

A report on Kondo physics in presence of Rashba SOI on the edge of a quantum spin Hall insulator

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Abstract

The main objective of this report is to examine the effect of a localized Kondo impurity on the edge of a quantum spin Hall insulator with Rashba spin-orbit interaction. A brief introduction to topological insulators and the spin-orbit interaction is provided in the beginning. Then we discuss the Kondo and the Rashba interactions. The technique of the Renormalization Group (RG) studied as a part of this project is presented with a few examples and then we use the perturbative RG approach to derive the scaling equations for XYZ the Kondo Hamiltonian. The last section provides a prelude to the conductance calculation which is of great theoretical and experimental interest - tuning the conductance in a quantum spin Hall insulator using Rashba interaction.

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1. Topological insulators

Right from microscopic physical laws to the behavior of the planets, human body to the DNA, one observes remarkable symmetry in the organization and behavior of systems. Indeed the concept of symmetry has become a powerful theoretical tool when it comes to understanding the complex systems and condensed matter is no exception. For example, while studying stable phases of matter, one observes that liquids and gases (classical and quantum) have translational and rotational symmetry unlike solids. Magnetically ordered solids - ferromagnets and anti-ferromagnets break rotation symmetry of the spin. The list extends even to quantum phenomena like superconductivity and superfluidity. This phenomena of 'symmetry breaking', finds its mathematical manifestation in the Ginzburg-Landau theory of phase transitions[1, 2]. It assigns an local order parameter which if non zero corresponds to a a stable ordered state of matter. This was the state of the art in Physics till 1980's.

The classical Hall effect was discovered a century ago and was very well understood, however it was in 1980's that the quantum version (QHE) was discovered. An excellent review on the subject is given in [3]. The QHE is observed in a 2D electron gas confined to a plane with a perpendicular magnetic field. The Hall conductance becomes quantized unlike its classical counterpart. It is an integral multiple of e^2/h for the Integer QHE and a rational multiple for Fractional QHE. This has been experimentally measured to a very great accuracy. It also turns out that although in such a setup the bulk becomes insulating, we have extended edge states which are robust to disorder. These states of matter are not understood by the symmetry breaking but instead require a notion of topology. Elementary notions of topology can be found in the text by Altland & Simons [4].

A topological state of matter in insulating in the bulk but supports edge states. These states are not characterized by a local order parameter but by the value of a topological invariant - TKNN or the Chern number. We say that these states are 'topologically protected'. Also the response function - the Hall conductance is related to the topological invariant. The topology results in the separation of spatial degrees of freedom in a QH system which means that the left- and right-moving electrons are present in opposite edges of the sample. Hence for this reason, QH edge channels are also termed 'chiral'. The important feature is that backscattering is not allowed since an electron can not reverse its direction of motion. It is important to note that the QH state breaks time-reversal symmetry.

Until few years back, QH systems were the only topological states of matter realized experimentally. In 2006 a new topological state of matter was predicted and was experimentally realized in 2007 - namely the Quantum Spin Hall effect (QSH). These belong to the topological class invariant under time-reversal. In simple terms, QSH can be understood as two copies of a QH state - one copy for each spin. We have counter-propagating states at each edge i.e. spin up and spin down move in opposite directions. Thus these states are also termed helical due to the correlation between spin and direction of propagation. Unlike QH systems, the helical liquid does not break TR symmetry. However, in analogy with a QH system, single - particle backscattering is suppressed in the absence of magnetic impurities, implying ballistic transport of charge along the edges. This has caused a lot of excitement, with speculations that topological insulators may serve as platform for future spintronic applications.

The QSH was first theoretically predicted in 2006 and experimentally realized in 2007 by Konig et al in HgTe/CdTe quantum wells (QW). The central feature is that due to large spin-orbit coupling (because of the heavy element Hg) there is band inversion. This important property can be exploited to vary the electronic structure when these QW structures are grown. When the thickness of the QW is greater than a critical thickness, these QW structures result in the QSH state. The detailed model is discussed in the review by Qi & Zhang [6]

It is of interest to note the features of the helical edge state in presence of impurities and interactions. The important aspect of the QSH edge state is invariance under TR. If we consider electron-electron interactions, there exist backscattering processes invariant under TR for example two-particle backscattering in which two rightmovers are back-scattered to two leftmovers. However this process transfers momentum $4k_F$ to the underlying lattice (where k_F is the Fermi momentum), is suppressed in semiconductor materials where QSH effect is expected to occur. Another potentially more important backscattering process is caused by a magnetic impurity via Kondo interaction. This serves as the basis for this project. With this brief introduction to topological insulators we conclude this section. For detailed reviews see [5, 6, 7].

2. Spin orbit interaction (SOI) and Rashba SOI

In this section, we provide an introduction to spin-orbit interaction (SOI) in general and also the Rashba SOI. We know that the electron has intrinsic angular momentum which gives rise to the concept of spin. Let us assume that an electron is subjected to an electric field $-e\vec{E} = -\nabla V$. We know through relativity, that an electron in its rest frame experiences a magnetic field $\vec{B} = -\vec{v} \times \vec{E}/c$. The interaction of a magnetic moment ($\vec{\mu}$) with this magnetic field yields the following potential energy term:

$$V_\mu = -\vec{\mu} \cdot \vec{B} = \frac{1}{ec} \vec{\mu} \cdot \vec{v} \times \nabla V \quad (2.1)$$

In an atom we consider the electric field due to the nucleus which is central, i.e. $V = V(r)$, hence the potential energy term is:

$$V_\mu \propto \vec{\mu} \cdot \vec{v} \times \vec{r} = \frac{1}{m} \vec{\mu} \cdot \vec{L} \quad (2.2)$$

where $\vec{L} = \vec{r} \times \vec{p}$ is the orbital angular momentum. Now the intrinsic magnetic moment ($\vec{\mu}$) is proportional to the intrinsic angular momentum (spin - \vec{S}) which is a multiple of the Pauli spin matrices ($\vec{\sigma}$). Hence the interaction is proportional to $\vec{\sigma} \cdot \vec{L}$. This gives us an elementary understanding of the spin-orbit coupling. A more rigorous approach is to take the nonrelativistic limit of the Dirac equation which results in the same spin orbit coupling term [8]. The equation thus obtained is also known as the Pauli equation:

$$\left(\frac{p^2}{2m} + V - \frac{p^4}{8m^3c^2} - \frac{e\hbar}{4m^2c^2} \vec{\sigma} \cdot \vec{p} \times \nabla V + \frac{\hbar^2}{8m^2c^2} \nabla^2 V \right) \psi = \epsilon \psi \quad (2.3)$$

V is the potential and we assume low energy dynamics i.e. $E = mc^2 + \epsilon$, where $\epsilon \ll mc^2$. The third term is a relativistic correction to the kinetic energy and the fourth term is the spin-orbit coupling term. The above equation describes a low-energy electron moving in vacuum in presence of a potential V . We employ the same equation also in the case of a solid, but we usually ignore the third and the fifth terms since these are much smaller. One is interested in obtaining an effective Hamiltonian which describes the motion of electrons in a semiconductor.

In solids, we have a periodic crystal potential and in addition to it we may also consider external electric fields from drain-source voltages, gate voltages etc. We are familiar with the concept of bands in a solid. Assuming we are interested in semiconductors, we have the conduction and valence band separated by a band gap ($E = E(k)$). The conduction band arises from the s orbitals and is twofold degenerate (due to spin) while the valence band arises from p orbitals and is sixfold degenerate. Let $|S\rangle$ and $|X\rangle, |Y\rangle, |Z\rangle$ denote the basis state for s - and p -orbitals respectively. The band structure so far requires no consideration of SOI.

If we include the effect of SOI on the band structure, the bands become mixed. The quantum numbers which describe the bands are the total angular momentum quantum numbers (j and m_j) along with the orbital quantum number (l). $l = 0$ denotes the conduction band and $l = 1$ denotes the valence band. The conduction band has $j = 1/2$ ($m_j = \pm 1/2$). Two of the valence bands have: $j = 1/2$ ($m_j = \pm 1/2$), and the other four: $j = 3/2$ ($m_j = \pm 1/2 \pm 3/2$). The $j = 3/2$ bands have different curvature and are called 'light hole' (lh) and 'heavy hole' (hh) bands.

Let us focus our attention on the SOI of the form: $H_{soi} = -\vec{B}(k) \cdot \vec{\sigma}$. This is valid if we assume that the terms that couple to spin are linear in \vec{k} . Time reversal symmetry (TR) requires that $\vec{B}(-k) = -\vec{B}(k)$. Now if the system has inversion symmetry then $\vec{B}(-k) = \vec{B}(k)$ which renders the interaction term zero. Hence inversion symmetry has to be broken so that the SOI is non-vanishing. In QW heterostructures, inversion symmetry is often broken. This leads to the Rashba term of the form: $H_R = \alpha(\vec{k} \times \vec{\sigma})_z$. The parameter α can be tuned using a gate voltage.

