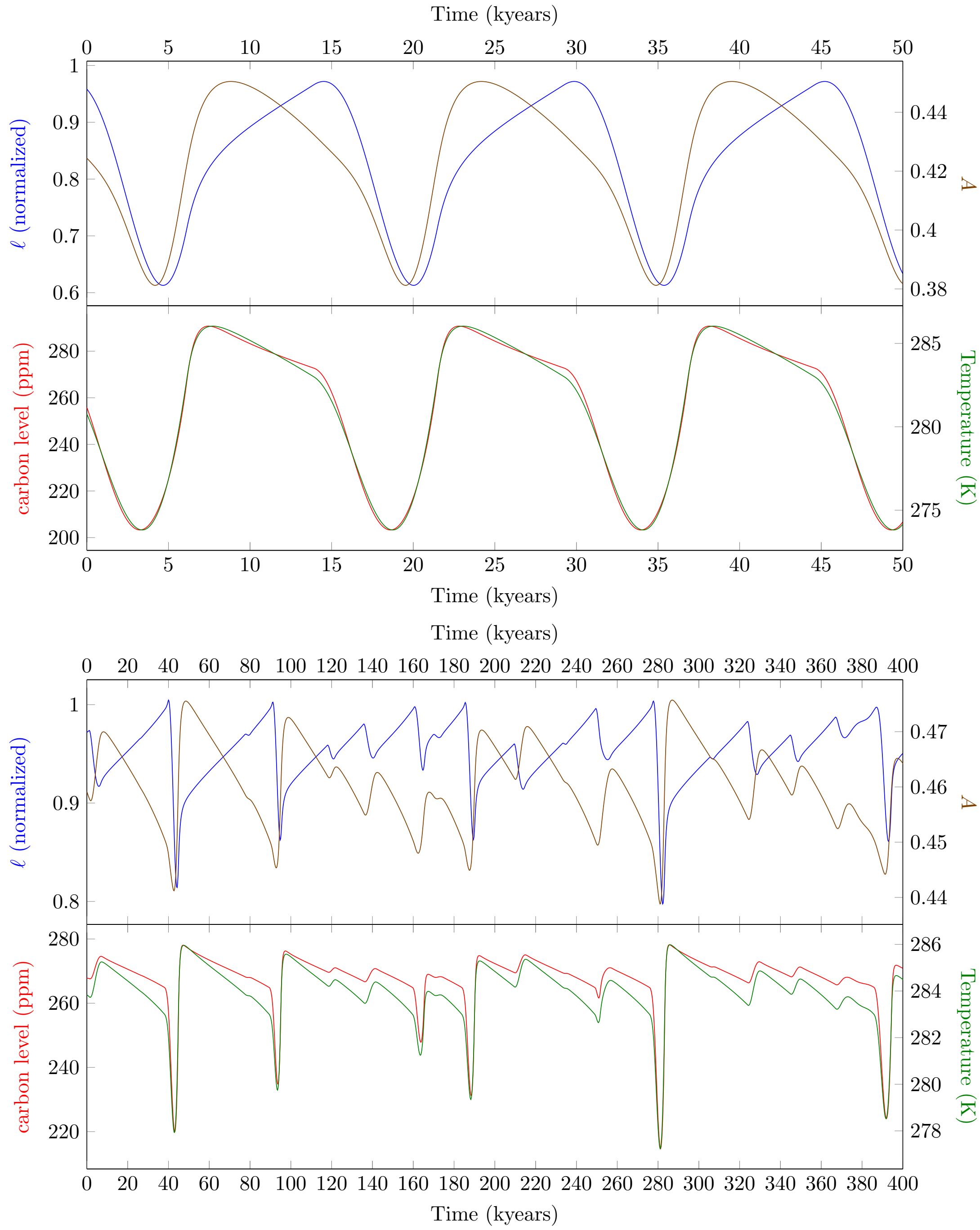

Figure 12: Plots for the four equation model. Above: not forced, $\gamma = 0.24$. Below: forced, $\gamma = 0.2016$.

Table 1: Parameter meaning and values for the two equation model.

