Chaos

Lorenz equations

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

$\sigma > 0$ **Prandtl number** (the ratio of momentum diffusivity to thermal diffusivity)

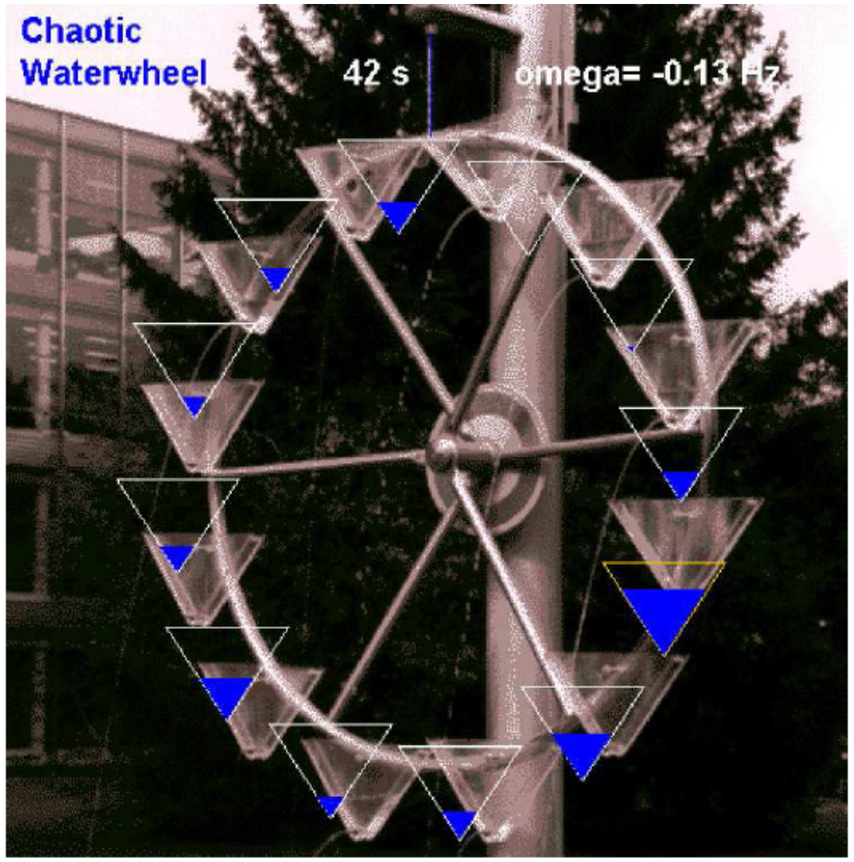
$r > 0$ **Rayleigh number** (associated with the heat transfer within the fluid)

$b > 0$

$\sigma = 10$

$b = 8/3$

A chaotic waterwheel



x proportional to the intensity of convective motion

y temperature difference between the ascending and descending currents

z distortion of the vertical temperature from linearity

Properties of Lorenz equations

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

Nonlinearity

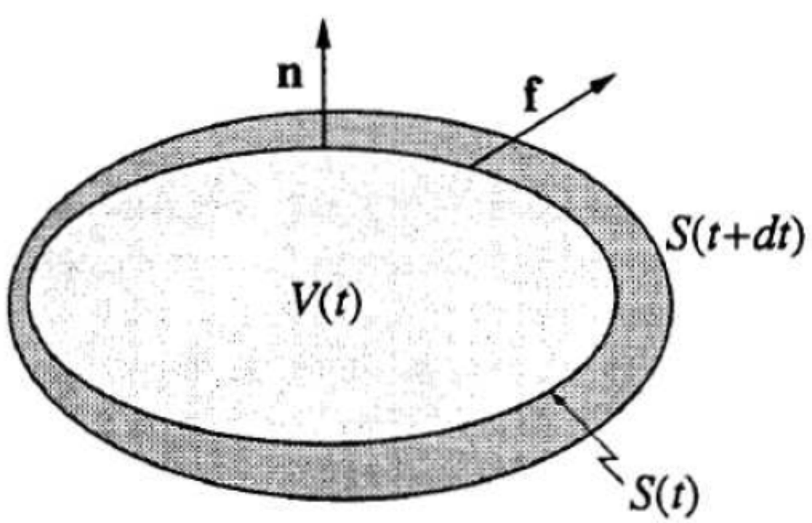
Symmetry $$(x(t), y(t), z(t)) \to (-x(t), -y(t), z(t))$$

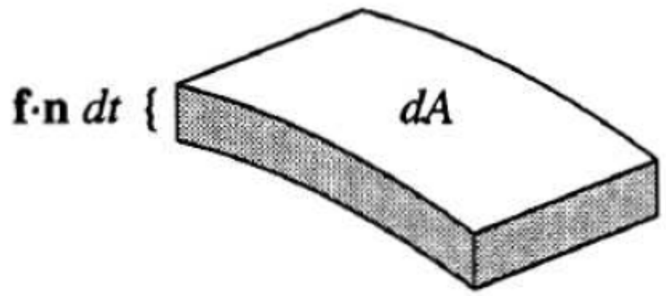
Volume contraction

The Lorenz system is **dissipative**: volumes in phase-space contract under the flow

