FRACTAL GEOMETRY

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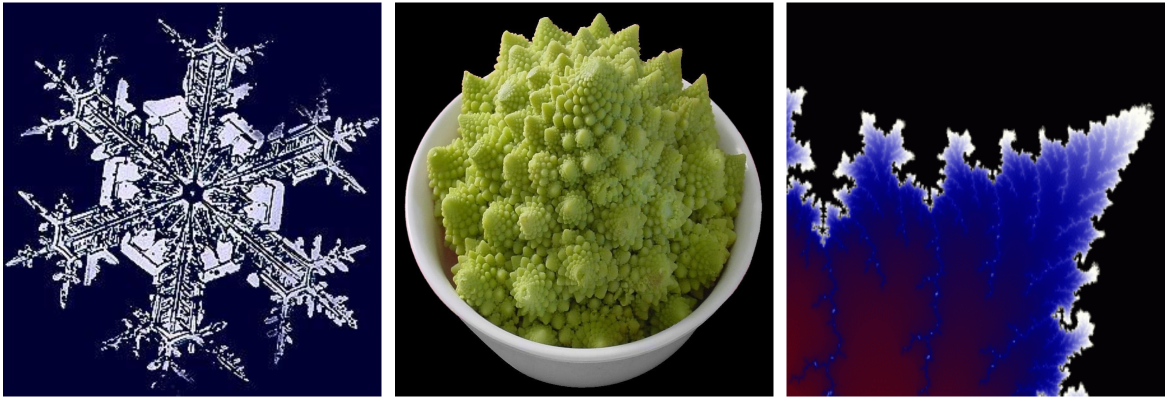
Topic:

- **fractal geometry**;
- and its applications.

□ Fractals: examples I

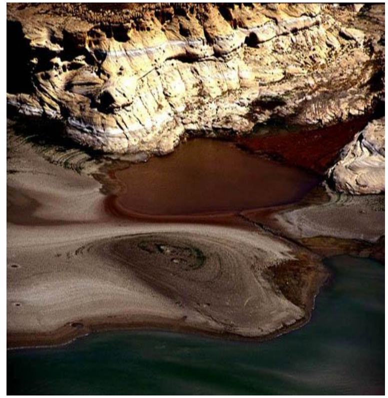
- **Example** Clouds profile

- **Example** Other fractals in nature

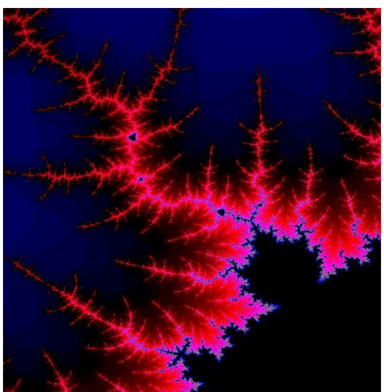
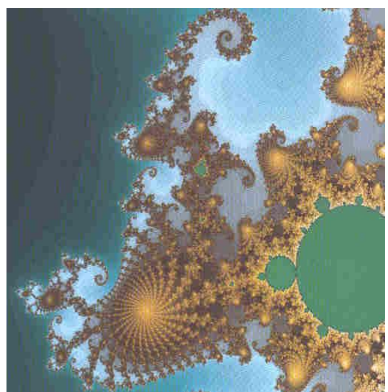
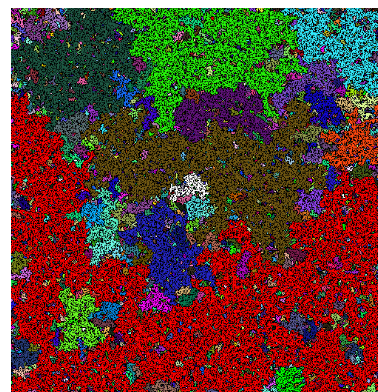