3. Rashba and Kondo Hamiltonians

In this section we introduce the Rashba and Kondo Hamiltonians. Before that let us briefly introduce the Kondo effect. It was experimentally observed decades back that metals in presence of a magnetic impurity exhibit a resistivity minimum as the temperature is decreased with an increase in resistivity as temperature is decreased further. A model Hamiltonian which describes the interaction of impurity spin and electron was introduced to investigate this further - the Kondo Hamiltonian. Second-order perturbation theory on this Hamiltonian explains the resistance minimum and the observed $\ln T$ dependence of the resistivity. This was first carried out back in the 1960s by J. Kondo. However as $T \rightarrow 0$, the $\ln T$ term diverges, indicating breakdown of perturbation theory. This remained a puzzle for a long time and it was only in the 1980s that with the ideas of Wilson's newly developed Renormalization Group the problem was fully solved [1].

The Kondo Hamiltonian describes the spin-spin interaction between impurity and electron, $H_K = \Psi^\dagger(0)\bar{\sigma}\Psi(0)\cdot\Phi^\dagger(0)\bar{\sigma}_i\Phi(0)$. Ψ is the electron field and Φ is the localized impurity field. σ represents the usual Pauli spin matrices. To avoid double occupancy of the impurity site, a large chemical potential term has to be added (Abrikosov pseudo-fermion formalism [17]). Since the impurity is localized at $x = 0$, $\phi^\dagger(0)\phi(0) = 1$, one is free to drop the impurity field when convenient.

3.1 Including Rashba SOI in QSH edge state

In a recent work [9], electrical manipulation and measurement of spin properties of QSH edge states was studied theoretically using Rashba SOI. A spatially uniform Rashba SOI was employed, controllable by a gate voltage. Hence this is of experimental interest. We consider the following QSH Hamiltonian:

$$H_{QSH} = v_F \int dx \Psi^\dagger (-i\partial_x \sigma_z) \Psi. \quad (3.1)$$

Here v_F is the Fermi velocity and Ψ is the two component spinor describing the helical edge state: $\Psi = (\psi_R e^{ik_F x}, \psi_L e^{-ik_F x})^T$. ψ_R and ψ_L are the slowly varying electron fields. We choose the right movers as spin up and left movers as spin down. The quick variations are encoded in the factors $e^{\pm ik_F x}$ which is not explicitly considered in [9]. For now drop these factors, i.e. consider $\Psi = (\psi_R, \psi_L)^T$ and proceed. Next we consider the Rashba SOI of the form:

$$H_R = \int dx \alpha(x) \Psi^\dagger (-i\partial_x \sigma_y) \Psi, \quad (3.2)$$

where $\alpha(x)$ is the coupling constant which can be tuned by the external gate voltage. We consider $\alpha(x) = \alpha$ as a real constant independent of x . Hence, if no other interactions are present the Hamiltonian is $H_0 = H_{QSH} + H_R$ which can be written on the form:

$$H_0 = v_\alpha \int dx \Psi^\dagger (\cos \theta \sigma_z + \sin \theta \sigma_y) (-i\partial_x) \Psi, \quad (3.3)$$

where $v_\alpha = \sqrt{\alpha^2 + v_F^2}$, $\sin \theta = \alpha/v_\alpha$ and $\cos \theta = v_F/v_\alpha$. Now we introduce rotation of the electron basis: $\Psi' = R(\theta)\Psi$ where $R(\theta) = e^{-i\sigma_x \theta/2}$. Hence the operator $(\cos \theta \sigma_z + \sin \theta \sigma_y)$ transforms as $R(\theta)(\cos \theta \sigma_z + \sin \theta \sigma_y)R^{-1}(\theta)$ such that H_0 remains invariant. With the above trick, H_0 transforms to:

$$H_0 = v_\alpha \int dx \Psi'^\dagger (-i\partial_x \sigma_z) \Psi'. \quad (3.4)$$

Note that the z axis above is rotated by an angle θ w.r.t the original z axis. In particular, the eigenvectors of σ_z are along the rotated z axis. Thus we infer that the effect of introducing a uniform Rashba SOI amounts to a renormalization of the Fermi velocity and a rotation of the spin axis.

Let us now consider the effect of including the fast phases. i.e. now considering $\Psi = (\psi_R e^{ik_F x}, \psi_L e^{-ik_F x})^T$. Substituting this in the QSH Hamiltonian (3.1), we obtain:

$$H_{QSH} = 2v_F k_F + v_F \int dx \Psi^\dagger (-i\partial_x \sigma_z) \Psi, \quad (3.5)$$

where the above spinor Ψ does not include the fast phase. The fast phase cancels out and H_{QSH} is thus invariant up to an additive constant. In a similar manner, the Rashba SOI in 3.2 becomes:

$$H_R = \alpha \left(ik_F \left(\psi_R^\dagger \psi_L e^{-2ik_F x} + \psi_L^\dagger \psi_R e^{2ik_F x} \right) - \left(\psi_R^\dagger e^{-2ik_F x} \frac{\partial \psi_L}{\partial x} - \psi_L^\dagger e^{2ik_F x} \frac{\partial \psi_R}{\partial x} \right) \right). \quad (3.6)$$

Writing the full Hamiltonian H_0 explicitly, including the fast phase factors, we have:

$$\begin{aligned} H_0 = & -iv_\alpha (2ik_F \cos \theta + \psi^\dagger \frac{\partial \psi_R}{\partial x} \cos \theta - \psi_L^\dagger \frac{\partial \psi_L}{\partial x} \cos \theta - \psi_R^\dagger \psi_L k_F \sin \theta e^{-2ik_F x} - \psi_L^\dagger \psi_R k_F \sin \theta e^{2ik_F x} \\ & - i\psi_R^\dagger \frac{\partial \psi_L}{\partial x} \sin \theta e^{-2ik_F x} + i\psi_L^\dagger \frac{\partial \psi_R}{\partial x} \sin \theta e^{2ik_F x}). \end{aligned} \quad (3.7)$$

We note that the inclusion of Rashba SOI does not cancel out the fast phase. However the terms coming from Rashba SOI tend to average to zero in the absence of the Kondo impurity. Conveniently one can always choose $x = 0$ as the impurity site for the Kondo problem. In such a case the presence of the fast phase factors $e^{\pm 2ik_F x}$ do not matter.

3.2 The Kondo term

Let us consider the following anisotropic Kondo Hamiltonian :

$$H_K = \Psi^\dagger(0) \bar{A} \cdot \bar{\sigma}^i \Psi(0). \quad (3.8)$$

Here $\Psi(x)$ is the electron field, and $\bar{\sigma}^i$ denotes the impurity spin. $\bar{A} = J_x \sigma_x \hat{x} + J_y \sigma_y \hat{y} + J_z \sigma_z \hat{z}$, and $\sigma_{x,y,z}$ denote the electron Pauli matrices. Hence the interaction is $J_x \sigma_x \sigma_x^i + J_y \sigma_y \sigma_y^i + J_z \sigma_z \sigma_z^i$. We have assumed that the impurity is located at $x = 0$.

We wish to make a unitary transformation (rotation) of the two-component spinor fields Ψ . Before that let us examine the Hamiltonian closely:

$$H_K = \Psi^\dagger(0) \bar{A} \cdot \bar{\sigma}^i \Psi(0) \equiv (\Psi^\dagger(0) \bar{A} \Psi(0)) \cdot (\Phi^\dagger(0) \bar{\sigma}^i \Phi(0)),$$

where on the right-hand side we have reintroduced the impurity field $\Phi(0)$ which is localized at $x = 0$. Going to a Fourier basis, $\Psi(x) = \sum_k c_k e^{ikx}$, we obtain:

$$\Psi^\dagger(0) \bar{A} \Psi(0) = \sum_{k,k'} c_k^\dagger \bar{A} c_{k'}. \quad (3.9)$$

Here c_k is an electron annihilation operator with wavevector k . On the other hand, the impurity field is localized at $x = 0$, hence $\Phi = a_d$, where a_d is the annihilation operator. Here d usually denotes the d^{th} subshell, where the impurity spin is generally located (s-d Hamiltonian). Hence $\Phi^\dagger(0) \bar{\sigma}^i \Phi(0) = a_d^\dagger a_d \bar{\sigma}^i$. Here $a_d^\dagger a_d$ is the number operator, with unit eigenvalue. As already commented upon, this allows for $\phi(0)$ and $\phi^\dagger(0)$ to be dropped.

Let us now introduce about the x -axis by an angle θ . The transformation can be represented as follows: $R(\theta) = R_{\bar{r}}(\theta) R_{spin}(\theta)$, where $R_{\bar{r}}$ is the rotation of spatial degrees of freedom and $R_{spin}(\theta)$ is for the spin.