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^3$$

$f \cdot n$ outward normal component of velocity





$$\dot{V} = \int_V \nabla \cdot f \, dV$$

$$\dot{V} = -(\sigma + 1 + b)V \qquad V(t) = V(0)e^{-(\sigma+1+b)t}$$

Fixed points $(x, y, z) = (0, 0, 0)$

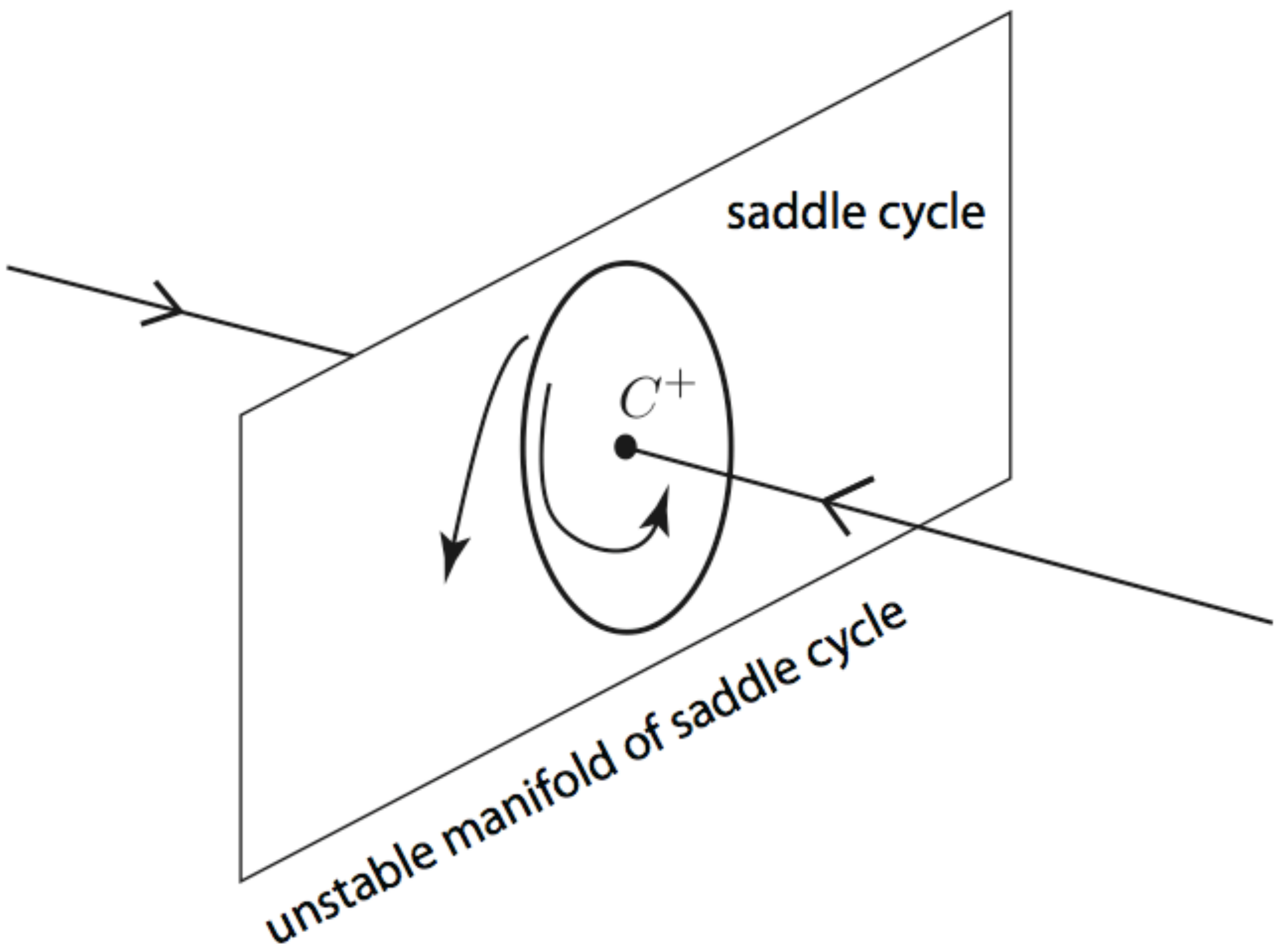
$$C^+ \equiv (x = y = \sqrt{b(r-1)}, z = r - 1)$$

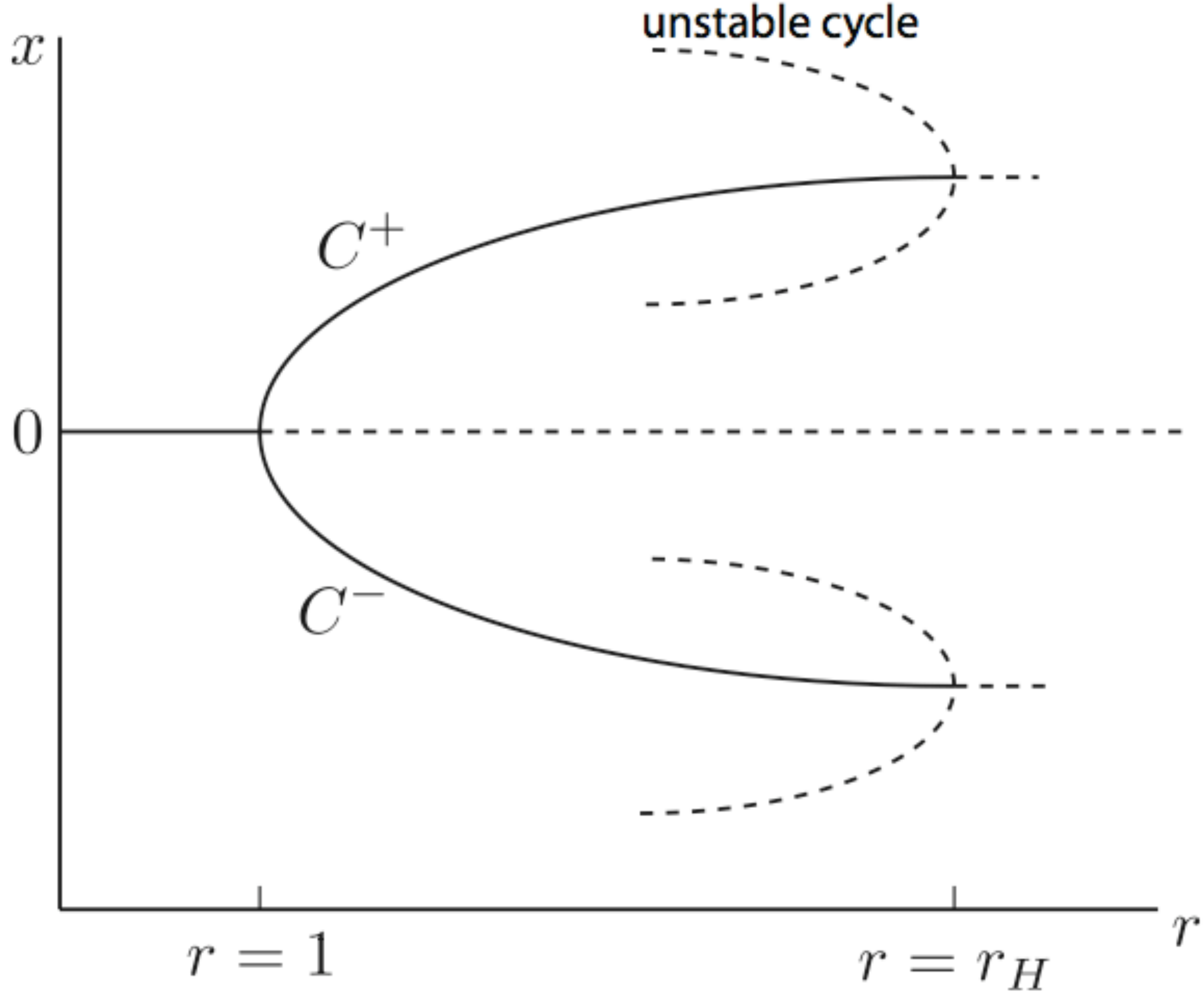
$$C^- \equiv (x = y = -\sqrt{b(r-1)}, z = r - 1)$$

