Complexity Science Doctoral Training Centre

CO903 Complexity and Chaos in Dynamical Systems

3 One-dimensional maps

Here we study a class of dynamical systems in which time is *discrete* rather than continuous (i.e. difference equations or iterated maps).
Consider a one-dimensional map

$$x_{n+1} = f(x_n),$$

where f is a smooth function from the real line to itself. The sequence $x_0,\ x_1,\ x_2, ...$ is called the orbit starting from x_0. Maps are useful in various ways:

- Tools for analysing differential equations (e.g., Poincaré maps, the Lorenz map).
- Models of natural phenomena (where discrete time is better to be considered, e.g., digitals electronics, in parts of economics and finance theory).
- Simple examples of chaos (Maps show a much wilder behaviour than differential equations).

Fixed points and linear stability

If $f(x^*) = x^*$, then x^* is a fixed point. The orbit remains at x^* for all future iterations ($x_n = x^* \Rightarrow x_{n+1} = f(x_n) = f(x^*) = x^*$).

To determine the stability of x^*, we consider a nearby orbit $x_n = x^* + \eta_n$. Then we have

$$x^* + \eta_{n+1} = f(x^* + \eta_n) = f(x^*) + f'(x^*)\eta_n + O(\eta_n^2).$$

This equation reduces to the equation of the linearised map

$$\eta_{n+1} = f'(x^*)\eta_n$$

with **multiplier** $\lambda = f'(x^*)$. The solution of the linear map can be found explicitly by writing a few terms: $\eta_1 = \lambda\eta_0$, $\eta_2 = \lambda\eta_1 = \lambda^2\eta_0$, ... , $\eta_n = \lambda^n\eta_0$.

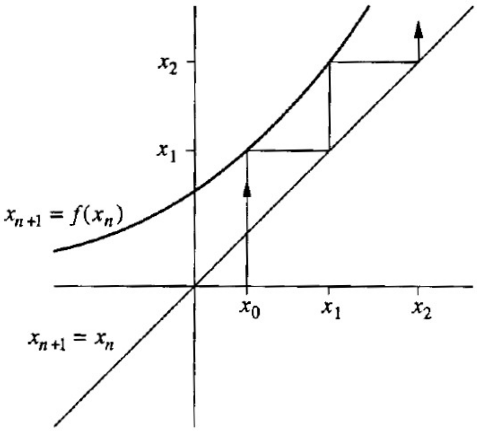
If $|\lambda| = |f'(x^*)| < 1$, $\eta_n \to 0$ as $n \to \infty \Rightarrow x^*$ is **linearly stable**

If $|\lambda| > 1 \Rightarrow x^*$ is **unstable**

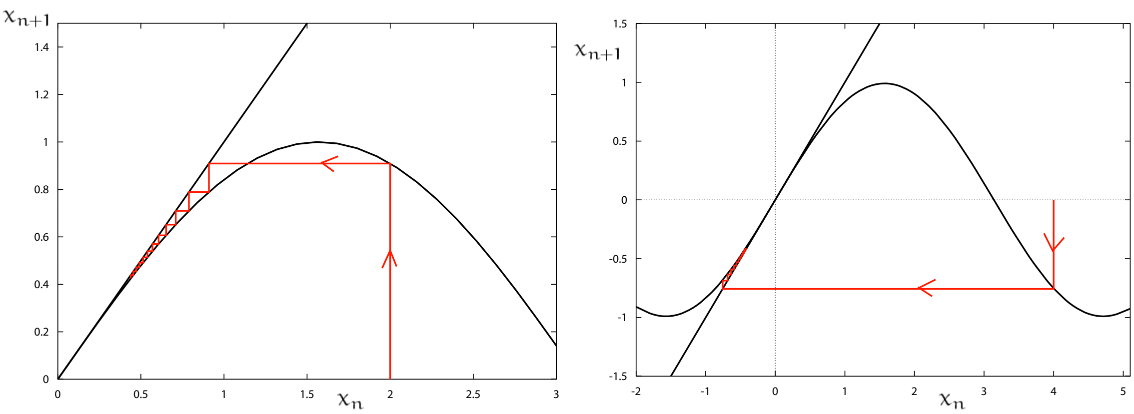
If $|\lambda| = 1 \Rightarrow$ **marginal** case (the neglected $O(\eta_n^2)$ terms determine the local stability)