Hence, $(\Psi^\dagger \bar{A} \Psi) \cdot (\Phi^\dagger \bar{\sigma}^i \Phi) = (\Psi^\dagger R^{-1} R \bar{A} R^{-1} R \Psi) \cdot (\Phi^\dagger R^{-1} R \bar{\sigma}^i R^{-1} R \Phi)$. The impurity field remains invariant, hence $R \Psi = \Psi'$, $R \bar{A} R^{-1} = \bar{A}'$ and $R \bar{\sigma}^i R^{-1} = \bar{\sigma}'^i$, and we have : $H_K = (\Psi'^\dagger \bar{A}' \Psi') \cdot \bar{\sigma}'^i$.

The x term in H_K is: $A_x = J_x \sigma_x$. Under the unitary transformation it becomes: $A'_x = R(\theta) J_x \sigma_x R^{-1}(\theta) = J_x \sigma_x = A_x$. This is of course expected since we are making a rotation about the x -axis.

The y term is $A_y = J_y \sigma_y$. Under the unitary transformation this becomes

$$A'_y = R(\theta) J_y \sigma_y R^{-1}(\theta) = J_y \begin{pmatrix} \sin \theta & -i \cos \theta \\ i \cos \theta & -\sin \theta \end{pmatrix} = J_y (\cos \theta \sigma_y + \sin \theta \sigma_z). \quad (3.10)$$

The z term is $A_z = J_z \sigma_z$. Under the unitary transformation this becomes

$$A'_z = R(\theta) J_z \sigma_z R^{-1}(\theta) = J_z \begin{pmatrix} \cos \theta & i \sin \theta \\ -i \sin \theta & -\cos \theta \end{pmatrix} = J_z (\cos \theta \sigma_z - \sin \theta \sigma_y). \quad (3.11)$$

We now have the new coefficients A'_x , A'_y and A'_z

If we rotate the impurity spin also, we will have the following relations :

$$\sigma_x'^i = \sigma_x^i \quad (3.12)$$

$$\sigma_y'^i = (\cos \theta \sigma_y^i + \sin \theta \sigma_z^i) \quad (3.13)$$

$$\sigma_z'^i = (\cos \theta \sigma_z^i - \sin \theta \sigma_y^i) \quad (3.14)$$

So rotating both impurity and electron spin, we arrive at:

$$H_K = \Psi'^\dagger (A'_x \sigma_x' + A'_y \sigma_y' + A'_z \sigma_z') \Psi \quad (3.15)$$

Expanding the above we get,

$$H_K = \Psi'^\dagger (J_x \sigma_x \sigma_x^i + (J_y \cos \theta \sigma_y + J_z \sin \theta \sigma_z)(\cos \theta \sigma_y^i + \sin \theta \sigma_z^i) + (J_z \cos \theta \sigma_z - J_y \sin \theta \sigma_y)((\cos \theta \sigma_z^i - \sin \theta \sigma_y^i)) \Psi' \quad (3.16)$$

Grouping similar terms:

$$H_K = \Psi'^\dagger ((J_x \sigma_x \sigma_x^i + (J_y \cos^2 \theta + J_z \sin^2 \theta) \sigma_y \sigma_y^i + (J_z \cos^2 \theta + J_y \sin^2 \theta) \sigma_z \sigma_z^i + \cos \theta \sin \theta (J_y - J_z) (\sigma_y \sigma_z^i + \sigma_z \sigma_y^i)) \Psi' \quad (3.17)$$

Thus inclusion of the Rashba SOI amounts to a renormalization of coupling constants to new values - J_x, J'_y, J'_z with $J'_y = J_y \cos^2 \theta + J_z \sin^2 \theta$ and $J'_z = J_z \cos^2 \theta + J_y \sin^2 \theta$. In addition there appears a new term proportional to $(\sigma \times \sigma^i)_x$. This term is identified as Dzyaloshinskii-Moriya (DM) spin interaction with DM vector $\bar{D} = \hat{x}$. This type of term is generally induced by relativistic spin-orbit interactions, as first realized by Dzyaloshinskii [18] and Moriya [19]

4. Renormalization Group (RG)

4.1 Introduction

In this section we briefly present some of the ideas of Renormalization Group (RG) to prepare for RG analysis for Kondo interaction. For an introduction to the subject, see [1, 4]

The partition function of a system governed by Hamiltonian H can be written as $Z = \text{tr} e^{-\beta H}$, where the trace means sum over all possible configurations. In the language of field theory, the field $\phi(\bar{x})$ is the governing quantity. Here \bar{x} can take both spatial and time variables. Now $Z = \int D\phi e^{-S[\phi]}$. Here $D\phi = \lim_{N \rightarrow \infty} \prod_{n=1}^N d\phi_n$ $S[\phi]$ is the action governed by the Hamiltonian. The continuum theory, which is valid under certain approximations, is governed by the above functional field integral.

Often it is the case that we are not interested in the physics at short length scales. Hence one could possibly think of integrating over the short-range fluctuations and obtain an effective theory of long-range fluctuations. Equivalently in Fourier space, one can integrate over 'fast' degrees of freedom and obtain an effective theory for 'slow' degrees of freedom. The meaning of the terms will become clear subsequently.

Let a be the microscopic cutoff, which limits the applicability of the theory, and let L denote the system size (longest scale). If we need to 'integrate over fast modes' to generate an effective theory, we require a cutoff which separates the two regions. Let $a^i = b > a$, be the artificially imposed cutoff. Thus $[a, a^i]$ defines the 'short wavelength' region and $[a^i, L^{-1}]$ is the 'long wavelength' region. Now we need to integrate out 'short range' fluctuations and this process will influence the action of 'long range' degrees of freedom. This is the key step in RG and is by no means trivial and often involves approximation. Having done so, this process can lead to various possibilities.

It can happen that the algebraic structure of the 'long range' degrees of freedom is completely different than the original theory, which isn't of much use to us. On the other hand we might obtain a structural similarity to the original theory. If this happens, then we can claim that we have an effective theory with us which is identical to the original theory but is different in two aspects - the coupling constants involved are renormalized and the cutoff increases from a to $a^i = ba$. Repeating the full procedure, we will have the new cutoff, $a^{ii} = b^2 a$ and so on. Each time the coupling constants will get 'renormalized' to a different value, which gives us an intuitive idea of a RG flow. So in a sense, RG is a continuous family of transformations. Mathematically these transformations form a semi-group, being different from a group because of the irreversibility of transformations. It is interesting to note that RG can be performed in both real space and momentum space. The latter approach is usually preferred, though not restricted to it.

The method of RG is a very powerful tool to explore the physics in the regime where perturbation theory breaks down. A classic example of this is the Kondo problem. Though perturbation theory could explain the resistivity minima, it required the method of RG to explain the zero temperature limit behavior. RG is closely related to the general theory of phase transitions and critical phenomena. It is worthwhile to mention a few words on the formulation of RG ideas. It initially started around 1960s, where the idea of flows of coupling constants arose in both condensed matter physics and particle physics. However all these ideas were formulated and developed by Wilson for which he received the Nobel Prize in 1982.

Interestingly RG has applications also in areas like chaos theory-dynamical systems where similar ideas are used to study for example the period-doubling route to chaos-Fiegenbaum constants. The common denominator is that the underlying system shows self similarity at different length scales.

Before we present the formal theory of RG, it is useful to consider a few examples. We shall discuss the classical 1D Ising model and quantum dissipative tunneling.

4.2 Ising Model - block spin RG

We consider a one-dimensional classical Ising model, which is governed by the Hamiltonian

$$H = -J \sum_{i=1}^N S_i S_{i+1} - H_{ext} \sum_{i=1}^N S_i. \quad (4.1)$$

Here i is the site index, and S_i represents the spin at the site i which can take the value ± 1 . H_{ext} is the external field. Hence the partition function Z can be written as:

$$Z = e^{-\beta H} = \exp \left(\beta J \sum_{i=1}^N S_i S_{i+1} + \beta H_{ext} \sum_{i=1}^N S_i \right) = \exp \left(K \sum_{i=1}^N S_i S_{i+1} + h \sum_{i=1}^N S_i \right). \quad (4.2)$$

We have introduced K and h for convenience. The summation inside the exponential can also be written as

$\sum_i (K S_i S_{i+1} + \frac{h}{2} (S_i + S_{i+1}))$. It is important to note that this is valid only if we assume periodic boundary conditions.

Now, $\exp(-\beta H)$ can be written as $\prod_{i=1}^N T(S_i, S_{i+1})$. T , usually called as the 'transfer matrix' is: $T(S, S') =$

$\exp \left(K S S' + \frac{h}{2} (S + S') \right)$. The partition function, $Z = \sum_{\{S_i\}} e^{-\beta H} = \sum_{\{S_i\}} \prod_{i=1}^N T(S_i, S_{i+1}) = \text{tr} T^N$, where the sum-

tion is over all spin configurations. One can easily convince oneself, by looking at a simple system of two spins that $Z = \sum_{S_1, S_2} (T(S_1, S_2) T(S_2, S_1)) = e^{2K+2h} + 2e^{-2K} + e^{2K-2h}$. The transfer matrix is

$$T = \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{pmatrix}, \quad (4.3)$$

implying that $\text{tr} T^2 = e^{2K+2h} + 2e^{-2K} + e^{2K-2h}$. Hence we have expressed the partition function, Z in terms of the N th power of the two-dimensional transfer matrix T . Thus all thermodynamic quantities can be evaluated.