Linear stability of the origin

Global stability of the origin

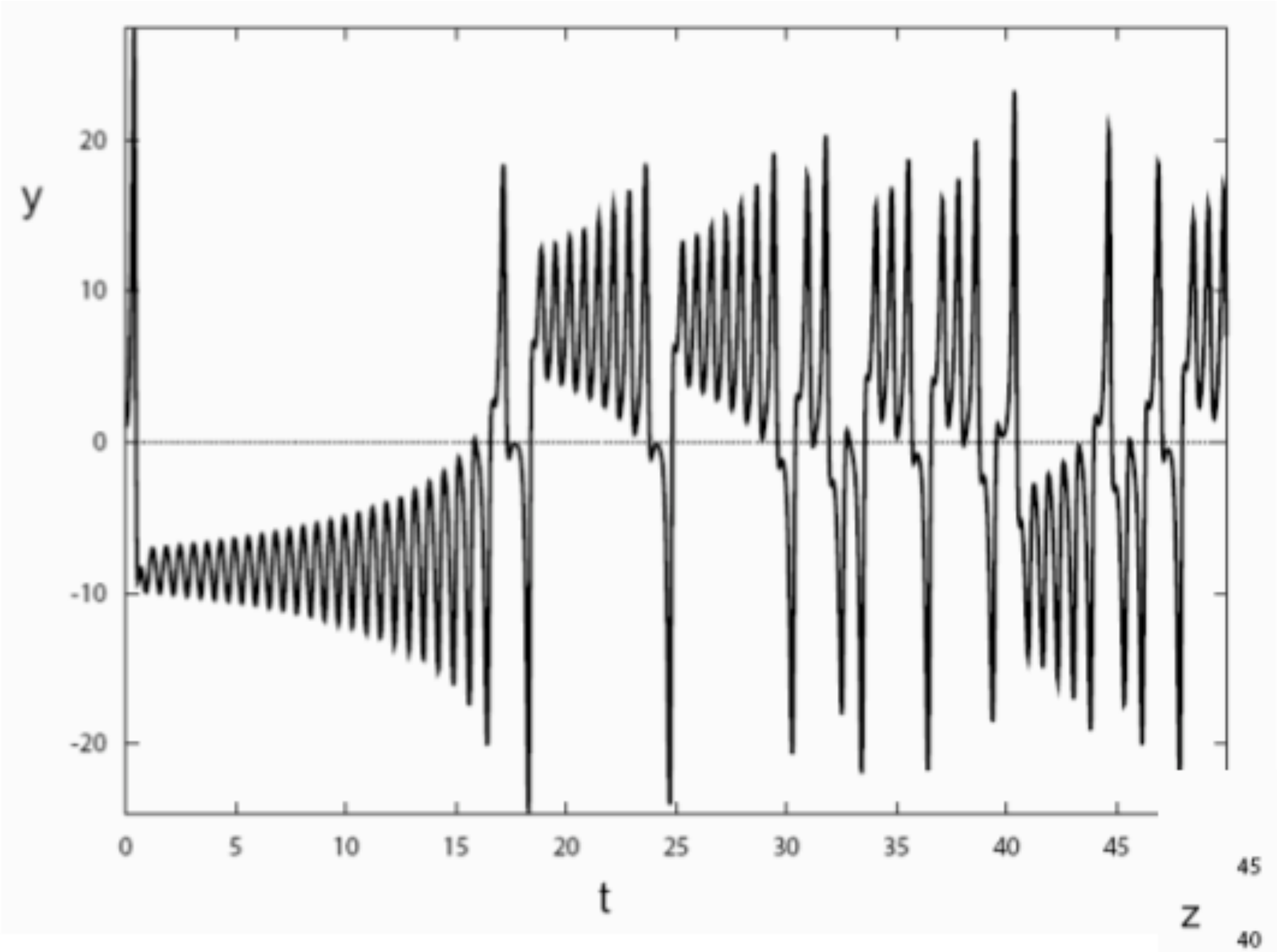
Stability of $C^+, \ C^-$

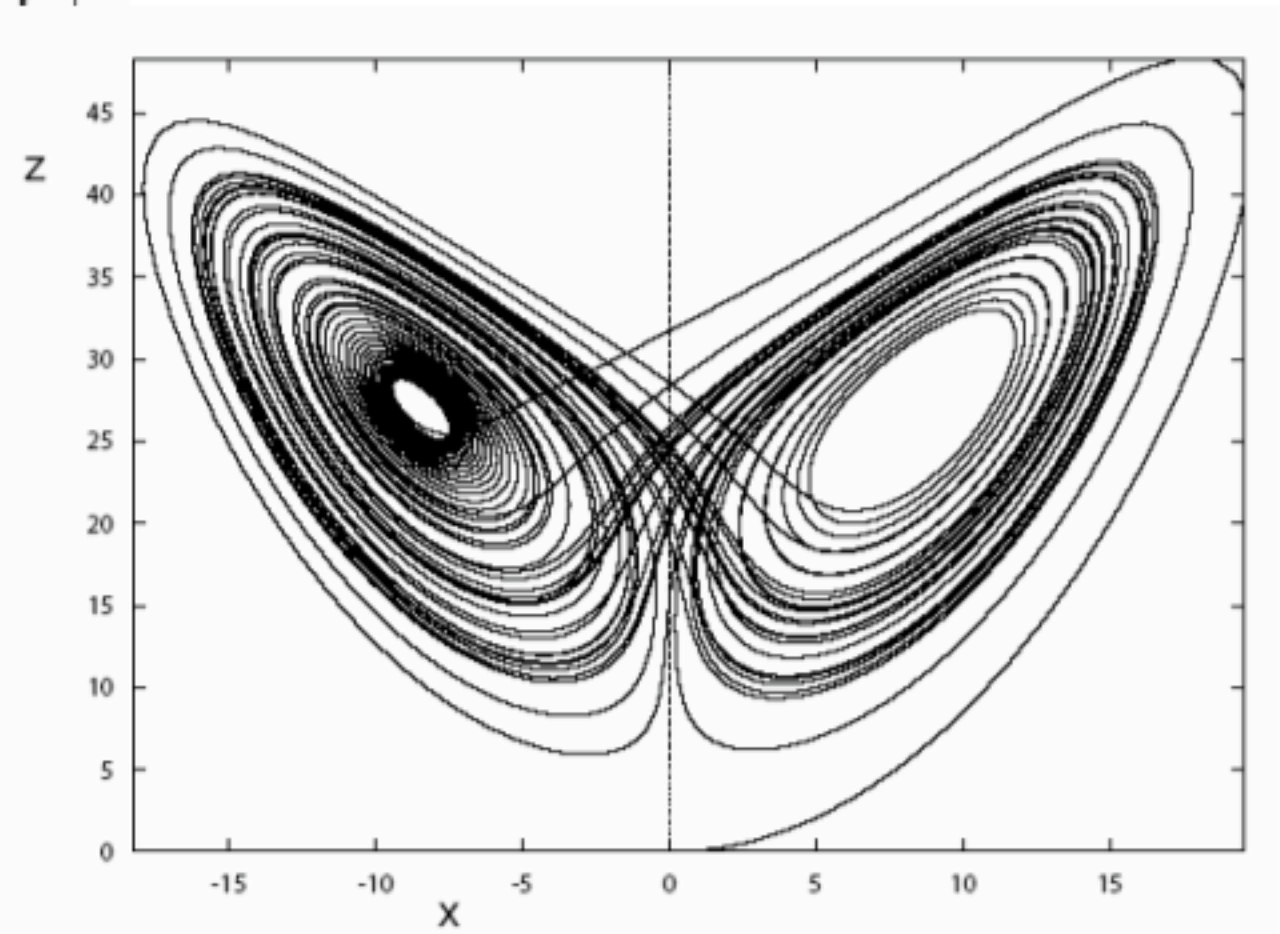




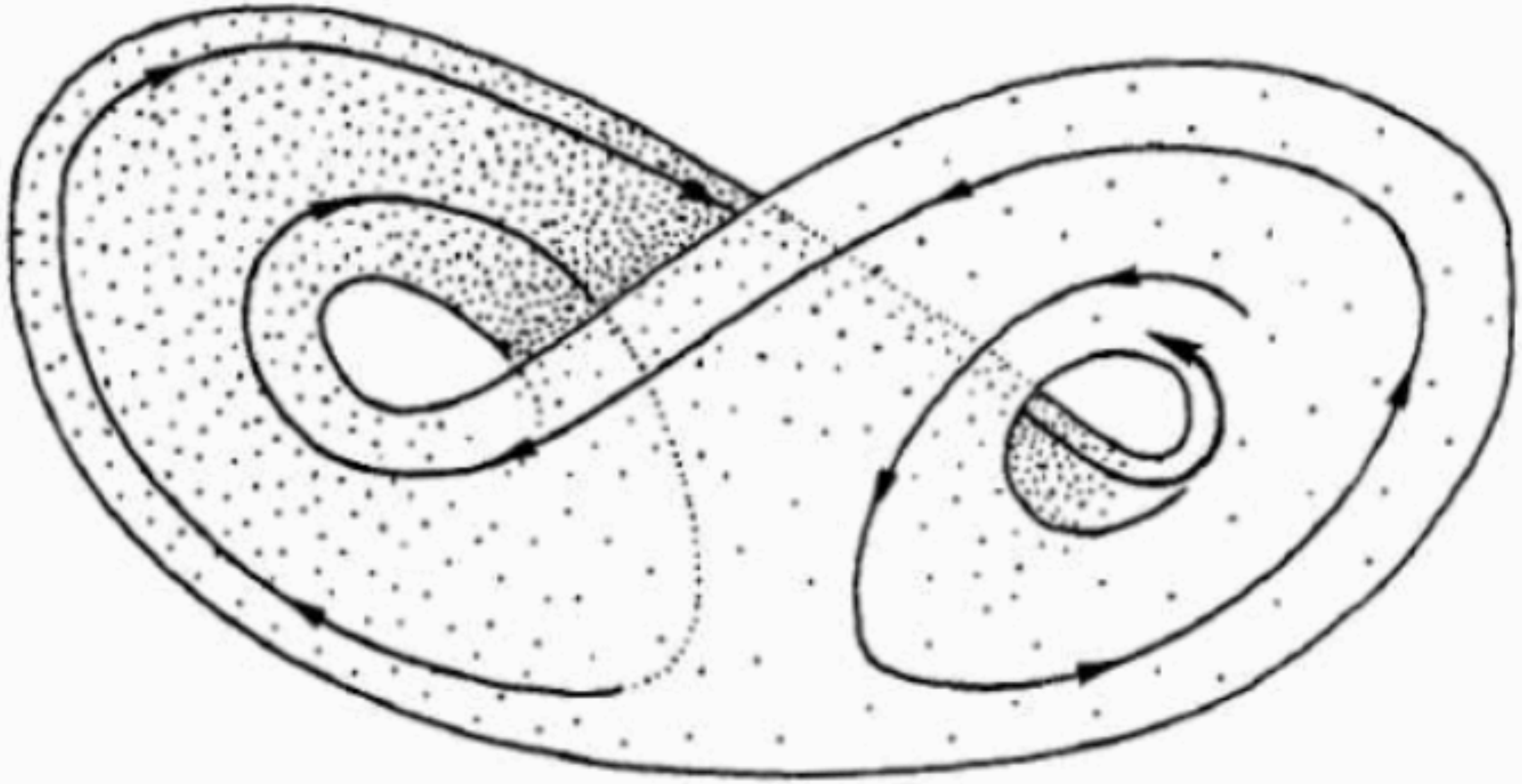
$r_H \approx 24.74$

Chaos on a strange attractor



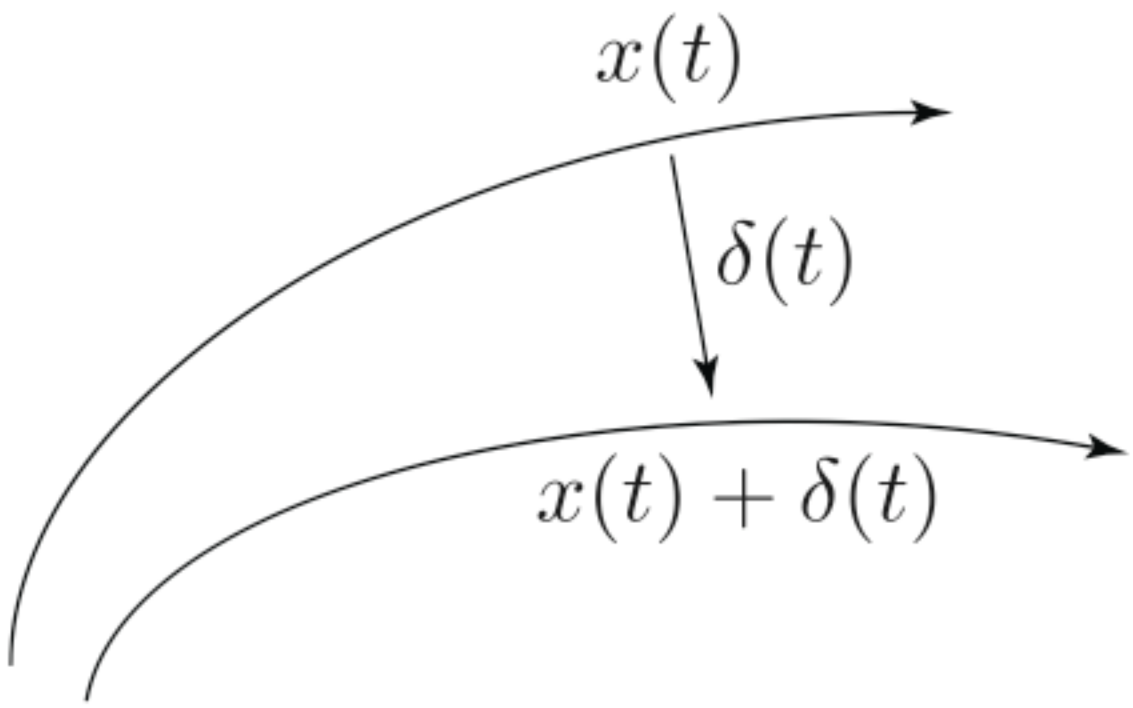