Symbol	Meaning	Value
C_T	Heat capacity	500 W yr/K/m^2
Q_0	Incoming solar energy	342.5 W/m^2
p	Fraction of land	0.3
α_v	albedo of vegetation	0.1
α_g	albedo of ground	0.4
α_{ma}	albedo of ice-covered ocean	0.85
α_{mi}	albedo of ice-free ocean	0.25
$T_{\alpha,\ell}$	Temperature below which ocean is ice-covered	263 K
$T_{\alpha,u}$	Temperature above which ocean is ice-free	300 K
B_0	constant in outgoing radiation	200 W/m^2
B_1	constant in outgoing radiation	2.5 W/K/m^2
T_{opt}	Optimal growth temperature	283 K
k	parameter for width of growth curve	0.004 1/yr/K^2
γ	death rate of vegetation	0.1 1/yr

Table 2: Parameter values and meaning for the KCG model. For the explanation of the model, see [Ghil and Childress, 1987]. Note that their γ is our p (fraction of planet that is land), to avoid confusion with γ being used for vegetation death rate. In our models, α_0 is changed to 0.4, because we introduce a partial vegetion cover, altering the albedo of the land.

Symbol	Meaning	Value
Q_0	Incoming solar radiation	342.5 W/m^2
p	Fraction of planet that is land	0.3
κ	linear coefficient of outgoing radiation	1.74 $W/m^2/K$
T_κ	constant in outgoing radiation	154 K
s	Slope of the 0 degree isotherm	$0.3\times10^{-3}m^{-1/2}$
T_{00}	Temperature at which the 0 degree isotherm would intersect the arctic coastline	283 K
$T_{\alpha,\ell}$	Temperature below which the ocean is ice-covered	217K
$T_{\alpha,u}$	Temperature above which the ocean is ice-free	283K
$T_{\epsilon,\ell}$	Lower temperature in temperature-precipitation feedback	273K
$T_{\epsilon,u}$	Upper temperature in temperature-precipiation feedback	283K
α_0	Albedo of ground (different in three-equation models)	0.25
α_1	Linear coefficient in land albedo dependence on ice sheet extent	4.1×10^{-7} 1/m
α_m	Minimal ocean albedo	0.25
α_M	Maximal ocean albedo	0.85
ϵ_m	Minimal ratio between accumulation and ablation	0.1
ϵ_M	Maximal ratio between accumulation and ablation	0.5
μ	Ratio of timescales for temperature and ice sheet evolution (control parameter in original model)	1.5
T_s	Temperature for normalizing ($\theta = T/T_s$)	276.93 K

Table 3: Parameters used in forcing. The frequencies are taken from [Ghil and Childress, 1987, p. 419]. The amplitudes are taken from the same book, page 432. We only used the forcings at these frequencies.

Symbol	Meaning	Value
τ_1	precession period	19 kyr
τ_2	precession period	23 kyr
τ_3	obliquity period	41 kyr
ϵ_1	precession amplitude	3×10^{-3}
ϵ_2	precession amplitude	4×10^{-3}
ϵ_3	obliquity amplitude	2×10^{-3}

Table 4: Parameter meaning and values for the carbon dioxide three equation model. Only those that are different from KCG are shown.

Symbol	Meaning	Value	Source
α_0	Albedo of bare ground (different from KCG)	0.4	
C_0	Pre-industrial carbon dioxide level, used for normalizing	280 ppm	Almost all the literature
B	Radiative forcing of greenhouse effect	5.35 W/m^2	[Hogg, 2008]
V	Volcanic input of carbon dioxide	0.028 ppm/yr	[Hogg, 2008]
W_0	Constant in weathering	0.013 ppm/yr	[Hogg, 2008]
W_1	Constant in weathering	1/12000 1/yr	[Hogg, 2008]
A_v	Vegetation cover (control parameter)		
τ_c	Characteristic timescale for ocean-atmosphere exchange	500 yr	
$C_{oc,0}$	Ocean carbon dioxide content for $T = 288$K	1000 ppm	Adjusted to better match data
b	Constant in temperature dependence of oceanic carbon content	0.029	[Fowler et al., 2013]
W_a	Uptake of carbon through vegetation	0.01 1/yr	Adjusted to match data

Table 5: Parameter meaning and values for the vegetation three equation model. Only those different from KCG are shown.

Symbol	Meaning	Value
α_0	Albedo of bare ground (different from KCG)	0.4
T_{opt}	Optimal growth temperature	283 K
k	parameter for width of growth curve	0.004 1/yr/K^2
α_v	Albedo of vegetation	0.1
TLL	Total land length	8000km
γ	Death rate of vegetation (control parameter)	

Table 6: Parameter meaning and values for the four equation model. Only those different from the three equation models are shown. These were all taken from [Lenton and Huntingford, 2003].