Keeping this in mind, we can explore a bit more about RG on this 1D Ising spin model. The analysis we discuss below was originally constructed by Kadanoff and thus is usually called Kadanoff's block spin RG. As we discussed earlier the aim of RG is to integrate out short length fluctuations and can be performed in real space or k space. Here we first group b neighboring spins together in a cluster, hence there are 2^b configurations. We need to sum over these 2^b configurations of each cluster and find an effective Z . Once we obtain that, we need to make a comparison to the original Ising structure. If there is resemblance to the original Ising structure, (it turns out that there is), then we say that the Ising model is renormalizable and the procedure can be iterated a number of times to obtain the behavior at large length scales.

Let us examine the case of $b = 2$. In the transfer matrix approach, this can be depicted by two T matrices (4.3). We need to take the partial trace over these and find an effective transfer matrix T' which turns out to be T^b in general for a cluster of b spins. For simplicity we denote e^{-K} as u and e^{-h} as v . Hence

$$T = \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{pmatrix} = \begin{pmatrix} u^{-1}v^{-1} & u \\ u & u^{-1}v \end{pmatrix}, \quad (4.4)$$

and

$$T' = \begin{pmatrix} u'^{-1}v'^{-1} & u' \\ u' & u'^{-1}v' \end{pmatrix}, \quad (4.5)$$

up to an over all multiplicative constant, where $u' = \frac{\sqrt{v+v^{-1}}}{(u^4+u^{-4}+v^2+v^{-2})^{1/4}}$ and $v' = \frac{\sqrt{u^4+v^2}}{\sqrt{u^4+v^{-2}}}$.

Note that the overall multiplicative constant just raises the free energy (which goes as $\ln Z$) by a constant. Thus the original structure of the Ising model is reproduced. We have arrived at an effective model with T replaced by T' . Note that the parameters u and v are renormalized. Let us examine this behavior closely.

After each RG step the parameters flow to new values, which we denote by $u' = f(u, v)$ and $v' = g(u, v)$. This is just like an iterated map. It turns out that this model has fixed points $(u, v) = (0, 1)$ and $(1, v)$, by which we mean that $u^* = f(u^*, v^*)$ and $v^* = g(u^*, v^*)$. These points remain invariant under the RG transformation. The case of $u = 1$

corresponds to $\beta = 0$ which is the infinite temperature case and $(u, v) = (0, 1)$ is the zero temperature case. These both are extreme cases of perfect order and disorder. However there is an important difference between these two fixed points. The high-T case is an attractive fixed point and the low-T case is a repulsive fixed point. This means that though we may start at an extremely low though non-zero temperature, the RG flow will drive us into a high temperature regime. The new block spin model will behave as if the original spin system was at a higher temperature. The high-T is an attractive fixed point, which implies that the RG trajectories will eventually approach this point after a large number of iterations. The nature of the fixed points is examined by linearizing around the fixed point and identifying the eigenvalues of the Jacobian matrix - a standard method employed in the treatment of dynamical systems.

4.3 Dissipative quantum tunneling

In the previous section we performed real space RG. We now discuss how momentum space RG is performed with the example of a quantum tunneling problem. We do not go into details about the underlying physics, rather start from the action and proceed further to demonstrate RG. The problem is basically to examine the effect of dissipation on quantum tunneling which can be captured by an action S given below. We consider a particle in a periodic potential of the form $V = c \cos\theta$, where c is a constant and θ is a field variable. The partition function is $Z = \int D\theta \exp(-S[\theta])$, where the action is :

$$S[\theta] = \frac{1}{4\pi g} \int_{|\omega| < \Lambda} \frac{d\omega}{2\pi} |\theta(\omega)|^2 |\omega| + c \int d\tau \cos(\theta(\tau)). \quad (4.6)$$

Here Λ is the upper cutoff of the theory. The first step is to divide the energy scales into slow and fast. We introduce a parameter b , thus $\Lambda/b < |\omega| < \Lambda$ are fast (f) degrees of freedom and $|\omega| < \Lambda/b$ are slow (s). Also we introduce:

$$\theta_s(\tau) = \int_s \frac{d\omega}{2\pi} e^{-i\omega\tau} \theta(\omega), \quad (4.7)$$

and

$$\theta_f(\tau) = \int_f \frac{d\omega}{2\pi} e^{-i\omega\tau} \theta(\omega), \quad (4.8)$$

where s and f denote the slow and fast degrees respectively. We next want to find an effective action for the slow degree of freedom. For this purpose let us write

$S[\theta] = S_s[\theta_s] + S_f[\theta_f] + S_U[\theta_s, \theta_f]$. The action contains a part S_U which contains the interaction between slow and fast degrees of freedom. In our case we have, $S_{s,f}[\theta_{s,f}] = \frac{1}{4\pi g} \int_{s,f} \frac{d\omega}{2\pi} |\theta(\omega)|^2 |\omega|$ and $S_U[\theta_s, \theta_f] = c \int d\tau \cos(\theta_s(\tau) + \theta_f(\tau))$.

$$\begin{aligned} Z &= \int D\theta \exp(-S[\theta]) = \int D\theta_s \int D\theta_f \exp(-S_s - S_f - S_U) \\ &= \int D\theta_s \exp(-S_s) \int D\theta_f \exp(-S_f - S_U) = \int D\theta_s \exp(-S_{eff}[\theta_s]) \end{aligned} \quad (4.9)$$

The effective action S_{eff} is defined as: $e^{-S_{eff}[\theta_s]} = e^{-S_s} \int D\theta_f e^{-S_f - S_U} = e^{-S_s} \langle e^{-S_U} \rangle_f$, where the average is defined as: $\langle X \rangle_f = \int D\theta_f e^{-S_f} X$. Up to first order we can write $\langle e^{-S_U} \rangle_f \approx e^{-\langle S_U \rangle}$. Thus we need to evaluate the average $\langle S_U[\theta_s, \theta_f] \rangle_f$, where

$$\langle S_U[\theta_s, \theta_f] \rangle_f = c \int D\theta_f e^{-S_f} \int d\tau \cos(\theta_s + \theta_f) = \frac{c}{2} \int D\theta_f e^{-S_f} \int d\tau (e^{i\theta} + e^{-i\theta}). \quad (4.10)$$

Substituting for S_f , $\theta = \theta_s(\tau) + \theta_f(\tau)$, and going to a Fourier basis, we obtain

$$\langle S_U[\theta_s, \theta_f] \rangle_f = \frac{c}{2} \int d\tau e^{i\theta_s(\tau)} \int D\theta_f e^{-\frac{1}{4\pi g} \int_f \frac{d\omega}{2\pi} \theta(\omega) |\omega| \theta(-\omega)} e^{i \int_f \frac{d\omega}{2\pi} e^{i\omega\tau} \theta(\omega)} + c.c., \quad (4.11)$$

where $c.c.$ denotes the complex conjugate. Now we note that the integral over the fast degree of freedom is gaussian and is straightforward to carry out. The integral yields a term of the form $e^{-\int_f \frac{d\omega}{2\pi} \frac{\pi g}{|\omega|}}$ up to a constant which comes from the gaussian integration. We arrive at the final expression:

$$\langle S_U[\theta_s, \theta_f] \rangle_f = c \int d\tau \cos(\theta_s) e^{-g \ln b} = cb^{-g} \int d\tau \cos(\theta_s). \quad (4.12)$$

Hence the effective action is:

$$S_{eff}[\theta_s] = \frac{1}{4\pi g} \int_s \frac{d\omega}{2\pi} |\theta(\omega)|^2 |\omega| + cb^{-g} \int d\tau \cos(\theta_s). \quad (4.13)$$

The effective action is identical in its structure to the original action hence is renormalizable. The next step in RG is rescaling. The effective action is restricted to fast degrees of freedom i.e. $\omega < |\Lambda|/b$. To make a comparison to the original action, we need to rescale the effective action such that it runs over all the degrees of freedom. Hence we replace ω and τ by ω' and τ' where $\omega' = b\omega$ and $\tau' = \tau/b$. We choose $\theta'(\tau') = \theta_s(\tau)$, thus $\theta'(\omega') = b^{-1}\theta(\omega)$. This substitution yields:

$$S_{eff}[\theta_s] = S'[\theta'] = \frac{1}{4\pi g} \int_{|\omega'| < \Lambda} \frac{d\omega'}{2\pi} |\theta(\omega')|^2 |\omega'| + cb^{1-g} \int d\tau' \cos(\theta'(\tau')) \quad (4.14)$$

Hence the effect of RG is to obtain a new coupling constant, which from renormalization is $c \rightarrow c(b) = cb^{1-g}$. It follows in this case $g = 1$ is the fixed point For $g > 1$, c decreases, implying ease of tunneling. If $g < 1$, c increases, implying that the particle becomes localized in the well.