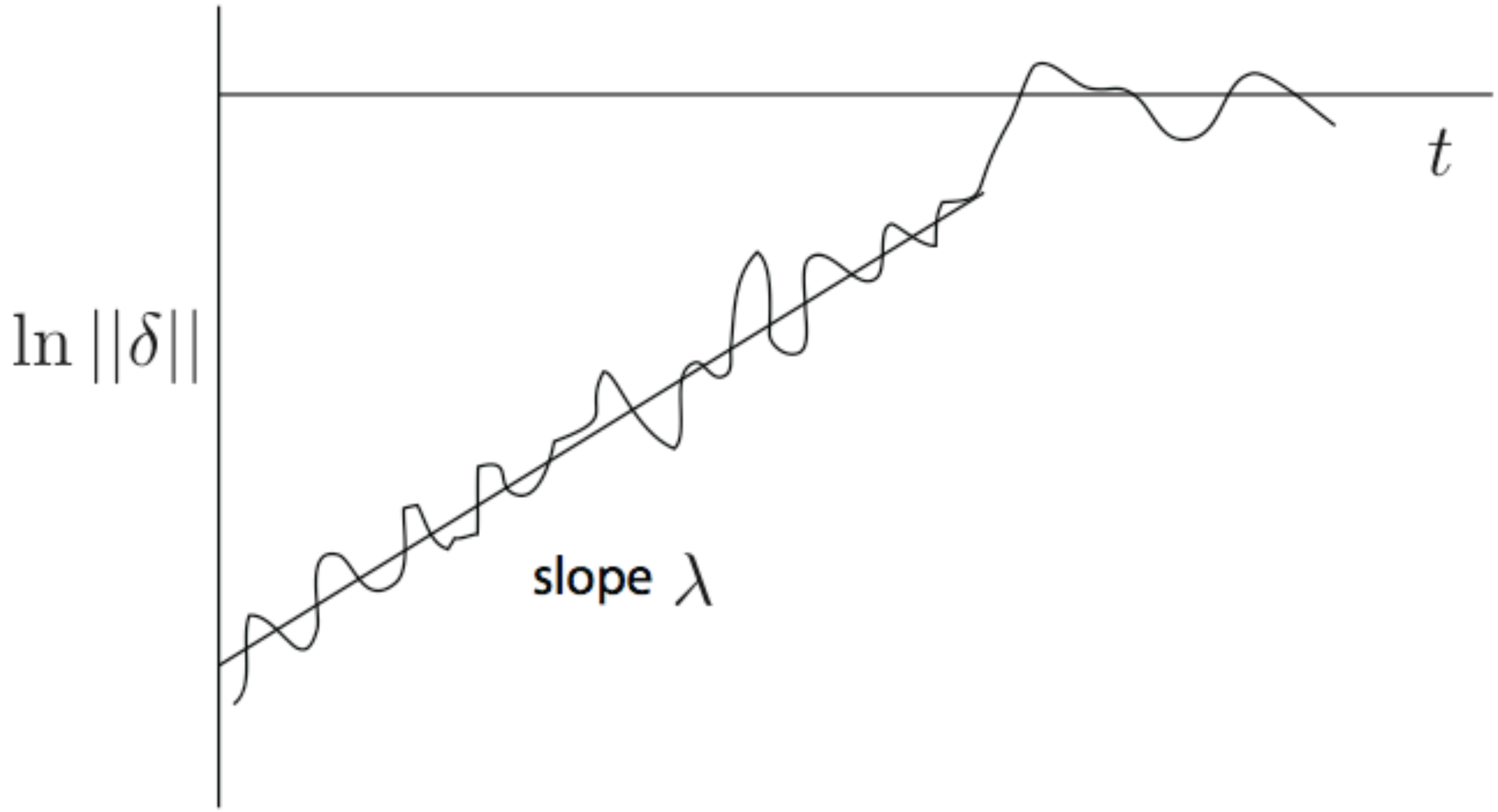
$$r > r_H$$



A schematic picture of the strange attractor in the Lorenz system

Exponential divergence of nearby trajectories





Defining Chaos

Chaos is aperiodic long-term behaviour in a deterministic system that exhibits sensitive dependence on initial condition

1. *Aperiodic long-term behaviour* means that there are trajectories which do not settle down to fixed points, periodic orbits, or quasi-periodic orbits as $t \to \infty$.

2. *Deterministic* means no noise.

3. *Sensitive dependence on initial conditions* means that nearby trajectories separate exponentially fast.