☐ Fractals: examples II

- **Example** Coastline profile


- **Example** Strange attractors and chaotic maps, percolating cluster

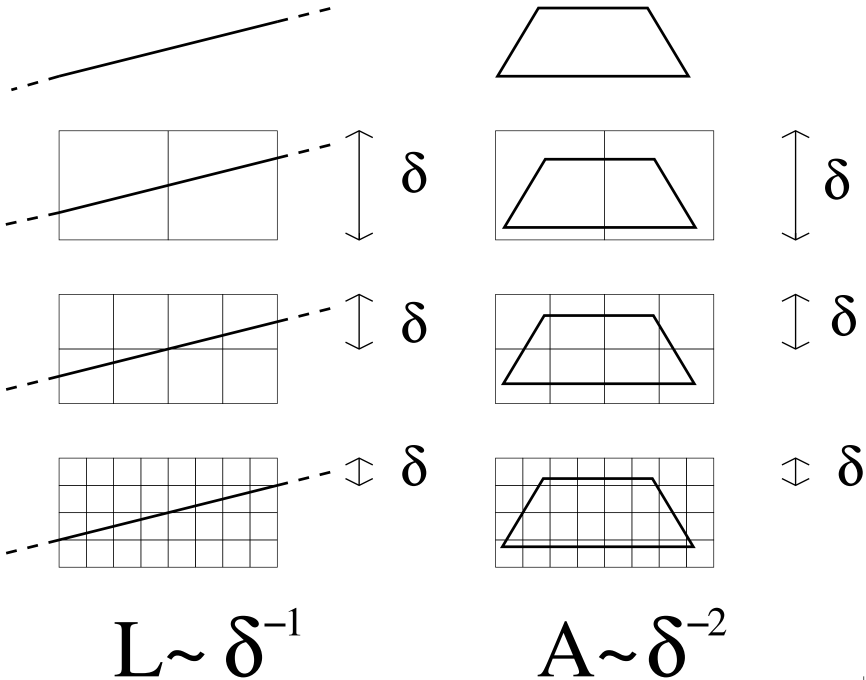




☐ Measure of Euclidean dimension, d

The number of "cells" of size δ necessary to cover the object (segment, polygon, ...) is (for $\delta \to 0$):

$N(\delta) \sim 1/\delta^d$

The exponent d is a measure of the dimensionality (named *Euclidean*) of the space filled by the object.

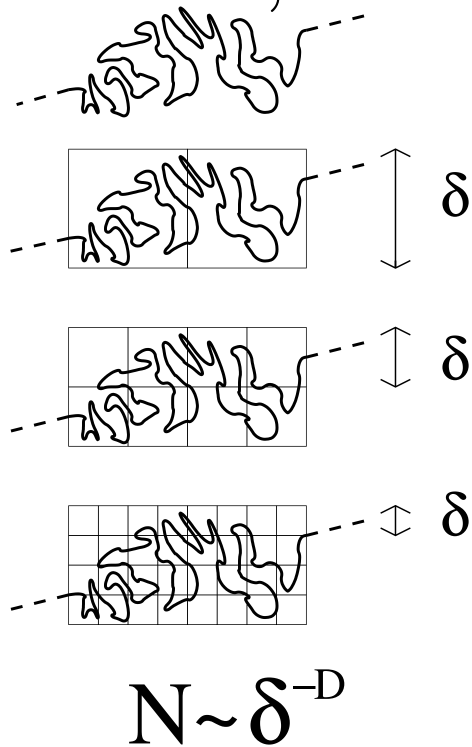


□ Measure of fractal dimension, D

Analogously, for a more complex geometric (fractal) object, for $\delta \to 0$, the relation