4.4 RG-general theory

So far, we have discussed briefly two examples of RG. We now formally summarize the general steps in a renormalization group procedure. We start with a field theory defined by an action $S[\phi]$, which can be a linear combination of operators, $\hat{O}(\phi)$ i.e. $S[\phi] = \sum_i g_i \hat{O}_i(\phi)$, where g_i are the coupling constants. RG captures the change in coupling constants after integrating over the fast degrees of freedom. The first step is to divide the field manifold ϕ into two sets - ϕ_f , which is to be integrated out - and the remaining, ϕ_s . One can perform this in real space as in the block spin method, in which case we integrate over degrees of freedom within a block of spins. Alternatively, working in momentum space, we define the fast degrees of freedom as those taking values $\Lambda/b < |p| < \Lambda$.

The second and the key step is to integrate over the fast sector, which often involves approximations. This gives us an effective action, $S'[\phi_s] = \sum_i g'_i \hat{O}'_i(\phi_s)$. Lastly we perform rescaling of frequency and field, which finally yields the renormalized action of the theory: $S[\phi] = \sum_i g'_i \hat{O}_i(\phi)$. Thus the effect of RG is to obtain new renormalized coupling constants. This effect can be captured in the mapping $\bar{g}' = \bar{F}(\bar{g})$. This relates the old ($\bar{g} = \{g_i\}$) and the new ($\bar{g}' = \{g'_i\}$) coupling constants. One can always choose $l = \ln b$ to be arbitrarily small and thus express $\bar{g}' - \bar{g} = R(\bar{g}) - \bar{g}$ in terms of the so called Gell-Mann-Low equation:

$$\frac{d\bar{g}}{dl} = R(\bar{g}) \quad (4.15)$$

The above equation is the central part of the RG analysis. It represents a mapping from the space of parameters to itself exactly like a dynamical system. One of the important property of the above equation is the set of 'fixed points'. These are the set of points (\bar{g}^*) which are invariant under the RG flow. Once the system is tuned to this set of parameter values it no longer changes under RG transformations, or, in other words remains invariant under all length or time scales. This behavior is also called 'self similarity' (as manifested for example in fractal structures).

To each physical system one can always assign a correlation length (ζ) which determines the exponential decay of correlations. If this has to be invariant under RG, it suggests that $\zeta = 0$ or ∞ . The case of $\zeta = \infty$ is interesting and corresponds to a second order phase transition enabling us to identify fixed points as potential transition points. Hence naturally it is quite interesting to examine the behavior in the vicinity of fixed points. Let \bar{g}^* be a fixed point and \bar{g} very close to it such that we can consider the following linear relation: $R(\bar{g}) \approx J(\bar{g} - \bar{g}^*)$, where J is the Jacobian, $J_{ab} = \partial R_a / \partial g_b$ evaluated at \bar{g}^* . To examine further we need to diagonalize J . Let λ_i and ϕ_i denote the eigenvalues and eigenvectors respectively such that $\phi_i^T W = \phi_i^T \lambda_i$. Note that we have used left eigenvectors. Now we express the i^{th} component of $\bar{g} - \bar{g}^*$ in the eigenbasis: $v_i = \phi_i^T (\bar{g} - \bar{g}^*)$, also called the scaling field. These components behave in a simple manner under RG: $dv_i/dl = \lambda_i v_i$ or $v_i(l) \sim \exp(l\lambda_i)$.

If $\lambda_i > 0$ then the RG flow is directed away from the fixed point and the associated field is said to be relevant. Such a fixed point is called an unstable fixed point, as we saw earlier in the $T = 0$ case of Ising model. If $\lambda_i < 0$, then the flow is attracted by the fixed point and the associated field is termed as irrelevant. Such a fixed point is called stable - $T = \infty$ case in the Ising model. If $\lambda_i = 0$, the associated field is called marginal. With this brief outlook we end the formal discussion on the general theory of RG.

5. Poor man's scaling for the XYZ Kondo model

Poor man's scaling was first introduced by Anderson in 1970 [13] who used a perturbative approach to derive scaling laws for the Kondo Hamiltonian. This has been of great interest in the past and finds mention in many standard texts[14, 15]. The approach is different compared to the standard RG method we discussed earlier. Here, we make an attempt to derive the poor man's scaling equations for the anisotropic Kondo Hamiltonian. We assume the general (XYZ) case i.e the couplings are different in all the three directions - x , y and z . The analysis performed here is similar to that in the text by Altland & Simons [4].

Let us begin by writing down the 1D helical liquid Hamiltonian (cf. 3.1)

$$H_L = -iv_F \left[\int_{-\infty}^{\infty} \psi_R^\dagger \frac{d}{dx} \psi_R - \int_{-\infty}^{\infty} \psi_L^\dagger \frac{d}{dx} \psi_L \right], \quad (5.1)$$

and the Kondo interaction

$$H_K = J_z \left[\psi_R^\dagger(0)\psi_R(0) - \psi_L^\dagger(0)\psi_L(0) \right] S_z + J_x \left[\psi_R^\dagger(0)\psi_L(0) + \psi_L^\dagger(0)\psi_R(0) \right] S_x + J_y \left[-i\psi_R^\dagger(0)\psi_L(0) + i\psi_L^\dagger(0)\psi_R(0) \right] S_y. \quad (5.2)$$

Here ψ_R and ψ_L are the right and left moving fermion fields respectively. Note that we are considering a helical liquid with the assumption that the right-moving electron is spin up and left-moving is spin down. However this does not affect the RG scheme used which can be seen from the analysis that follows. In k-space, we can write the Hamiltonian as:

$$H = \sum_{k,\sigma} \epsilon_{k\sigma} c_{k\sigma}^\dagger c_{k\sigma} + \sum_{k,k'} \left[J_z S_z (c_{k'\uparrow}^\dagger c_{k\uparrow} - c_{k'\downarrow}^\dagger c_{k\downarrow}) + J_x S_x (c_{k'\uparrow}^\dagger c_{k\downarrow} + c_{k'\downarrow}^\dagger c_{k\uparrow}) + J_y S_y (-ic_{k'\uparrow}^\dagger c_{k\downarrow} + ic_{k'\downarrow}^\dagger c_{k\uparrow}) \right], \quad (5.3)$$

where σ means the spin and $\epsilon_{k\sigma}$ is the energy, which is $\pm v_F k$ for the right(left) mover. J_x , J_y and J_z are the coupling parameters in the three directions and \hat{S} denotes impurity spin.

The RG scheme used here captures the effect of the fast degrees of freedom on the slow degrees - like evaluating the effective action in the conventional RG method. Under RG, the coupling constants are altered as: $J_i \rightarrow J_i + \delta J_i$. We divide the conduction band into high energy states, which have the energy in the range: $D/b < |\epsilon_k| < D$ and low energy states $0 < |\epsilon_k| < D/b$. D denotes the bare bandwidth and b is the cutoff parameter which is greater than 1. We write an eigenstate of the Hamiltonian, $|\psi\rangle$, as the sum of three terms, $|\psi_0\rangle$, $|\psi_1\rangle$ and $|\psi_2\rangle$. $|\psi_1\rangle$ represents the component in which there are no electrons (holes) in upper (lower) band edge, $|\psi_0\rangle$ ($|\psi_2\rangle$) has at least one hole (electron) in the lower (upper) band edge. By lower (upper) band edges we mean the high energy electron and hole states ($D/b < |\epsilon| < D$). Let us examine $H_{21} = P_2 H P_1$, where P_2 and P_1 are the projection operators onto the $|\psi_2\rangle$ and $|\psi_1\rangle$ subspaces respectively: $H_{21} = |\psi_2\rangle\langle\psi_2|H|\psi_1\rangle\langle\psi_1|$. This term accounts for the scattering of conduction electron into the band edge. Similarly defining other projection operators, we can write the following relation:

$$\begin{pmatrix} H_{00} & H_{01} & H_{02} \\ H_{10} & H_{11} & H_{12} \\ H_{20} & H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \end{pmatrix} = E \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \end{pmatrix} \quad (5.4)$$

We neglect the contribution by the terms H_{02} and H_{20} , since we are interested in low energy excitations. Hence we can easily arrive at the following relation:

$$\left[H_{10} \frac{1}{E - H_{00}} H_{01} + H_{11} + H_{12} \frac{1}{E - H_{22}} H_{21} \right] |\psi_1\rangle = E |\psi_1\rangle. \quad (5.5)$$