Defining Attractor and Strange Attractor

An attractor Λ is a closed set A with the following properties

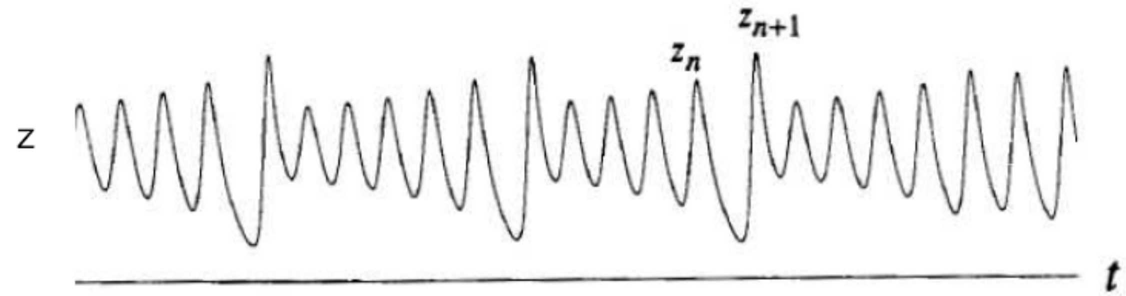
1. A is an invariant set: any trajectory $x(t)$ that starts in A stays in A for all time.

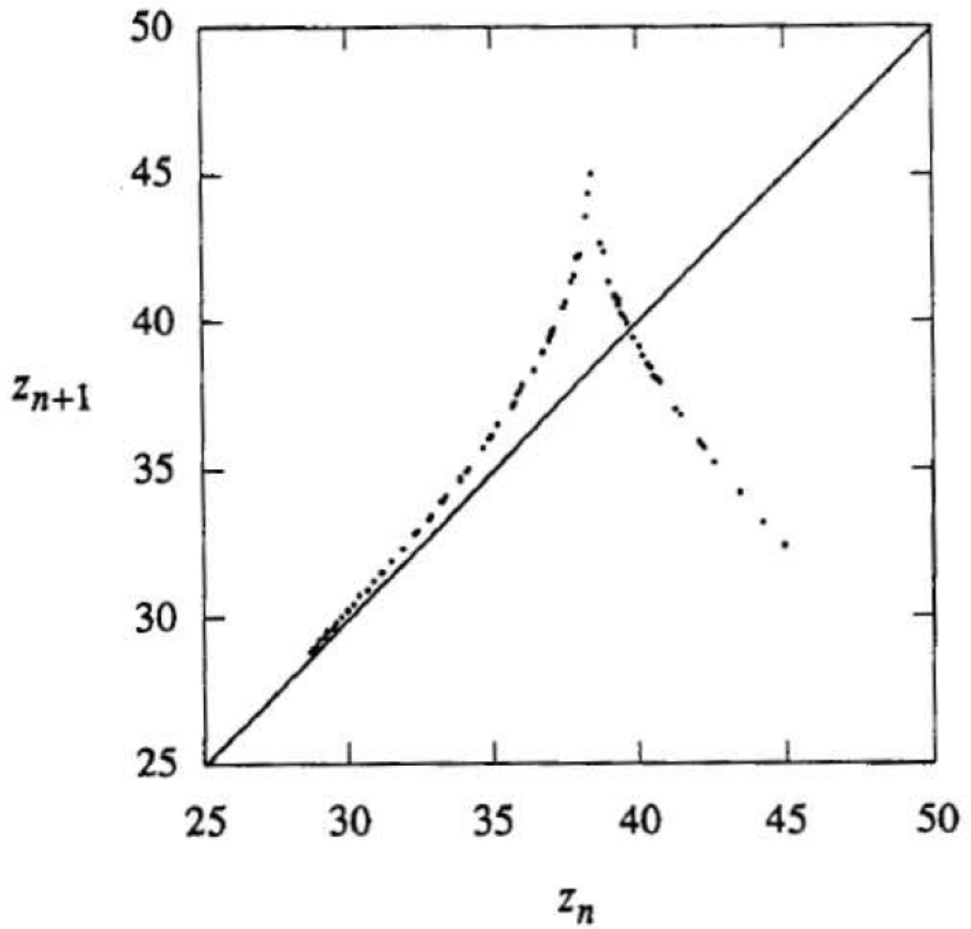
2. A *attracts an open set of initial conditions*: there is an open set U containing A such that if $x(0) \in U$, then the distance from $x(t)$ to A tends to zero as $t \to \infty$. This means that A attracts all trajectories that start sufficiently close to it. The largest such U is called the *basin of attraction* of A.

3. A *is minimal*: there is no proper subset of A that satisfies conditions 1 and 2.

Strange attractor: an attractor that exhibits sensitive dependence on initial conditions (i.e. positive Lyapunov exponents). Trajectories are called chaotic if at least one Lyapunov exponent is positive.