Symbol	Meaning	Value
k_c	CO2 compensation point for photosynthesis	29 ppm
k_μ	CO2 half-saturation point for photosynthesis	145 ppm
μ_1	Normalizing constant for photosynthesis CO2 response	1.478

Oscillations in a simple climate-vegetation model

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Abstract

We analyze a simple dynamical systems model for climate-vegetation interaction. The planet we consider consists of a large ocean and a land surface on which vegetation can grow. The temperature affects vegetation growth on the surface and the amount of sea ice on the ocean. On the other hand, vegetation and sea ice change the albedo of the planet, which in turn changes the energy balance and temperature evolution. Two stable states are observed, as well as oscillatory behaviour. The oscillations arise through a Hopf bifurcation when the parameter γ, death rate of vegetation, is low enough. The model can be compared on the one hand to the climate-vegetation models in the style of Daisyworld, and on the other hand can be put in line with simple models trying to explain oscillations in the climatic system. Some mathematical results are obtained and the relevance of the model is discussed.

1 Introduction

Climate has an important effect on vegetation. Plant growth is affected by temperature, carbon dioxide levels and availability of different nutrients. Also the available space is important: ice covered parts of land are not suitable for vegetation growth. It also works the other way around, though: vegetation plays an important role in the regulation of the climate. Many different effects are observed. One of the most important is the albedo effect: vegetation is darker than bare ground or ice and therefore absorbs more solar radiation and warms the planet. This phenomenon appears to be important in desert regions, where it interacts with the hydrological cycle. Charney [Charney, 1975] was the first to include this in a model, but others have followed since [Claussen et al., 1999; Zeng et al., 1999; Zeng and Neelin, 2000]. An other important region where albedo feedback is important are the high latitudes, where boreal forests mask snow in winter, causing an effective warming of the surface [Bonan, 2008; Brovkin et al., 2003].

Another effect is the uptake of carbon dioxide by plants, which in turn attenuates the greenhouse effect and cools the surface. In addition to the more obvious effects of albedo change and carbon dioxide uptake, there are many mechanisms through which vegetation influences climate. Examples are the effect of plankton on cloud formation (the CLAW hypothesis [Ayers and Cainey, 2007]), evapotranspiration or more exotic feedbacks, such as the so called lightning-biota feedback as studied in [Shepon and Gildor, 2008]. A review of different mechanisms can be found in [Meir et al., 2006]. The article [Claussen, 2009] gives an introduction on the vegetation-climate interactions on long timescales.

Although many examples are known where vegetation plays an essential role in the climate system, it has only been rather recently that vegetation is included as an active player in climate models. Climatic models range from simple, conceptual ODE models up to full scale GCMs (*general circulation models* or *global climate models*). Across the whole range, vegetation can be included to better explain various climatic phenomena and trends. In some cases, predictions for models that couple atmosphere, ocean and vegetation dynamics (sometimes refered to as *Earth system models*) differ radically from models excluding vegetation [Meir et al., 2006], showing the need to include vegetation in our models to obtain better understanding of the climate.

The simplest models for climate are conceptual models, which usually use differential equations to model a simple climate system. These models are not realistic, but allow to study basic underlying mechanisms. They are also useful for exploring bifurcations, which are related to tipping points [Lenton et al., 2008] in the climate system. Bifurcations or tipping points correspond to situations in which a small parameter change can have a large effect on the behaviour of the whole system. Studying conceptual models can also provide guidance in interpreting results from larger, more complicated models [Brovkin et al., 1998; Brovkin et al., 2003]. A popular application of dynamical systems modeling for climate is the explanation of the ice age cycles, which are often represented as oscillations of a climatic oscillator [Crucifix, 2012; Ghil and Childress, 1987; Ghil, 1994]. In these climatic oscillator models, vegetation is not usually included. There are however some simple models exploring the interaction between climate and vegetation, and we will briefly review some of them in the next nextion.

2 ODE models for vegetation - climate

One of the most famous models dealing with vegetation and climate is probably Daisyworld [Watson and Lovelock, 1983]. This model was conceived as an example of how vegetation acts to regulate planetary temperature through the albedo feedback, for a wide range of parameters (in this case, the incoming solar radiation). The model has been thorougly studied and extended. A review can be found in [Wood et al., 2008].

Another series of ODE models for vegetation-climate interactions is described by Svirezhev and von Bloh [Svirezhev and von Bloh, 1996; Svirezhev and von Bloh, 1997]. They include an equation for temperature evolution (absent from Daisyworld) and look at only one type of vegetation (whereas Daisyworld has two). In their 0-dimensional model, they find multiple steady states, which is a feature regularly recurring in climate-vegetation models, both in simple [Aleina et al., 2013] and more complex [Claussen, 1998] models.