Fixed points with multiplier $\lambda = 0$ are called **superstable** (perturbations decay much faster)

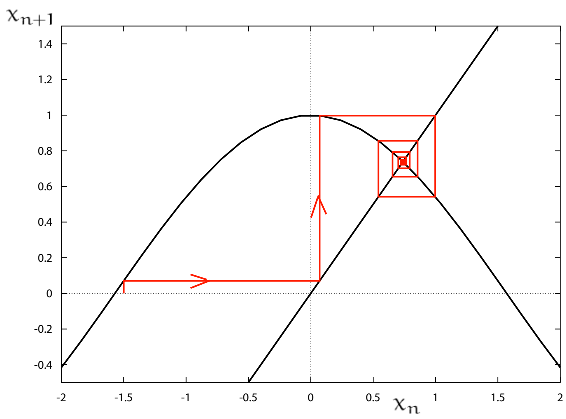
Cobwebs



Example 1. A cobweb for the map $x_{n+1} = \sin(x_n)$ helps to show that $x^* = 0$ is globally stable



Example 2. Given $x_{n+1} = \cos(x_n)$ we can show that a typical orbit spirals into the fixed point $x^* = 0.739...$ as $n \to \infty$ ($x = 0.739...$ is the unique solution of $x = \cos(x)$).



The spiral motion implies that x_n converges to x^* through damped oscillations (typically if $\lambda < 0$). If $\lambda > 0$ the convergence is monotonic.

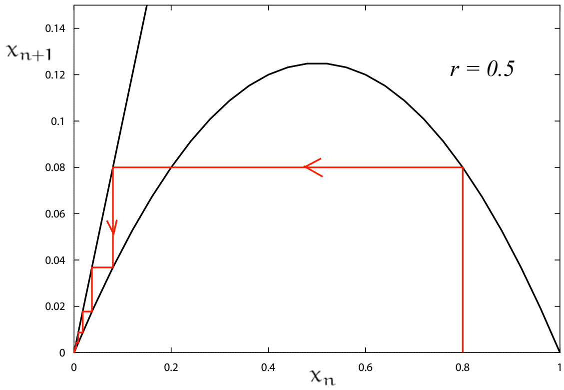
Logistic map

Consider the **logistic map**

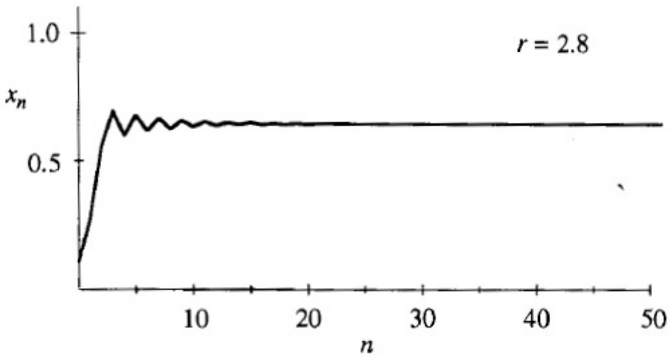
$$x_{n+1} = rx_n(1 - x_n),$$

a discrete time analog of the logistic equation for population growth studied earlier. $x_n \geq 0$ is a dimensionless measure of the population in the nth generation and $r \geq 0$ is the intrinsic growth rate. The graph of the logistic map is a parabola with a maximum value of $r/4$ at $x = 0.5$. Here we restrict the control parameter $0 \leq r \leq 4$ so that the equation maps the interval $0 \leq x \leq 1$ into itself.