Lorenz map

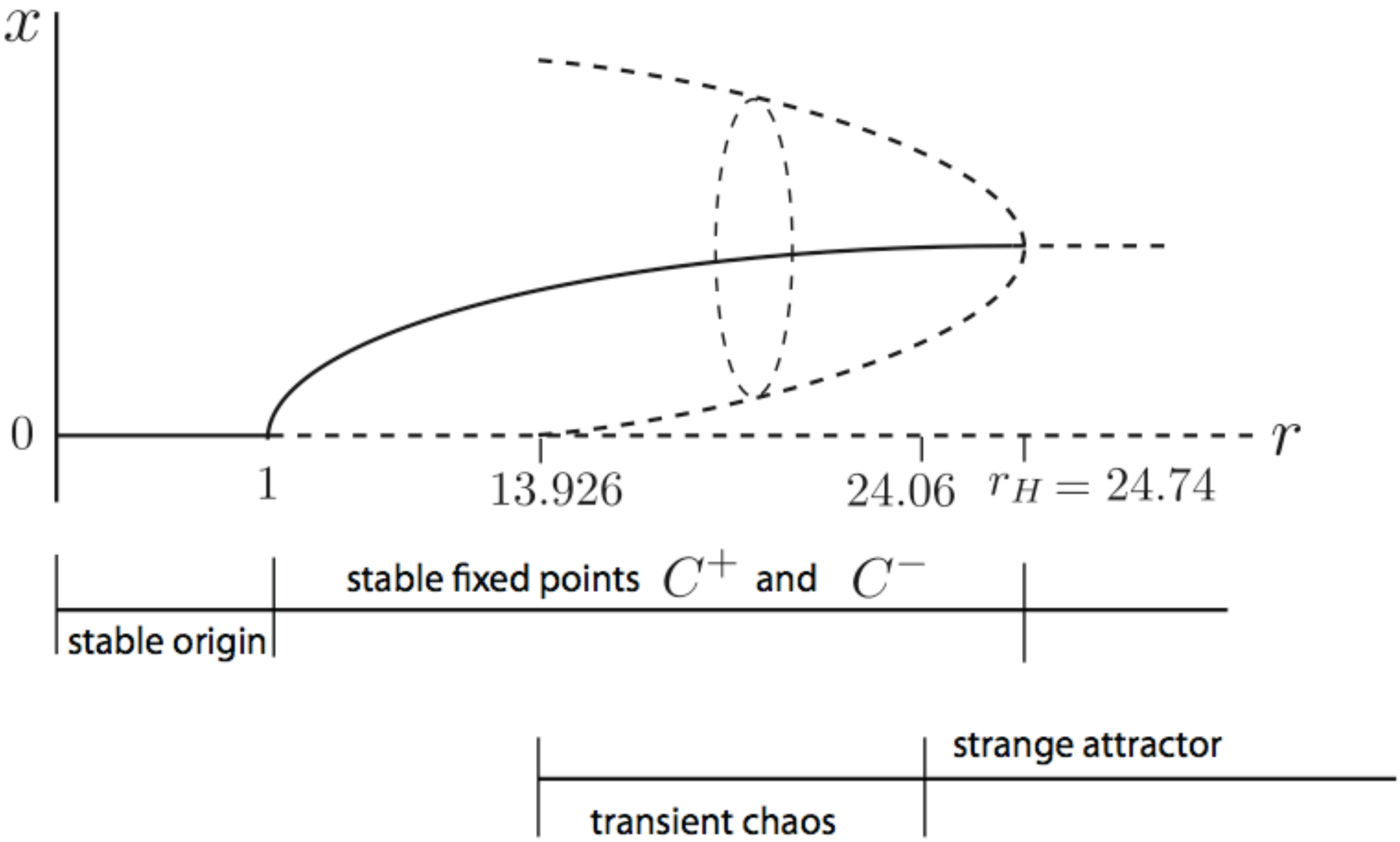


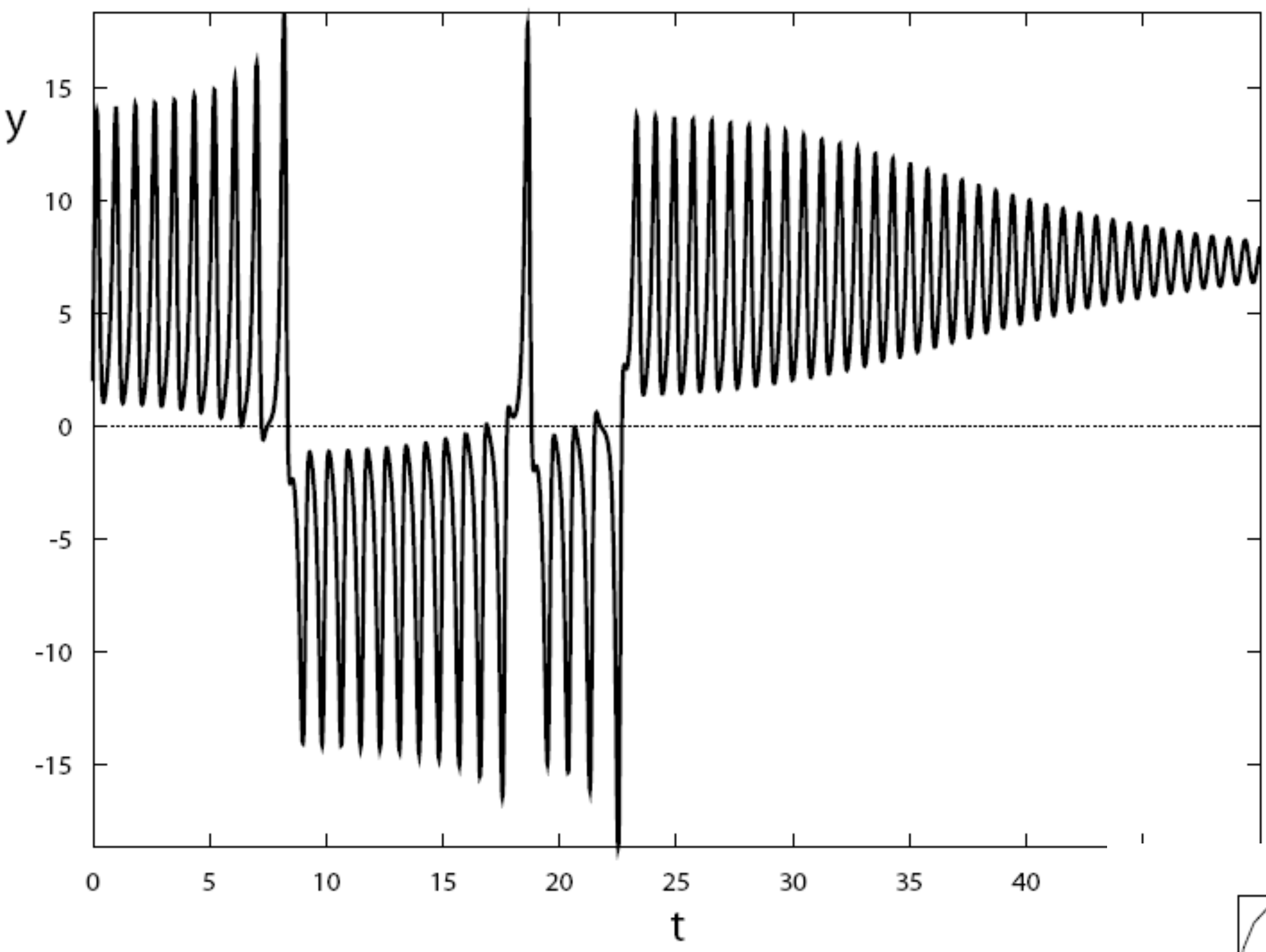


$$z_{n+1} = f(z_n)$$

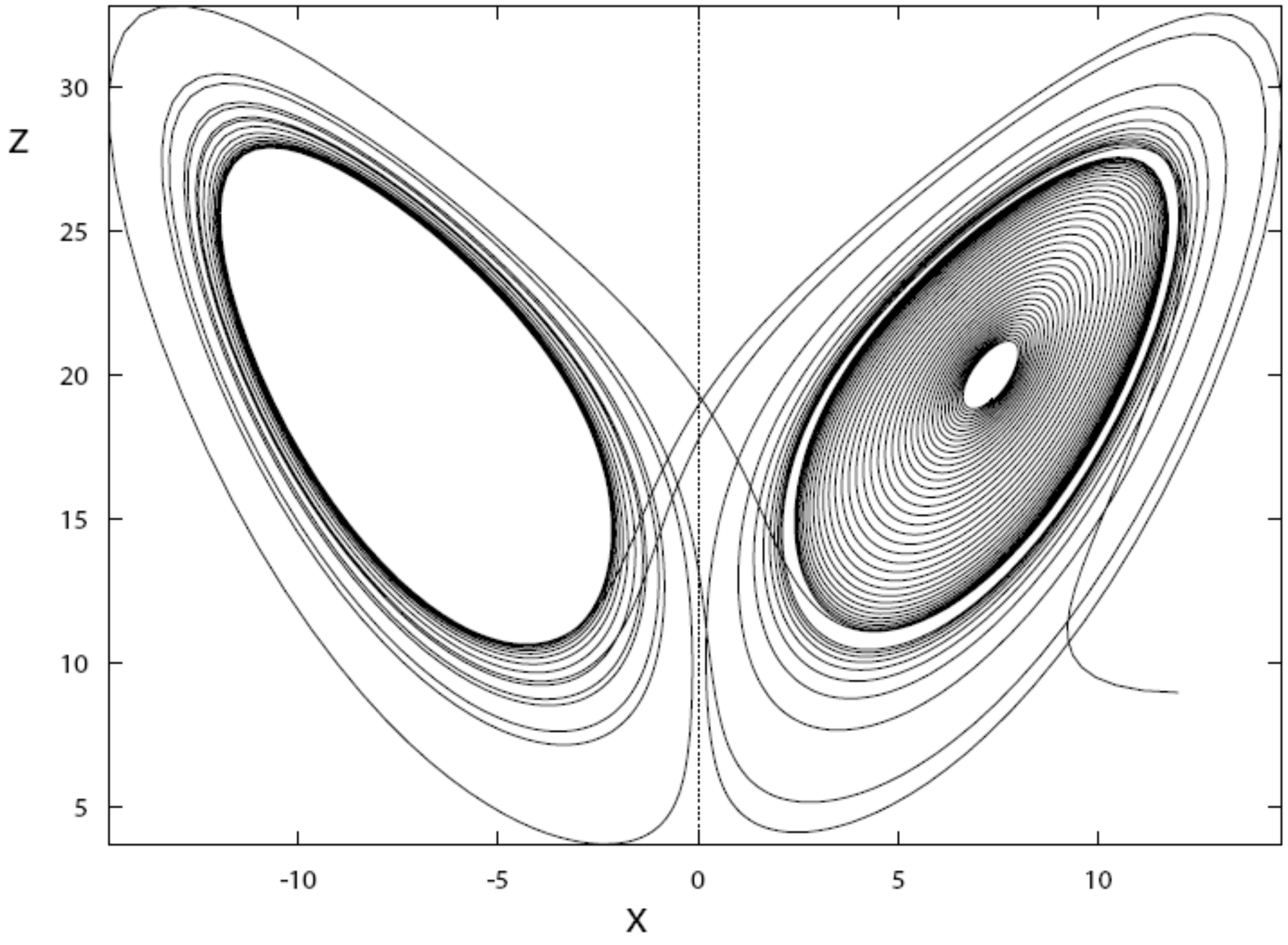