An interesting feature to examine in these simple models, is the possibility of internal oscillations. As mentioned before, climate often acts as a huge oscillator, and ODE models are used as a tool to describe this behaviour, but vegetation is not often included. Oscillatory behaviour has been observed in Daisyworld-like models, for example when an explicit temperature equation is added [Nevison et al., 1999], or when delays are introduced [Gregorio et al., 1992].

Something that is usually missing from these models is the inclusion of an ocean. Earth's ocean constitute about 70% of the area of the planet, and are a very important factor in determining the climate. The ocean is usually included in ODE models for glacial cycles. Our model will include an ocean and its corresponding sea ice, and will fit between the Daisyworld-like models and the climatic oscillator models.

3 Description of the model

The climate system contains a numerous amount of subsystems, all working together to produce highly nonlinear behaviour through its many feedback mechanisms. One of the simplest and most important feedback effects is through the albedo of the planet. The most important factor determining global climate is the energy we receive from the sun. Some of this energy is reflected, and the amount depends on the albedo. Darker areas absorb more energy, warming the planet, and lighter areas (such as snow and ice) tend to cool down the planet. The albedo effect has since long been included in climate models, is the driver behind Daisyworld and will constitute the main mechanism in our present model.

The model's equations are given in system (1). The variable T denotes global average temperature, while A denotes the fraction of land that is covered by vegetation.

$$\begin{cases} C_T \dfrac{dT}{dt} = (1-\alpha(T,A))Q_0 - R_o(T) \\ \dfrac{dA}{dt} = \beta(T)A(1-A) - \gamma A \end{cases} \tag{1}$$

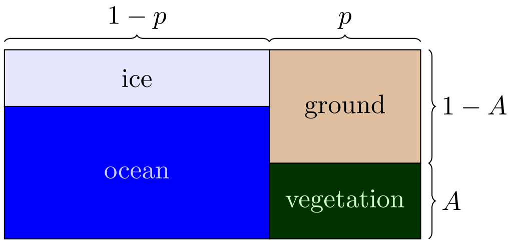

Figure 1: Schematic representation of the planet, including a fraction p land, $1-p$ ocean. Ocean can be covered by sea ice, and the land is covered by a fraction A vegetation.

The temperature equation expresses that temperature changes as a result of the balance between incoming and outcoming energy. The variable Q_0 is the incoming solar energy, which is equal to 342.5 W/m^2. The function α denotes albedo and is given by $(1-p)\alpha_o(T) + p(\alpha_v A + \alpha_g(1-A))$. Here p is the fraction of the planet that is land, taken to be 0.3, as on Earth. The values α_o, α_v and α_g denote the albedo of the ocean, albedo of vegetation and albedo of bare ground respectively. The latter two are constant, and the essential thing is that $\alpha_v < \alpha_g$ (forests are darker and absorb more energy than bare ground). The albedo of the ocean will be taken as a function of temperature, to take into account the presence of sea ice. We will use a ramp function, as in [Sellers, 1969] and [Ghil and Childress, 1987]. The function is given by equation (2):

$$\alpha_o(T) = \begin{cases} \alpha_{ma} & \text{if} \quad T \leq T_{\alpha,\ell} \\ \alpha_{ma} + \frac{\alpha_{mi}-\alpha_{ma}}{T_{\alpha,u}-T_{\alpha,\ell}}(T - T_{\alpha,\ell}) & \text{if} \quad T_{\alpha,\ell} < T \leq T_{\alpha,u} \\ \alpha_{mi} & \text{if} \quad T_{\alpha,u} < T \end{cases}. \tag{2}$$

Here $\alpha_{ma} = 0.85$ (ice-covered ocean), $\alpha_{mi} = 0.25$ (ice-free ocean), $T_{\alpha,\ell} = 263$ and $T_{\alpha,u} = 300$. The value of $T_{\alpha,u}$ is rather high (almost 27 degrees Celsius), which means a tiny bit of sea ice will be present even for very high global temperature. Figure 1 shows a schematical representation of the proportions on our planet.

The function $R_o(T)$ denotes the outgoing energy from the planet. Often the quartic Stefan-Boltzmann law is used, or a linearization thereof, but we opt to take into account the fact that increasing temperature entrains increasing carbon dioxide levels and thus greenhouse effect, which tends to decrease the outgoing radiation. The form for $R_o(T)$ we choose is

$$R_o(T) = B_0 + B_1(T - T_{\text{opt}}),$$

where B_0, B_1 are constants and T_{opt} is the optimal growth temperature for the vegetation. There is a huge uncertainty on the amount of these values, especially in B_1, the linear radiative forcing of temperature, since it all depends on which effects are taken into account and which are not. We will not attempt to obtain this values as realistic representations of reality, but for definiteness we will use $B_0 = 200$ and $B_1 = 2.5$. The exact value of these parameters doesn't play an important role for the behaviour of the model, but we can note that they correspond rather well to a linearization of an outgoing energy term of the form $g\sigma T^4$, where g is a constant denoting the grayness, which is usually taken between 0.5 and 0.6.