$$N(\delta) \sim 1/\delta^D$$

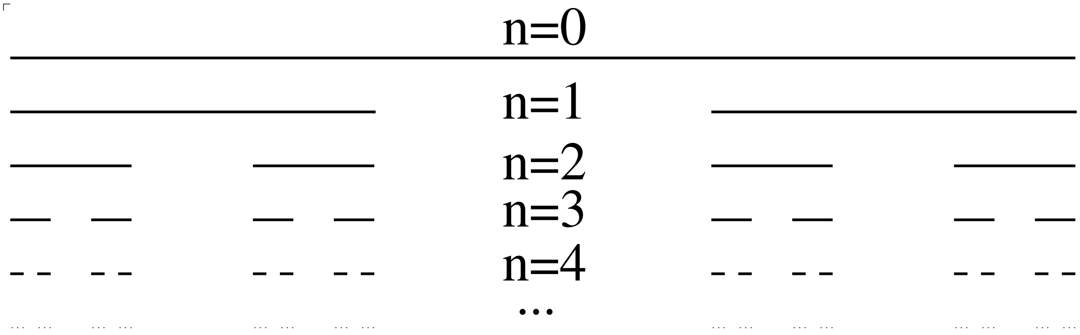
defines its (fractal) dimensionality D (named from *Hausdorff-Besicovitch*).



☐ Examples of deterministic fractals

$$D = \lim_{\delta \to 0} \frac{\ln N(\delta)}{\ln 1/\delta}$$

- **Triadic Cantor set**



After the nth iteration, the length of a single element is:

$$\delta = 3^{-n}$$

The number of “cells” of size δ necessary to cover the object is:

$$N(\delta) = 2^n$$

Thus, its fractal dimensionality is:

$$D = \lim_{n \to \infty} \frac{\ln N(\delta)}{\ln 1/\delta} = \frac{\ln 2}{\ln 3} \simeq 0.6309$$

□ Triadic Koch curve

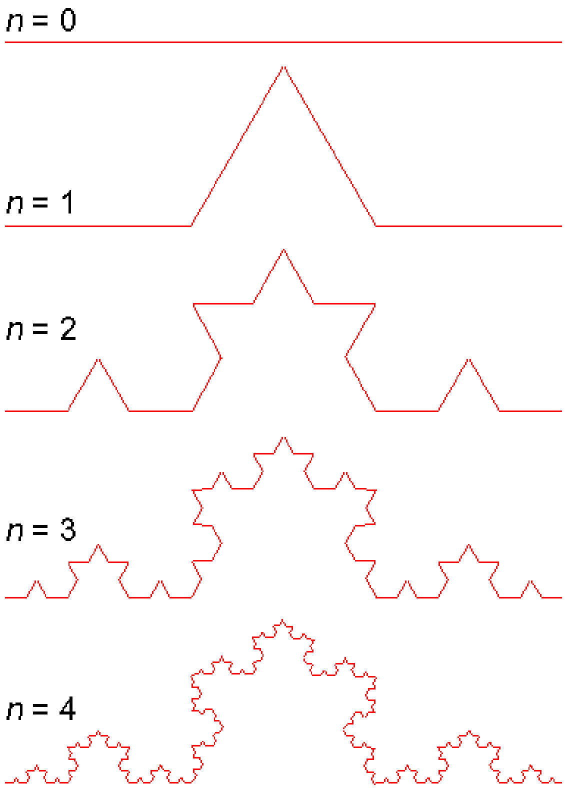
After the nth iteration, the length of a single element is:
$\delta = 3^{-n}$
The number of "cells" of size δ necessary to cover the object is:
$N(\delta) = 4^n$

Its fractal dimensionality is:

$$D = \lim_{n\to\infty} \frac{\ln N(\delta)}{\ln 1/\delta} = \frac{\ln 4}{\ln 3} \simeq 1.2628$$