can show that no stable limit cycles

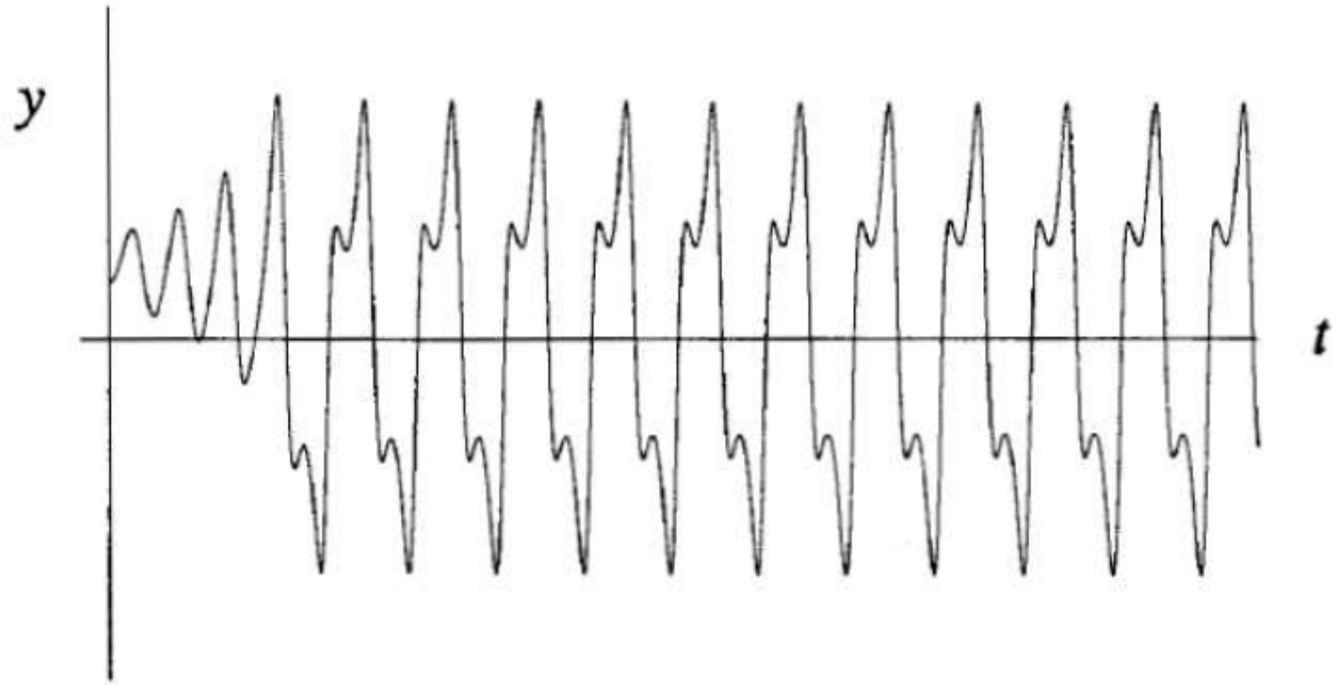
Parameter space





r = 21





$r = 350$