The second equation expresses that vegetation grows logistically, with a temperature-dependent growth rate $\beta(T)$. The shape of the function β is taken to be the same as in Daisyworld and related models (3):

$$\beta(T) = \max(0, 1 - k(T - T_{\text{opt}}))^2. \tag{3}$$

This means that growth rate is zero, except for in a certain interval, in which the dependence is parabolic with a maximum at T_{opt}, which is 283K in our model. The parameter γ is the death rate of plants and its value is 0.1 in our model. Later on γ will serve as a control parameter. Table 1 contains an overview of the parameters and their values.

Table 1: Parameter meaning and values.

Symbol	Meaning	Value
C_T	Heat capacity	500 W yr/K/m^2
Q_0	Incoming solar energy	342.5 W/m^2
p	Fraction of land	0.3
α_v	Albedo of vegetation	0.1
α_g	Albedo of ground	0.4
α_{ma}	Albedo of ice-covered ocean	0.85
α_{mi}	Albedo of ice-free ocean	0.25
$T_{\alpha,\ell}$	Temperature below which ocean is ice-covered	263 K
$T_{\alpha,u}$	Temperature above which ocean is ice-free	300 K
B_0	Constant in outgoing radiation	200 W/m^2
B_1	Constant in outgoing radiation	2.5 W/K/m^2
T_{opt}	Optimal growth temperature	283 K
k	Parameter for width of growth curve	0.004 1/yr/K^2
γ	Death rate of vegetation	0.1 1/yr

4 Results

4.1 Fixed points and stability

Let us start by looking at the fixed points of the system (1). There is a fixed point for $A = 0$. The temperature is the solution of $Q_0(1 - \alpha(T, 0)) = R_o(T)$. For our parameter values, there is only a solution where $T < T_{\alpha,\ell}$: the solution is $T \approx 243$K. This means a planet without vegetation will go into a very cold "snowball Earth" state. Note that this is different from other EBMs (energy balance models) where another stable state is found [Ghil and Childress, 1987]. A quick check shows that if we put $p = 0$, such that there is only ocean, three fixed points are found where $A = 0$, one with a high temperature. So the inclusion of the relatively high ground albedo pushes the system to a low temperature, if no vegetation is present.

If $A \neq 0$, in a fixed point we must have

$$A = 1 - \frac{\gamma}{\beta(T)}.$$

From the temperature equation, we see that also

$$A = \frac{1}{\alpha_v - \alpha_g}\left[\frac{1}{p}\left(1 - \frac{R_o(T)}{Q_0} - (1-p)\alpha_o(T)\right) - \alpha_g\right].$$

Figure 2 show pictures of those two curves. They intersect in two points, which means we have two fixed points where $A > 0$.

The Jacobian matrix for the system (1) is

$$\begin{pmatrix} \frac{-1}{C_T}\left[Q_0\frac{\partial\alpha}{\partial T} + \frac{\partial R_o}{\partial T}\right] & \frac{-1}{C_T}\frac{\partial\alpha}{\partial A} \\ \frac{\partial\beta}{\partial T}A(1-A) & \beta(T)(1-2A) - \gamma \end{pmatrix}. \tag{4}$$

For the fixed point where $A = 0$, the eigenvalues are $\frac{-1}{C_T}\left[Q_0\frac{\partial\alpha}{\partial T} + \frac{\partial R_o}{\partial T}\right]$ and $\beta(T) - \gamma$. These are both negative, since T is so low in this fixed point that $\partial\alpha/\partial T = 0$ and $\beta(T) = 0$. This state is thus stable.

Let us look at the other two fixed points, where $A > 0$. Note that since $A = 1 - \gamma/\beta(T)$, the temperature must be in the range where $\beta(T) > 0$. With our choice of parameters, this range

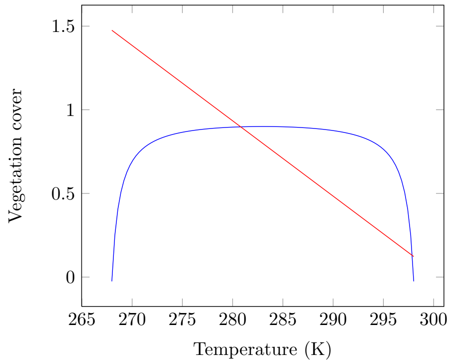

Figure 2: The fixed points are the intersections of these curves.