Let us consider the term which scatters electrons to the conduction band edge and back to the $|\psi_1\rangle$ subspace i.e. $H_{12} \frac{1}{E - H_{22}} H_{21}$. Writing down H_{21} and H_{12} explicitly:

$$H_{21} = \sum_{k_s, k_f} \left[J_z S_z (c_{k_f \uparrow}^\dagger c_{k_s \uparrow} - c_{k_f \downarrow}^\dagger c_{k_s \downarrow}) + J_x S_x (c_{k_f \uparrow}^\dagger c_{k_s \downarrow} + c_{k_f \downarrow}^\dagger c_{k_s \uparrow}) + J_y S_y (-i c_{k_f \uparrow}^\dagger c_{k_s \downarrow} + i c_{k_f \downarrow}^\dagger c_{k_s \uparrow}) \right] \quad (5.6)$$

$$H_{12} = \sum_{k'_s, k_f} \left[J_z S_z (c_{k'_s \uparrow}^\dagger c_{k_f \uparrow} - c_{k'_s \downarrow}^\dagger c_{k_f \downarrow}) + J_x S_x (c_{k'_s \uparrow}^\dagger c_{k_f \downarrow} + c_{k'_s \downarrow}^\dagger c_{k_f \uparrow}) + J_y S_y (-i c_{k'_s \uparrow}^\dagger c_{k_f \downarrow} + i c_{k'_s \downarrow}^\dagger c_{k_f \uparrow}) \right] \quad (5.7)$$

In 5.7 we have k'_s assuming that when electrons are scattered back to the low-energy subspace it can have a different energy. Here k_f and k_s represent the fast and slow degrees of freedom respectively. In order to evaluate the contribution one needs to identify all the possible processes. Note that the band edge states are virtual hence are constrained by conservation of k_f and also the spin. Just taking into account $H_{12}H_{21}$, we see that there are 8 possible processes. We summarize the processes and their amplitudes in the following table:

(1)	$c_{k'_s \uparrow}^\dagger c_{k_f \uparrow} c_{k'_s \uparrow}^\dagger c_{k_f \uparrow} c_{k_s \uparrow}$	$J_z^2 S_z^2$
(2)	$c_{k'_s \downarrow}^\dagger c_{k_f \uparrow} c_{k'_s \downarrow}^\dagger c_{k_f \uparrow} c_{k_s \uparrow}$	$(J_x + i J_y S_y) J_z S_z$
(3)	$c_{k'_s \downarrow}^\dagger c_{k_f \uparrow} c_{k'_s \downarrow}^\dagger c_{k_f \uparrow} c_{k_s \downarrow}$	$J_x^2 S_x^2 + i J_x J_y S_y S_x - i J_x J_y S_x S_y + J_y^2 S_y^2$
(4)	$c_{k'_s \uparrow}^\dagger c_{k_f \uparrow} c_{k'_s \uparrow}^\dagger c_{k_f \uparrow} c_{k_s \downarrow}$	$J_z S_z (J_x S_x - i J_y S_y)$
(5)	$c_{k'_s \uparrow}^\dagger c_{k_f \downarrow} c_{k'_s \uparrow}^\dagger c_{k_f \downarrow} c_{k_s \uparrow}$	$J_x^2 S_x^2 - i J_x J_y S_y S_x + i J_x J_y S_x S_y + J_y^2 S_y^2$
(6)	$c_{k'_s \downarrow}^\dagger c_{k_f \downarrow} c_{k'_s \downarrow}^\dagger c_{k_f \downarrow} c_{k_s \uparrow}$	$-J_z S_z (J_x S_x + i J_y S_y)$
(7)	$c_{k'_s \uparrow}^\dagger c_{k_f \downarrow} c_{k'_s \uparrow}^\dagger c_{k_f \downarrow} c_{k_s \downarrow}$	$(J_x - i J_y S_y) (-J_z S_z)$
(8)	$c_{k'_s \downarrow}^\dagger c_{k_f \downarrow} c_{k'_s \downarrow}^\dagger c_{k_f \downarrow} c_{k_s \downarrow}$	$J_z^2 S_z^2$

Hence, writing down the first term in the expression explicitly, one has

$$H_{12} \frac{1}{E - H_{22}} H_{21} |\psi_1\rangle = J_z^2 \sum_{k'_s k_f k_s} S_z c_{k'_s \uparrow}^\dagger c_{k_f \uparrow} \frac{1}{E - H_{22}} S_z c_{k'_s \uparrow}^\dagger c_{k_f \uparrow} |\psi_1\rangle + \text{remaining terms} \quad (5.8)$$

Note that $\sum_{k_f} = \int_{D/b}^D d\epsilon \nu(\epsilon) \approx \nu_0 D(1 - 1/b)$. Here we have set the density of states $\nu(\epsilon)$ as roughly constant (ν_0). Since the band edge occupancy of ψ_1 is zero, we can write the term 5.8 as:

$J_z^2/4 \sum_{k_s k'_s} \nu_0 D(1 - 1/b) c_{k'_s \uparrow}^\dagger c_{k_s \uparrow} \frac{1}{E - D + \epsilon_{k_s} + H_0}$, where H_0 is the single-particle Hamiltonian of the band electrons. We have approximated H_{22} as H_0 to lowest order. Also we have assumed that $\epsilon_{k_f} \approx D$. H_0 can be put to zero if we measure our energies w.r.t the ground state. Also E and ϵ_{k_s} are very small compared to D , hence we may ignore them. Using the identity $S_z^2 = 1/4$ for a spin-half electron, we arrive at our final (though approximate) expression for the first term:

$$(1) \frac{J_z^2}{4} M, \text{ where } M = \sum_{k_s k'_s} \left[\nu_0 D(1 - 1/b) c_{k'_s \uparrow}^\dagger c_{k_s \uparrow} \left(\frac{-1}{D} \right) \right]$$

Similarly, we obtain for the other terms:

$$(2) (-i J_x J_z S_y / 2 - J_y J_z S_x / 2) N, \text{ where } N = \sum_{k_s k'_s} \left[\nu_0 D(1 - 1/b) c_{k'_s \downarrow}^\dagger c_{k_s \uparrow} \left(\frac{-1}{D} \right) \right];$$

$$(3) \left(\frac{J_x^2}{4} + \frac{J_y^2}{4} + J_x J_y S_z \right) Q \text{ where } Q = \sum_{k_s k'_s} \left[\nu_0 D(1 - 1/b) c_{k'_s \downarrow}^\dagger c_{k_s \downarrow} \left(\frac{-1}{D} \right) \right];$$

$$(4) (i J_x J_z S_y / 2 - J_z J_y S_x / 2) P \text{ where } P = \sum_{k_s k'_s} \left[\nu_0 D(1 - 1/b) c_{k'_s \uparrow}^\dagger c_{k_s \downarrow} \left(\frac{-1}{D} \right) \right];$$

$$(5) \left(\frac{J_x^2}{4} + \frac{J_y^2}{4} - J_x J_y S_z \right) M;$$

$$(6) (-i J_x J_z S_y / 2 - J_z J_y S_x / 2) N;$$

$$(7) (i J_x J_z S_y / 2 - J_y J_z S_x / 2) P;$$

(8) $J_z^2/4Q$

We have thus seen all possible contributions arising from the terms (1) - (8). In obtaining the above we have used certain straightforward spin identities. These were contributions arising from the term which took into account scattering of electrons to the higher band edge and back to the $|\psi_1\rangle$ subspace. We also have a similar term which takes into account scattering of a hole to the lower band edge and back, i.e. $H_{10} \frac{1}{E - H_{00}} H_{01}$. The contributions which arise due to this terms are exactly the same as above. Hence, in the end, the change in the coupling constants which result from taking into account electron scattering alone, can be multiplied by the factor 2 to get the net change in coupling constants. This is valid in the limit of the approximations we have made earlier. We rearrange the above eight terms (1) - (8) and write down the terms associated with S_x , S_y and S_z as follows:

$$-J_x J_y S_z (M - Q) = J_x J_y S_z \nu_0 (1 - 1/b) \left(\sum_{k_s, k'_s} c_{k'_s \uparrow}^\dagger c_{k_s \uparrow} - c_{k'_s \downarrow}^\dagger c_{k_s \downarrow} \right), \quad (5.9)$$

$$-J_y J_z S_x (N + P) = J_x J_y S_z \nu_0 (1 - 1/b) \left(\sum_{k_s, k'_s} c_{k'_s \uparrow}^\dagger c_{k_s \downarrow} + c_{k'_s \downarrow}^\dagger c_{k_s \uparrow} \right), \quad (5.10)$$

$$-J_x J_z S_y (-iP + iN) = J_x J_y S_z \nu_0 (1 - 1/b) \left(\sum_{k_s, k'_s} -i c_{k'_s \uparrow}^\dagger c_{k_s \downarrow} + i c_{k'_s \downarrow}^\dagger c_{k_s \uparrow} \right). \quad (5.11)$$

The above equations, when combined with contributions from the lower band edge, give us precisely the $\delta J'_i$'s. For example, $\delta J_z = 2\nu_0 J_x J_y (1 - 1/b)$ and so on. Hence we arrive at the following relations:

$$\frac{dJ_z}{dl} = 2\nu_0 J_x J_y, \quad (5.12)$$

$$\frac{dJ_x}{dl} = 2\nu_0 J_z J_y, \quad (5.13)$$

$$\frac{dJ_y}{dl} = 2\nu_0 J_x J_z, \quad (5.14)$$

where ν_0 is the density of states and $b = e^l$. Substituting for $J_x = J_y$ gives the usual RG equations for XXZ Kondo. With the above differential recursion relations, one can derive the following relations: $J_x^2 - J_y^2 = c_1$, $J_y^2 - J_z^2 = c_2$, $J_z^2 - J_x^2 = c_3$. c_1, c_2 and c_3 are constants depending on the initial conditions. In the XXZ case it is straightforward to examine the flow in $J_\perp - J_z$ space. However solving the above three coupled equations turns out to be more challenging. However upon inspection of the equations 5.12-5.14, one can infer that the coupling constants J_x, J_y, J_z flow to strong coupling regime upon renormalization implying that the perturbation theory breaks down.