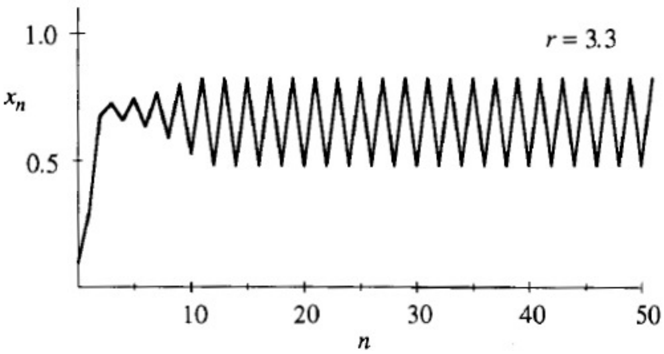
If $r < 1$, $x_n \to 0$ as $n \to \infty$



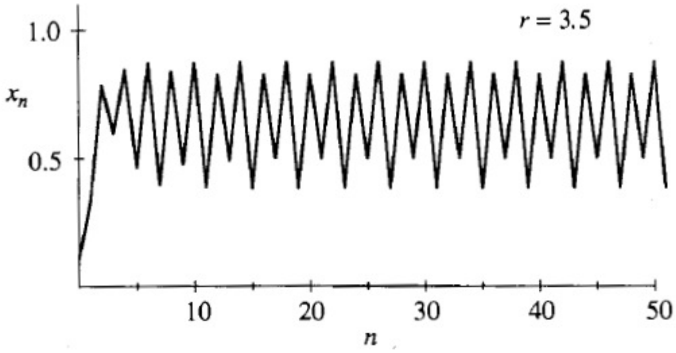
If $1 < r < 3$ the population grows and eventually reaches a nonzero steady state:



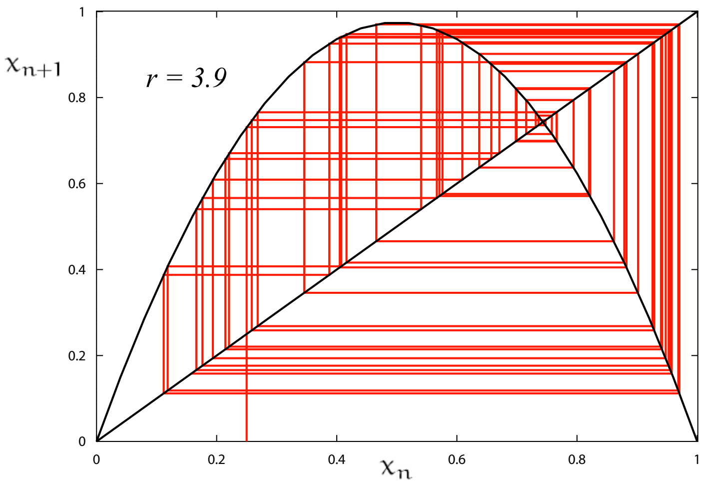
For larger r we observe oscillations in which x_n repeats every two iterations, i.e. a **period-2 cycle**:



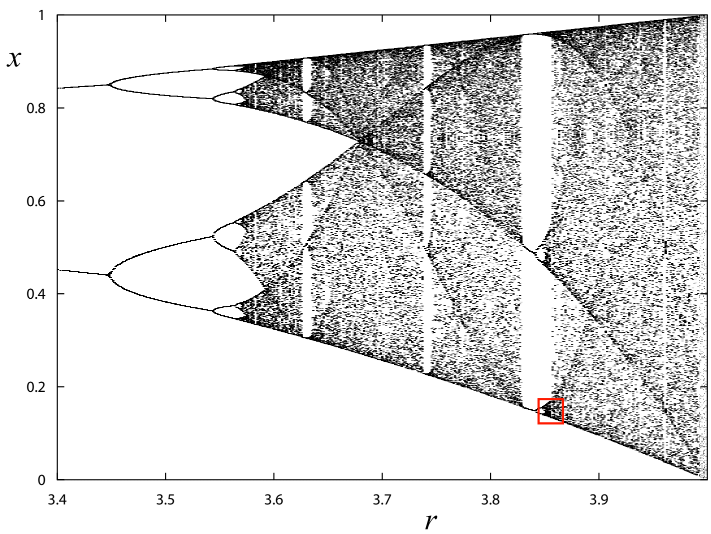
At still larger r, a cycle repeats every four generations, i.e. a **period-4 cycle**:



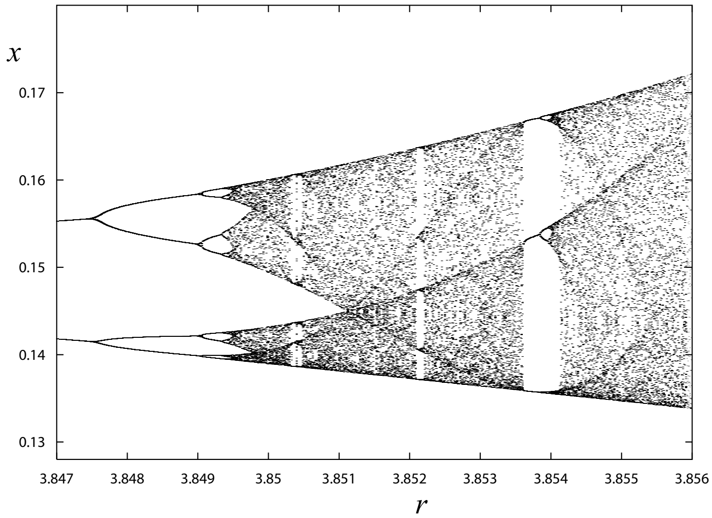
For many values of r, the sequence x_n never settles down to a fixed point or a periodic orbit, i.e. the long-term behaviour is aperiodic



To see the long-term behaviour for all values of r at once, we can plot the **orbit diagram** (the system's attractor as a function of r).



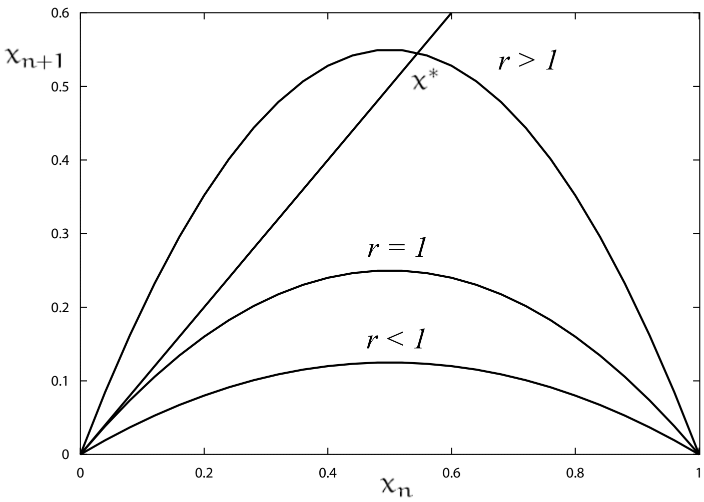
We observe a cascade of period-doublings until at $r \approx 3.57$, where the map becomes chaotic. For $r > 3.57$ the orbit diagram reveals a mixture of order and chaos. The large periodic window beginning near $r \approx 3.83$ contains a stable period-3 cycle. A blow-up of part of the period-3 window is shown below (a copy of the orbit diagram reappears in miniature):