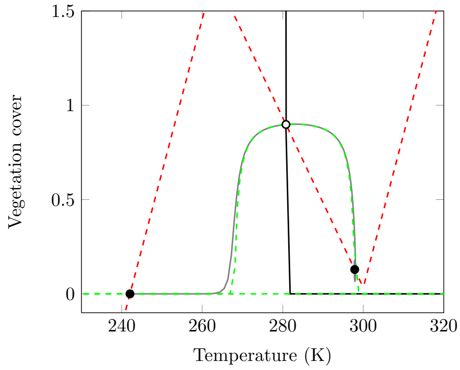

Figure 3: Phaseplane with nullclines (dashed lines) and fixed points (dots). The stable manifold of the saddle is shown in black and the unstable in gray. Coordinates of the fixed points: $(242.04, 0)$, $(280.8, 0.898)$ and $(297.9, 0.129)$.

also lies in between $T_{\alpha,\ell}$ and $T_{\alpha,u}$, which is the "interesting" range, since here we have the sea ice feedback. This corresponds with the decreasing part of the nullcline, as shown in the phaseplane in Figure 3.

The leftmost of these two fixed points where $A > 0$ is a saddle. This can be seen from numerical computations, but we can also deduce this analytically. The determinant of the Jacobian matrix is

$$\frac{-1}{C_T}\left[\left(Q_0\frac{\partial\alpha}{\partial T}+\frac{\partial R_o}{\partial T}\right)(\beta(T)(1-2A)-\gamma)-\frac{\partial\alpha}{\partial A}\frac{\partial\beta}{\partial T}A(1-A)\right].$$

Note that, in our "interesting" range of temperature, the nullcline for the temperature equation decreases. The equation for this nullcline is

$$A=\frac{1}{\alpha_v-\alpha_g}\left[\frac{1}{p}\left(1-\frac{R_o(T)}{Q_0}-(1-p)\alpha_o(T)\right)-\alpha_g\right],$$

and the derivative of this expression is less than zero iff $\partial_T R_o(T)+Q_0\partial_T\alpha(T,A)<0$. In addition, the fixed point lies on the other nullcline, so we know that $A=1-\gamma/\beta(T)$. This gives us

$$\beta(T)(1-2A)-\gamma=-\beta(T)+\gamma<0,$$

where the inequality follows from $0<A=1-\gamma/\beta(T)$.

Furthermore, $\partial_A\alpha(T,A)=p(\alpha_v-\alpha_g)<0$. Putting this together, we have the following:

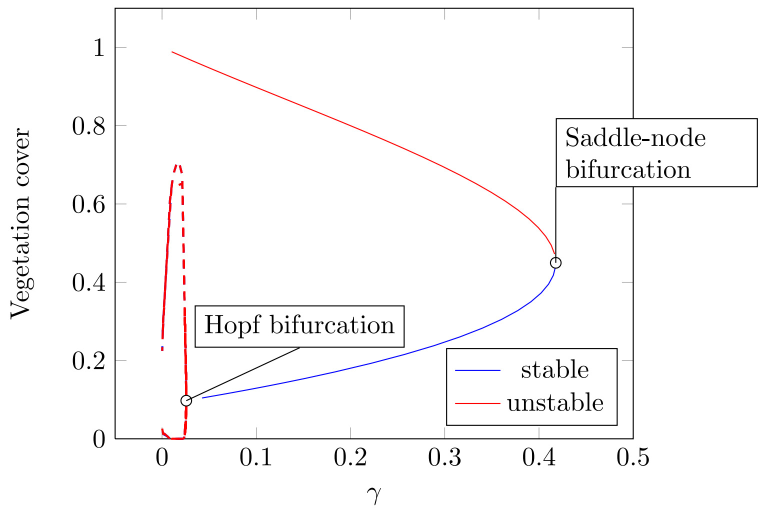

Figure 4: Bifurcation diagram. The curves were computed with XPPAUT.

Proposition 1. *A fixed point* (T, A) *where* $A > 0$ *is a saddle if* $\beta'(T) > 0$.