An exact treatment of Kondo effect in a helical liquid was carried out by Schiller and Ingersent [16]. These authors considered a helical interacting liquid with a Kondo term: $H = H_{Lutt} + H_K$. H_{Lutt} is the Luttinger liquid Hamiltonian of the following form:

$$H_{Lutt} = -iv_F \sum_{s=\pm} s \int dx \Psi_s^\dagger(x) \partial_x \Psi_s(x) + U a \int dx: \Psi_+^\dagger(x) \Psi_+(x) : : \Psi_-^\dagger(x) \Psi_-(x) :,$$

where $: \dots :$ indicates normal ordering [4]. This describes an interacting Luttinger liquid with Fermi velocity v_F and interaction strength U . H_K is the usual XXZ Kondo term. The key part is to bosonize the above Hamiltonian through which we can map the interacting Hamiltonian to a free Hamiltonian with renormalization of U and v_F . Poor man's scaling on this reveals that for the XXZ anisotropic Kondo interaction, a large U drives the system to a strong coupling regime. We expect the same to be true for the XYZ Kondo interaction also. For a discussion of a poor man's scaling approach to the XYZ Kondo interaction with the added Dzyaloshinskii-Moriya term, like in 3.17, see the recent work by M. Pletyukhov and D. Schuricht [20]

6. Correction to the conductance - foundation

In this section we wish to discuss the conductance of a helical edge state electron liquid in presence of a localized magnetic impurity. The conductance of a perfect QSH device is $2e^2/h$. This is twice the conductance of a normal QH liquid because we have two counter-propagating states on each side of the device. The presence of a magnetic impurity allows for the possibility of backscattering of a single electron, as we discussed earlier. This correction to the conductance of a QSH device in presence of a magnetic impurity has been studied recently by Maciejko et al. [10] and Tanaka et al. [11]. Here we follow the approach in [11] to explore how the presence of Rashba SOI changes the picture. Earlier in Chapter 3, we saw the effect of including Rashba SOI on a QSH edge: a renormalization of the Fermi velocity and rotation of the spin quantization axis. We also found out how this influences the Kondo Hamiltonian. Our motivation in the present chapter is to lay the ground for an experimental prediction of how the conductance in presence of a magnetic impurity changes when Rashba SOI which is turned via an external gate voltage.

It turns out that the language of bosonization is more convenient in treating one dimensional problems. There are several conventions which can be employed, however we stick to that used in [11]. The bosonized Hamiltonian for the QSH edge state is:

$$H_0 = \frac{\hbar v}{2\pi} \int dx [K(\partial_x \theta)^2 + K^{-1}(\partial_x \phi)^2]. \quad (6.1)$$

We identify v with v_α - the renormalized Fermi velocity when one includes the Rashba SOI. The parameter K accounts for electron-electron interactions at the edge. Repulsive interactions correspond to $K < 1$. ϕ is the bosonic field and θ is the dual field, which satisfy the following commutation relation: $[\phi(x), \partial_y \theta(y)] = i\pi \delta(x-y)$. Also, we employ the following bosonized description of fermion fields: $\psi_{R,L} = e^{-i(\theta \pm \phi)} / \sqrt{2\pi\alpha}$, where the right movers are spin up and the left movers are spin down. Here α is a short-range cutoff and $+$ ($-$) corresponds to R (L). In our analysis we consider an impurity which carries spin-1/2. The XYZ Kondo Hamiltonian after the transformation used for diagonalizing Rashba and H_{QSH} is:

$$H_K = \Psi^\dagger J_x S_x \Psi S_x^i + \Psi^\dagger J'_y S_y \Psi S_y^i + \Psi^\dagger J'_z S_z \Psi S_z^i + D \Psi^\dagger S_y \Psi S_z^i + D \Psi^\dagger S_z \Psi S_y^i. \quad (6.2)$$

$S_{x,y,z}$ denote the Pauli spin matrices for spin 1/2 electron and $S_{x,y,z}^i$ denote the same for the localized impurity. As we found in Chapter 3, $J'_y = J_y \cos^2 \theta + J_z \sin^2 \theta$ and $J'_z = J_z \cos^2 \theta + J_y \sin^2 \theta$ and $D = \frac{J_y - J_z}{2} \sin(2\theta)$. Recall that $J_{x,y,z}$ are the original coupling constants for the XYZ Kondo Hamiltonian. Also note the distinction between the dual field $\theta(x)$ and the Rashba control parameter θ . It is instructive to write the above in bosonized form also:

$$H_K = \frac{J_x}{2} \left(\frac{e^{2i\phi(0)} + e^{-2i\phi(0)}}{2\pi\alpha} \right) \left(\frac{S^+ + S^-}{2} \right) + \frac{J'_y}{2} \left(\frac{-ie^{2i\phi(0)} + ie^{-2i\phi(0)}}{2\pi\alpha} \right) \left(\frac{S^+ - S^-}{2i} \right) \\ - \frac{J'_z}{\pi} \partial_x \theta(0) S_z + \frac{D}{2} \left(\frac{-ie^{2i\phi(0)} + ie^{-2i\phi(0)}}{2\pi\alpha} \right) S_z - \frac{D}{\pi} \partial_x \theta(0) \left(\frac{S^+ - S^-}{2i} \right) \quad (6.3)$$

We have dropped the superscript i for the impurity. In order to discuss the conductance, we add a bias term to the Hamiltonian which can be accounted for by assigning different chemical potentials to left and right movers: $\pm eV/2$. Earlier we discussed that backscattering of electrons in presence of a magnetic impurity is responsible for corrections to the conductance. Backscattering flips the spin of the electron and also of the impurity. Hence equivalently, the bias can be replaced by assigning different energy values to up and down components of the impurity spin, i.e. we have the following term: $H_V = -eV S_z$. However note that after the inclusion of the Rashba SOI, the full Hamiltonian has been rotated. Employing the previous Rashba transformation on this term also, we have: $H_V = -eV \cos \theta S_z + ieV \sin \theta S_y$.

Next we rescale the bosonic fields in the following manner: $\{\phi, \theta\} \rightarrow \{\sqrt{K}\phi, \theta/\sqrt{K}\}$ after which H_0 assumes the form:

$$H_0 = \frac{\hbar v}{2\pi} \int dx [(\partial_x \theta)^2 + (\partial_x \phi)^2]. \quad (6.4)$$

Following [11], we perform the following unitary transformation $U = e^{i\lambda\phi(0)S_z}$ on the total Hamiltonian, i.e. $H \rightarrow UHU^{-1}$. As for H_0 , this implies:

$$\begin{aligned} H_0 &\rightarrow UH_0U^{-1} = H_0 + [i\lambda\phi(0)S_z, H_0] = \\ &H_0 + i\lambda S_z \frac{\hbar v}{2\pi} 2\partial_x \theta(0) [\phi(0), \partial_x \theta(0)] = \\ &H_0 - \lambda \hbar v S_z \partial_x \theta(0). \end{aligned} \quad (6.5)$$

Choosing $\lambda = -J'_z/\pi\hbar v\sqrt{K}$, the additional term in the last line of 6.5 becomes $\frac{J'_z S_z \partial_x \theta(0)}{\pi\sqrt{K}}$ which exactly cancels with $\frac{J'_z}{\pi} \partial_x \theta(0) S_z$ in H_K after field rescaling.