Some analysis of logistic map

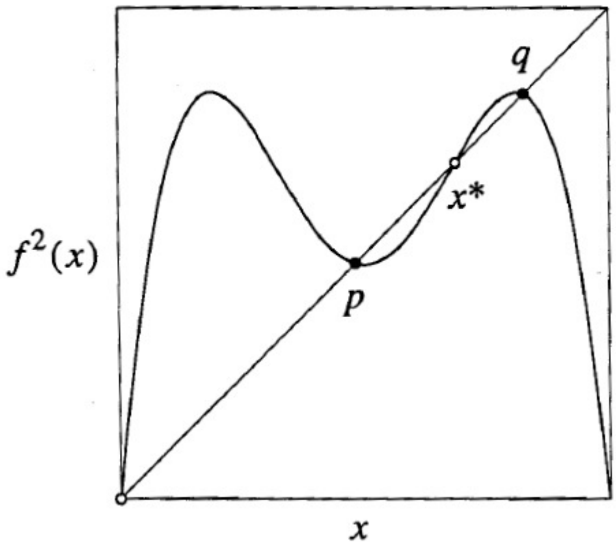
The fixed points satisfy $x^* = f(x^*) = rx^*(1 - x^*)$. Hence $x^* = 0$ for all r and $x^* = 1 - 1/r$ for $r \geq 1$ (from the condition $0 \leq x^* \leq 1$). Stability depends on multiplier $f'(x^*) = r - 2rx^*$.

- $f'(0) = r \Rightarrow x^*$ - stable if $r < 1$ and unstable if $r > 1$
- $f'(1 - 1/r) = 2 - r \Rightarrow x^* = 1 - 1/r$ is stable if $|2 - r| < 1$, i.e. $1 < r < 3$ and unstable if $r > 3$



x^* bifurcates from the origin in a transcritical bifurcation at $r = 1$. As r increases beyond 1, the slope at x^* gets steeper. The critical slope $f'(x^*) = -1$ is attained when $r = 3$. The resulting bifurcation is called a **flip bifurcation** (often associated with period-doubling).

Here we will show that the logistic map has a 2-cycle for $r > 3$. A 2-cycle exists if and only if there are two points p and q such that $f(p) = q$ and $f(q) = p$. Equivalently, such a p must satisfy $f(f(p)) = p \Rightarrow p$ is a fixed point of the second-iterate map $f^2(x) \equiv f(f(x))$.



To find p and q we have to solve $f^2(x) = x$, i.e. $r^2x(1-x)[1-rx(1-x)] - x = 0$. Since the fixed points $x^* = 0$ and $x^* = 1 - 1/r$ are solutions of this equation we can reduce the equation to a quadratic one by factoring out the fixed points. Solving the resulting quadratic equation we get

$$p,\ q = \frac{r + 1 \pm \sqrt{(r-3)(r+1)}}{2r}.$$

For $r > 3$ the roots p and q are real and we have a 2-cycle. For $r < 3$ the roots are complex and a 2-cycle doesn't exist.

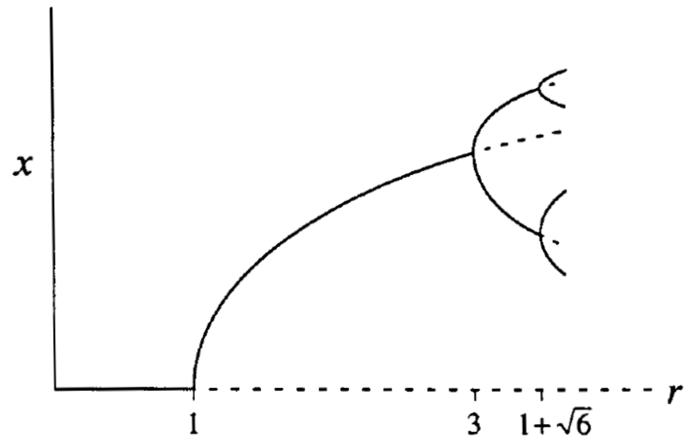
For analysing the stability of a cycle we can reduce the problem to a question about the stability of a fixed point. Both p and q are solutions of $f^2(x) = x \Rightarrow p$ and q are fixed points of the second-iterate map $f^2(x)$. The original 2-cycle is stable if p and q are stable fixed points. To determine whether p is a stable fixed point of f^2 we compute the multiplier