Proof. From the estimations we made above, we see that under this condition the determinant is negative. A negative determinant of the Jacobian implies that the fixed point is a saddle. □

The stable and unstable manifolds of the saddle are also shown in Figure 3. The stable manifold acts as a separatrix, and is almost vertical. This means that initial conditions with low temperature will lead to the low temperature, no vegetation state and initial conditions with a high temperature will lead to the steady state with some vegetation present.

For the rightmost fixed point, we cannot use the criterium above since it says nothing when $\beta'(T) < 0$. Computation of the eigenvalues shows that this point is a stable focus. The trace of the Jacobian matrix is

$$\tau = \frac{-1}{C_T}\left[Q_0\frac{\partial\alpha}{\partial T} + \frac{\partial R_o}{\partial T}\right] + \beta(T)(1-2A) - \gamma = \frac{-1}{C_T}\left[Q_0\frac{\partial\alpha}{\partial T} + \frac{\partial R_o}{\partial T}\right] - \beta(T) + \gamma.$$

The first term is larger than zero, but the term $-\beta(T)$ compensates and for our standard parameter values, the whole is negative such as to make the focus stable. We can however suspect that, for changing parameter values, the point loses its stability. This is the next thing we will investigate.

4.2 Hopf Bifurcation and oscillations

We are going to vary the parameter γ, the death rate of the vegetation, and look what happens. For decreasing γ, or longer-living vegetation, the focus loses its stability in a Hopf bifurcation and gives rise to a limit cycle. This occurs for $\gamma = 0.02572$, so when the overturning time of vegetation $(1/\gamma)$ is about 40 years. The results can be seen in the bifurcation diagram in Figure 4. Also note that when γ is higher than 0.41, no fixed point with non-zero vegetation exists. The stable fixed point, together with the saddle, are created in a saddle node bifurcation. We mentioned above that the parameter change needs to make the trace positive in order to change the stability. This also works if we, instead of decreasing γ, decrease C_T, the thermal heat capacity. It is not one of these parameters separately, but rather their product which determines the behaviour of the system. We can regard C_T as a typical timescale for temperature adjustments, and analogously $1/\gamma$ as the typical timescale for vegetation. The oscillations therefore occur when these timescales have a certain ratio. This is resemblant of other Hopf bifurcations where the time scales need to match, as for example in [Ghil and Tavantzis, 1983].

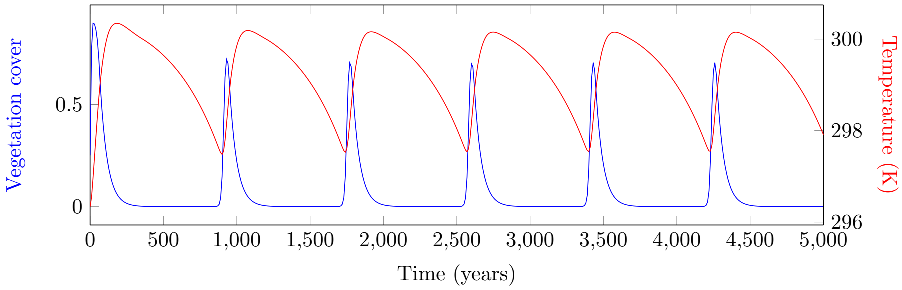

Figure 5: Trajectories for $\gamma = 0.02$, so $\gamma/C_T = 1/10$

Figure 5 shows plots of temperature and vegetation cover versus time.

The oscillations have an amplitude of a few degrees for the temperature, but are very large for the vegetation going from almost no vegetation to 70% of the land which is covered by plants. In addition, the vegetation plot show a sawtooth-like shape. It is also noteworthy that the vegetation cover stays at almost zero for long times, after which it shoots back up. Since, as noted before, the sea ice feedback is very important, we also look at a plot of the sea ice present on the planet. This is shown in Figure 6. It is interesting to see that sea ice varies little, from no sea ice to about 6%. But, as mentioned before, without the sea ice albedo feedback, the model would not have this behaviour. Sea-ice has been noted to be a possible determining factor in the inception of glacial cycles [Gildor and Tziperman, 2001], which is resemblant of the importance in our present model. We can remark here that our model essentially only differs from [Svirezhev and von Bloh, 1996] in the inclusion of the ocean and the sea ice (and the parametrization of the outgoing radiation), and that in that paper the authors actually prove the absence of limit cycles using Dulac's criterion. This reinforces the argument that sea ice plays an import role in climatic oscillations.