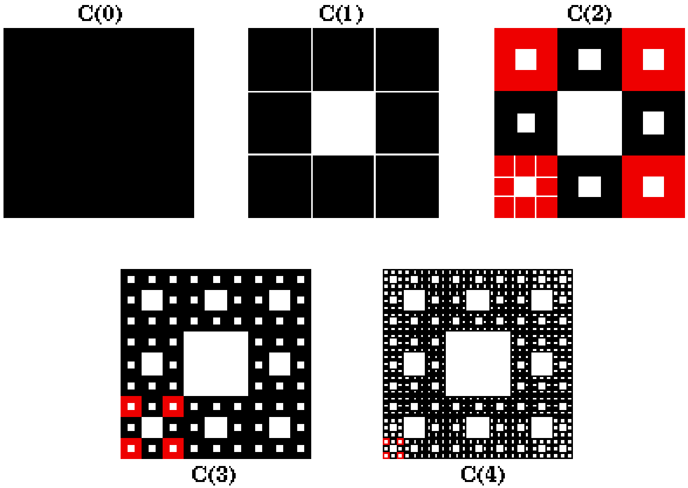
Note that the Koch curve total length tends to infinity:

$$L(\delta) = \left(\frac{4}{3}\right)^n$$

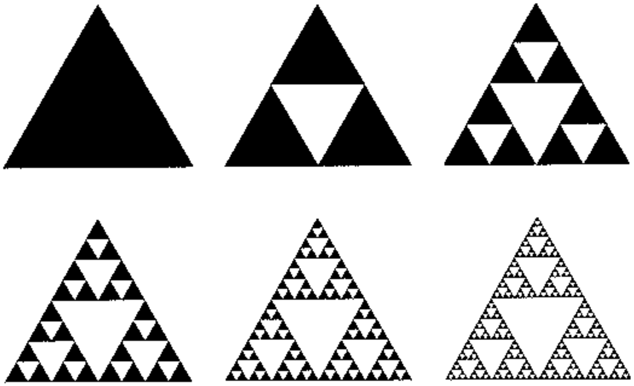


☐ Sierpinski carpet and gasket

After the nth iteration, the length of a single element is: $\delta = 3^{-n}$
The number of "cells" of size δ necessary is: $N(\delta) = 8^n$
Its fractal dimensionality is:
$D = \lim_{n\to\infty} \frac{\ln N(\delta)}{\ln 1/\delta} = \frac{\ln 8}{\ln 3} \simeq 1.89$



After the nth iteration, the length of a single element is: $\delta = 2^{-n}$
and the number of "cells" of size δ necessary is: $N(\delta) = 3^n$
Its fractal dimensionality is:
$D = \lim_{n\to\infty} \frac{\ln N(\delta)}{\ln 1/\delta} = \frac{\ln 3}{\ln 2} \simeq 1.58$



□ Definition and fractal proprieties

- The formal **definition** of a **fractal**, proposed by **Mandelbrot**, is:

  a set of points whose Hausdorff-Besicovitch *dimensionality, D, is different from the Euclidean dimension of its embedding space, d.*

- The number of points (mass) of a fractal in a sphere of radius R and volume $V \sim R^d$ is:

$$M(R) \sim R^D$$

- Given a point of the fractal in position 0 we define the *correlation function*, $C(r)$, as the probability to find an other point at a distance r. This relation holds:

$$M(R) \sim \int_0^V C(r)dV \sim \int_0^R C(r)r^{d-1}dr$$

  and from the above relations it follows that $C(r)$ must be a *power law*:

$$C(r) \sim r^{-(d-D)}$$

  ($C(r)$ is quite different in usual geometric objects)

□ Scaling laws

- Fractals are characterised by a very important feature, named *scale invariance*:

  > if you take a subset of points, S', of the fractal S, and you expand it of a scale r (i.e., its points coordinates are multiplied by a factor r) you obtain a set rS which is equal (congruent) to the starting set S.

  In such a case, the fractal is called **self similar**.

  More generally, the coordinates of a point $\vec{x} = (x_1, x_2)$ in S' could be multiplied for (e.g.) two different factors r_1, r_2 (i.e., $(x_1, x_2) \longrightarrow (r_1 x_1, r_2 x_2)$) to get a set which is congruent with S.

  In the latter case, the fractal is called **self affine**.