Similarly the transformation U takes $S^+ \rightarrow e^{i\lambda\phi(0)} S^+$ and $S^- \rightarrow e^{-i\lambda\phi(0)} S^-$, thus

$$H_V \rightarrow -eV \cos \theta S_z + \frac{ieV \sin \theta}{2i} \left(e^{i\lambda\phi(0)} S^+ - e^{-i\lambda\phi(0)} S^- \right). \quad (6.6)$$

In H_K , terms involving the vertex operator remain invariant under U , except for the impurity spin transformation. However, the term $\partial_x \theta(0) S_y$ transforms as

$$U : \partial_x \theta(0) S_y \rightarrow (e^{i\lambda\phi(0)S_z} \partial_x \theta(0) e^{-i\lambda\phi(0)S_z}) (e^{i\lambda\phi(0)S_z} S_y e^{-i\lambda\phi(0)S_z}) \quad (6.7)$$

The first factor can be written as: $\partial_x \theta(0) + [i\lambda\phi(0)S_z, \partial_x \theta(0)] = \partial_x \theta(0) - \lambda\pi S_z$ while the second factor becomes $(1/2i)(e^{i\lambda\phi(0)} S^+ - e^{-i\lambda\phi(0)} S^-)$. Hence the term assumes the form

$$(\partial_x \theta(0) - \lambda\pi S_z) \frac{1}{2i} (e^{i\lambda\phi(0)} S^+ - e^{-i\lambda\phi(0)} S^-) = \left(\frac{\partial_x \theta(0)}{2i} - \frac{\lambda\pi}{4i} \right) e^{i\lambda\phi(0)} S^+ - \left(\frac{\partial_x \theta(0)}{2i} + \frac{\lambda\pi}{4i} \right) e^{-i\lambda\phi(0)} S^- \quad (6.8)$$

It follows that the transformed Kondo term is given by:

$$\begin{aligned} H_K &= \frac{1}{8\pi\alpha} (J_x e^{2i\phi(0)\sqrt{K}} + J_x e^{-2i\phi(0)\sqrt{K}} - J'_y e^{2i\phi(0)\sqrt{K}} + J'_y e^{-2i\phi(0)\sqrt{K}}) e^{i\lambda\phi(0)} S^+ \\ &+ \frac{1}{8\pi\alpha} (J_x e^{2i\phi(0)\sqrt{K}} + J_x e^{-2i\phi(0)\sqrt{K}} + J'_y e^{2i\phi(0)\sqrt{K}} - J'_y e^{-2i\phi(0)\sqrt{K}}) e^{-i\lambda\phi(0)} S^- \\ &+ \frac{D}{2} \left(\frac{-ie^{2i\phi(0)\sqrt{K}} + ie^{-2i\phi(0)\sqrt{K}}}{2\pi\alpha} \right) S_z - \frac{D}{\pi} \left(\left(\frac{\partial_x \theta(0)}{2i\sqrt{K}} - \frac{\lambda\pi}{4i} \right) e^{i\lambda\phi(0)} S^+ - \left(\frac{\partial_x \theta(0)}{2i\sqrt{K}} + \frac{\lambda\pi}{4i} \right) e^{-i\lambda\phi(0)} S^- \right), \end{aligned} \quad (6.9)$$

where we have used 6.3, 6.5, 6.6, 6.7 and 6.8 in deriving it.

Our aim is to calculate the correction δG to the conductance in presence of the magnetic impurity. As we discussed earlier, this can be accounted for by considering backscattering of electron due to the impurity, which is always accompanied by a spin flip of the impurity. Hence the correction to the current is proportional to the time derivative of S_z . More precisely, $\delta I = -e\partial_t S_z$. However, again note that in our case, due to the Rashba SOI, we will have $\delta I = -e\partial_t S'_z = -e\partial_t (\cos \theta S_z - i \sin \theta S_y)$. This can be readily computed from the Heisenberg equation of motion: $\partial_t S'_z(t) = (i/\hbar)[H, S'_z(t)]$, where $S'_z(t) = e^{iHt} S'_z e^{-iHt}$. Here $H = H_V + H_K + H_0$. Staying in the weak-coupling regime where $T > T_K$ (where T_K is the Kondo temperature [15]), we can perform a perturbative analysis for evaluating δG . This involves the methodology of linear response, where we take the limit $V \rightarrow 0$. Also, to lowest order in perturbation theory, we neglect higher-order terms, i.e. from J^2 onwards. Under these guidelines, it is enough to evaluate the following commutator: $[H, S'_z]$.

Noting that H_0 and H_V commute with S'_z , the job entails to evaluate $[H_K, S'_z] = [H_K, \cos \theta S_z - i \sin \theta S_y] = \cos \theta [H_K, S_z] - i \sin \theta [H_K, (S^+ - S^-)/2i] = \cos \theta [H_K, S_z] - \sin \theta [H_K, S^+]/2 + \sin \theta [H_K, S^-]/2$.

Denoting:

$$T_1 = \frac{1}{8\pi\alpha} (J_x e^{2i\phi(0)\sqrt{K}} + J_x e^{-2i\phi(0)\sqrt{K}} - J'_y e^{2i\phi(0)\sqrt{K}} + J'_y e^{-2i\phi(0)\sqrt{K}}) e^{i\lambda\phi(0)}, \quad (6.10)$$

$$T_2 = \frac{1}{8\pi\alpha} (J_x e^{2i\phi(0)\sqrt{K}} + J_x e^{-2i\phi(0)\sqrt{K}} + J'_y e^{2i\phi(0)\sqrt{K}} - J'_y e^{-2i\phi(0)\sqrt{K}}) e^{-i\lambda\phi(0)}, \quad (6.11)$$

$$T_3 = \frac{D}{2} \left(\frac{-ie^{2i\phi(0)\sqrt{K}} + ie^{-2i\phi(0)\sqrt{K}}}{2\pi\alpha} \right), \quad (6.12)$$

$$T_4 = -\frac{D}{\pi} \left(\left(\frac{\partial_x \theta(0)}{2i\sqrt{K}} - \frac{\lambda\pi}{4i} \right) e^{i\lambda\phi(0)} \right), \quad (6.13)$$

$$T_5 = \frac{D}{\pi} \left(\left(\frac{\partial_x \theta(0)}{2i\sqrt{K}} + \frac{\lambda\pi}{4i} \right) e^{-i\lambda\phi(0)} \right), \quad (6.14)$$

we have

$$H_K = (T_1 + T_4)S^+ + (T_2 + T_5)S^- + T_3 S_z \quad (6.15)$$

Using standard commutation relations, let us write down all the terms that arise in δI :

$$I_1 = \cos\theta(-(T_1 + T_4)S^+ + (T_2 + T_5)S^-), \quad (6.16)$$

$$I_2 = -\frac{\sin\theta}{2} ((T_2 + T_5)(-2S_z) + T_3 S^+), \quad (6.17)$$

$$I_3 = \frac{\sin\theta}{2} ((T_1 + T_4)(2S_z) - T_3 S^-). \quad (6.18)$$

Thus, finally we obtain the expression for the change in the current δI :

$$\delta I = -\frac{ie}{\hbar} (I_1 + I_2 + I_3). \quad (6.19)$$

The above expression for δI forms the basis for evaluation of the correction to the conductance δG . This requires the extensive methodology of linear response theory, retarded Green's functions, correlations functions and conformal field theory and are left to future work. However, we conclude this section by summarizing briefly some of these aspects. These topics are covered in a great detail in the text by G.D. Mahan [12].

Experiments in condensed matter perturb the system and measure the response, which can be either linear or nonlinear. However if the perturbation is small enough, the nonlinear effects are usually negligible. The perturbation can be an electrical field, optical field or anything else. Hence in this limit one often approximates that the response of the physical system is proportional to the intensity of the perturbing field. The quantity which describes the response is called the response function and is a real experimental observable quantity. This is directly related to the correlation function and can be obtained from a Kubo formul. We do not go through the lengthy derivation of the Kubo formalism, but simply state the Kubo formula for the electrical conductivity:

$$\sigma(\omega) = \frac{1}{\omega V} \int_0^\infty dt e^{i\omega t} \langle \psi | [j^\dagger(t), j(0)] | \psi \rangle + \frac{in_0 e^2}{m\omega} \quad (6.20)$$

Here j is the current operator and V is the volume of the system. In our case, up to lowest order in perturbation theory, we will need to evaluate the following commutator: $[\delta I(t), \delta I(0)]$. It turns out that we can write the Kubo formula in terms of time-ordered Green's function and the formula for conductivity then becomes:

$$\sigma(\omega) = \frac{i}{\omega} [\Pi(\omega) + \frac{n_0 e^2}{m}] \quad (6.21)$$

$\Pi(\omega)$ is the current-current correlation function which can be evaluated using the Matsubara formalism: $\Pi(\tau) \propto \langle T_\tau j^\dagger(\tau) j(0) \rangle$, where T_τ represents time ordering. This is valid at zero temperature. For $T > 0$, one can use conformal field theory to carry out the further analysis, in exact analogy to the study of finite-temperature tunneling in a QSH device [21].

Conclusions

We have found that the Rashba SOI can play an important role in the physics of the QSH edge state. While the Rashba term can be absorbed in the helical liquid through a simple unitary transformation, its effect on the Kondo interaction between an electron and a localized spin-1/2 edge impurity is more profound. We find that the coupling constants get renormalized, and in addition a non-collinear Dzyaloshinskii-Moriya interaction is generated.

To obtain a more complete understanding of the Rashba rotated Kondo interaction, the technique of RG was studied and we arrived at scaling equations for XYZ Kondo through a perturbative RG approach.

Lastly we arrived at the expression for δI which is the change in current due to impurity on the edge of a QSH insulator. This in the presence of a Rashba SOI serves as the basis for calculation of the conductance which is left to future work. Importantly, this will provide a prediction of how the conductance changes when tuning the Rashba SOI via a gate voltage and would be of a great experimental interest.

The field of topological insulators is relatively new and is developing fast hence there is a wide scope for lots of theoretical and experimental research. In conclusion it is important to note that topological insulators are not only of interest for fundamental physics but also could serve as potential materials for new technology in the future.

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