Note that temperature jumps up quickly, but that cooling down and growth of sea ice is slower. This is reminiscent of the typical behaviour of temperature over glacial-interglacial cycles, where temperature rises quickly (so-called terminations) but the buildup of ice sheets and cooling of the planet is slower. It indicates that inclusion of vegetation in the "ice age oscillators" might be an interesting direction to pursue, possibly combined with a simple carbon cycle. As far as we know this hasn't been done for ODE models. There are some results however from more complicated models that show that vegetation plays an important role in glacial cycles [Meissner et al., 2003; Horton et al., 2010].

Because of Proposition 1 and the shape of the growth curve, it is essential that the fixed point for which the Hopf bifurcation occurs has a temperature higher than T_{opt} (since $\partial_T \beta(T)$ needs to be negative), and lower than $T_{\alpha,u}$ (to have the sea ice feedback). This means, the optimal growth temperature for plants needs to be relatively low and there must be ice present for relatively high temperatures. This is one of the main questionable points of our model, since we used an optimal temperature of only 10 degrees Celsius for plant growth. The model shows however how even a very simple, 2 equation model can exhibit interesting behaviour through a simple feedback mechanism. The conditions $\partial_T \beta < 0$ and $\partial_T \alpha < 0$, necessary for the fixed point to become unstable, could be perhaps be found in a different kind of model, using not sea ice but some other mechanism as the feedback. Into mind comes for example the hydrological cycle, which has been studied already in some conceptual models [Aleina et al., 2013; Brovkin et al., 1998], but for which no oscillations were observed. We can note in this respect that limit cycles are also observed when changing for example the parametrization of outgoing radiation into a modified Stefan-Boltzmann law, or if we change the parameters $T_{\alpha,u}$ and T_{opt}. The important thing is that the conditions above on the derivatives are satisfied.

A direction to pursue could be the extension of our zerodimensional model to a spatial model, where latitude-dependence is included. This would allow for example to have desert vegetation interacting with the sea ice in higher latitudes, through the global temperature. There have already been some studies of larger models [Zeng and Neelin, 2000; Zeng et al., 1999] which conclude that the variable vegetation in the Sahel region provides strong feedbacks, and interacts with global

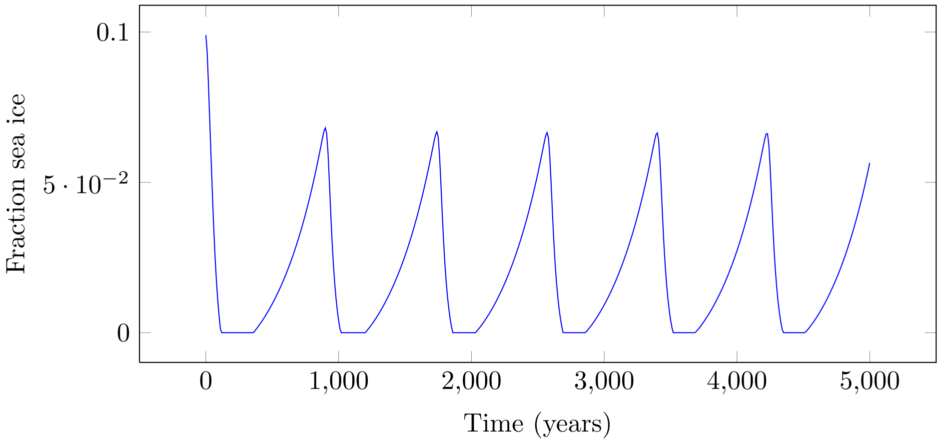

Figure 6: The fraction of ocean that is covered by ice as a function of time ($\gamma = 0.02$)

temperatures.

Another extension that comes into mind is the inclusion of vegetation in the ocean, plankton. Plankton interacts with climate in different ways, through its possible effect on cloud formation [Ayers and Cainey, 2007], but also through its albedo.

5 Conclusion

We described a simple dynamical systems model for climate-vegetation interaction. The planet has a large ocean which can be covered by sea ice, and a ground area which can be covered by vegetation. The coupling between temperature and vegetation is given by the growth rate and the albedo. The system exhibits two stable states, one without and one with vegetation. The state with vegetation can lose its stability through a Hopf bifurcation and give rise to a limit cycle. This happens when the typical time scale for vegetation overturn becomes high enough. The influence of the sea ice in reinforcing the albedo feedback is essential to have this behaviour. Although some parameter values are not entirely realistic, the results add to the evidence that vegetation, in combination with other feedback effects, can play an important role in the modeling of climate. The model is also interesting because it is one of the simplest ODE models for climate-vegetation interactions that exhibits oscillatory behaviour. It opens up some questions as well: could the model be extended to a (more realistic) spatial model? Or, how can vegetation be included in ODE models for ice-ages cycles?

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