Problem sheet 1

1. If X is a random variable (RV) and a and b are constants, show that:

   (a) $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$

   (b) $\mathrm{VAR}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

   (c) $\mathrm{VAR}(aX + b) = a^2\mathrm{VAR}(X)$

   where $\mathbb{E}[\cdot]$ denotes the expectation of its argument and $\mathrm{VAR}(\cdot)$ the variance.

2. (*Conditional sum and product rules*) For three RVs X, Y and Z show that:

   (a) $P(Y \mid Z) = \sum_{X \in \mathcal{X}} P(X, Y \mid Z)$ (where $\mathcal{X}$ is the range of X)

   (b) $P(X, Y \mid Z) = P(X \mid Y, Z)P(Y \mid Z)$

3. *Bayes' rule* is often written in the form

   $$P(X \mid Y) = \frac{P(Y \mid X)P(X)}{P(Y)}$$

   Show that Bayes' rule can also be written as

   $$P(X \mid Y) = \frac{P(Y \mid X)P(X)}{\sum_{X \in \mathcal{X}} P(Y \mid X)P(X)}$$

   where $\mathcal{X}$ is the range of X.

4. Let $X_1, X_2 \ldots X_n$ be any set of RVs. Show that:

   $$\begin{aligned} &P(X_1, X_2 \ldots X_n) \\ &\quad = P(X_1)P(X_2 \mid X_1)P(X_3 \mid X_1, X_2) \ldots P(X_n \mid X_1, X_2 \ldots X_{n-1}) \end{aligned}$$

   where $P(X_1, X_2 \ldots X_n)$ is the joint distribution over all n RVs and $P(\cdot \mid \cdot)$ are conditionals.
   (Hint: use the product rule)

5. Bill works at the marketing firm *Arbitrary Promotions Plc*. One morning, his boss calls him into her office and invites him to pick one from a set of three small boxes lying on her desk. Exactly one of the three boxes contains

the key to the executive suite: if Bill ends up with the correct box, he will win a promotion. Bill picks a box, say #1, and his boss (who knows which box contains the key) opens *another* box, say #3, and reveals it to be empty. She then offers Bill the chance to change his choice to box #2. Use Bayes' rule to decide whether or not Bill should accept the offer and switch. (Note that an intuitive answer alone is *not* sufficient: you should work from first principles to obtain your result, making clear any assumptions you are making along the way.) (This is of course the Monte Hall problem, but you must be able to formally show the correct result).