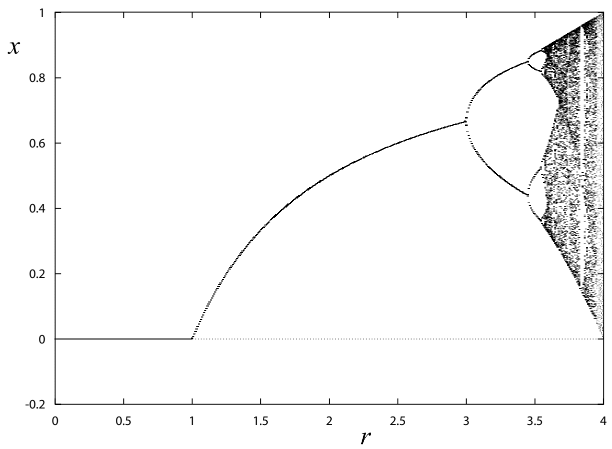
$$\lambda = \frac{d}{dx}(f(f(x)))\,|_{x=p} = f'(f(p))f'(p) = f'(q)f'(p).$$

The multiplier is the same at $x = q$. After carrying out the differentiations and substituting for p and q we obtain

$$\lambda = r(1-2q)r(1-2p) = 4 + 2r - r^2.$$

The 2-cycle is linearly stable if $|4 + 2r - r^2| < 1$, i.e. for $3 < r < 1 + \sqrt{6}$.





Lyapunov exponent

To be called *chaotic*, a system should also show sensitive dependence on initial conditions, in the sense that neighbouring orbits separate exponentially fast. The definition of the Lyapunov exponent for a chaotic differential equation can be extended to one-dimensional maps.

Given some initial condition x_0, consider a nearby point $x_0 + \delta_0$, where $\delta_0 \ll 1$. Let δ_n be the separation after n iterates. If $|\delta_n| \approx |\delta_0| e^{n\lambda}$, then λ is called the Lyapunov exponent. A positive Lyapunov exponent is a signature of chaos.

A more precise and computationally useful formula for λ can be derived. We note that $\delta_n = f^n(x_0 + \delta_0) - f^n(x_0)$. Then by taking logarithms

$$\lambda \approx \frac{1}{n} \ln \left| \frac{\delta_n}{\delta_0} \right| = \frac{1}{n} \ln \left| \frac{f^n(x_0 + \delta_0) - f^n(x_0)}{\delta_0} \right| = \frac{1}{n} \ln |(f^n)'(x_0)|$$

in the limit $\delta_0 \to 0$. Using the chain rule we have

$$(f^n)'(x_0) = \prod_{i=0}^{n-1} f'(x_i)$$

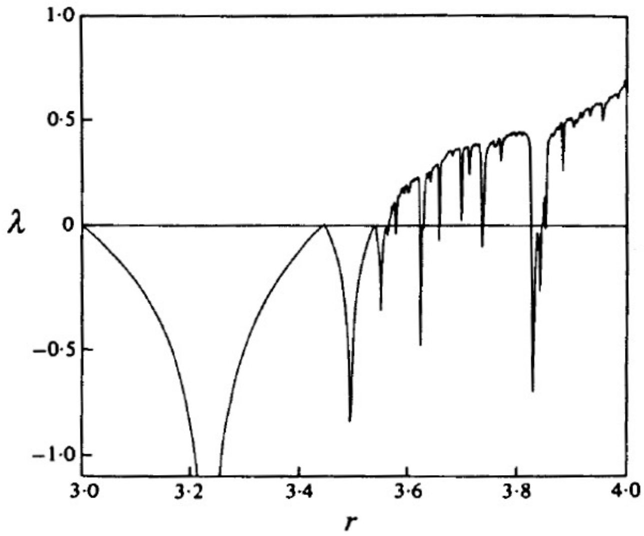
and

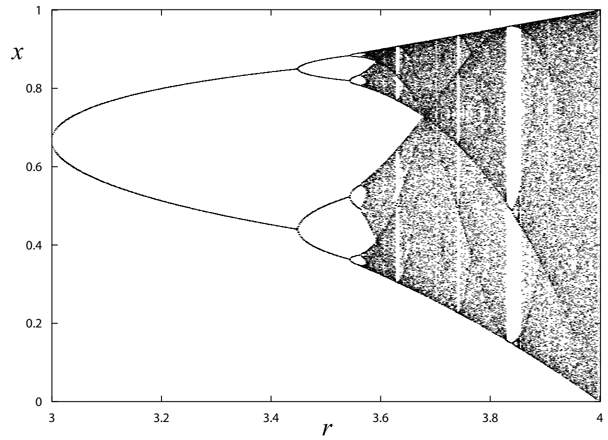
$$\lambda \approx \frac{1}{n} \ln \left| \prod_{i=0}^{n-1} f'(x_i) \right| = \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)| \,.$$

Then the **Lyapunov exponent** for the orbit starting at x_0 is defined as

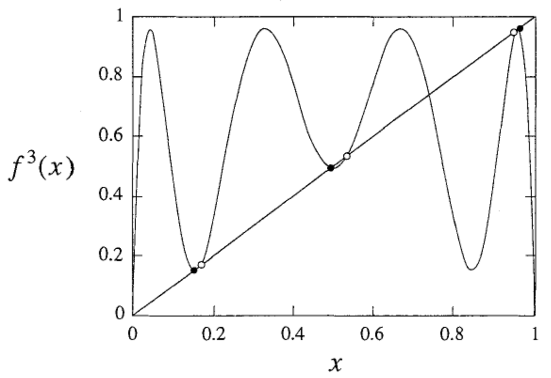
$$\boxed{\lambda = \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=0}^{n-1}\ln|f'(x_i)|\right]}$$

The Lyapunov exponent for the logistic map found numerically:



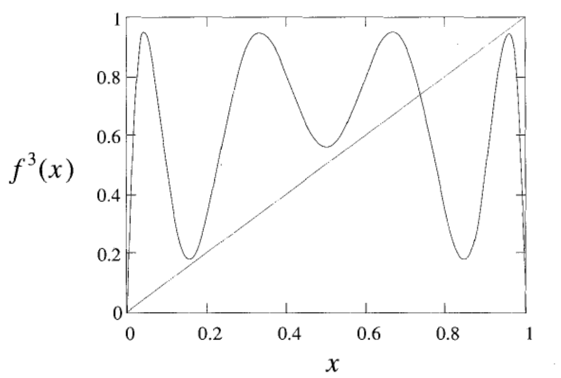


The bifurcation diagram of the logistic map $x_{n+1} = rx_n(1 - x_n)$ demonstrates the presence of the period-3 window near $3.8284... \le r \le 3.8415....$ The third-iterate map $f^3(x)$ is the key to understand the birth of the period-3 cycle (note that the notation $f^3(x)$ here means $x_{n+3} = f^3(x_n)$). Any point p in a period-3 cycle repeats every three iterates, so such points satisfy $p = f^3(p)$, and are therefore fixed points of the third-iterate map. Consider $f^3(x)$ for $r = 3.835$:



The black dots correspond to a stable period-3 cycle (can see by the slope) and the open dots correspond to an unstable 3-cycle (the slope exceeds 1).

If we decrease r the graph changes shape and the marked intersections have vanished (see the figure for $r = 3.8$):



At some critical r the graph $f^3(x)$ must have become tangent to the diagonal (the stable and unstable period-3 cycle coalesce and annihilate in a **tangent bifurcation**).