Complexity Science Doctoral Training Centre

CO903 Complexity and Chaos in Dynamical Systems

Routes to chaos

If a nonlinear system has chaotic dynamics then it is natural to ask how this complexity develops as parameters vary. For example, in the logistic map

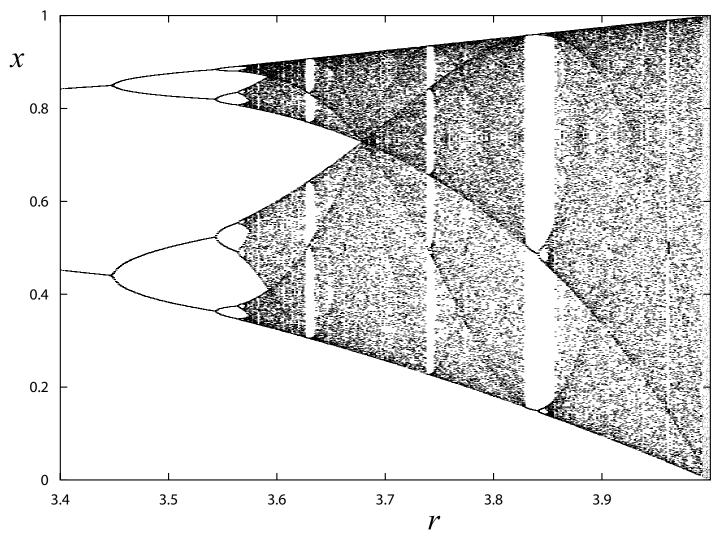
$$x_{n+1} = rx_n(1 - x_n)$$

it is easy to show that if $r = 1/2$ then there is a fixed point at $x = 0$ which attracts all solutions with initial values x_0 between 0 and 1, while if $r = 4$ the system is chaotic. How, then, does the transition to chaos occur as the parameter r varies? The identification and description of routes to chaos has had important consequences for the interpretation of experimental and numerical observations of nonlinear systems. If an experimental system appears chaotic then it can be very difficult to determine whether the experimental data comes from a truly chaotic system, or if the results of the experiment are unreliable because there is too much external noise.

Period doubling

The period doubling route to chaos is found, for example, in the logistic map:

$$x_{n+1} = f_r(x_n) \equiv rx_n(1 - x_n)$$



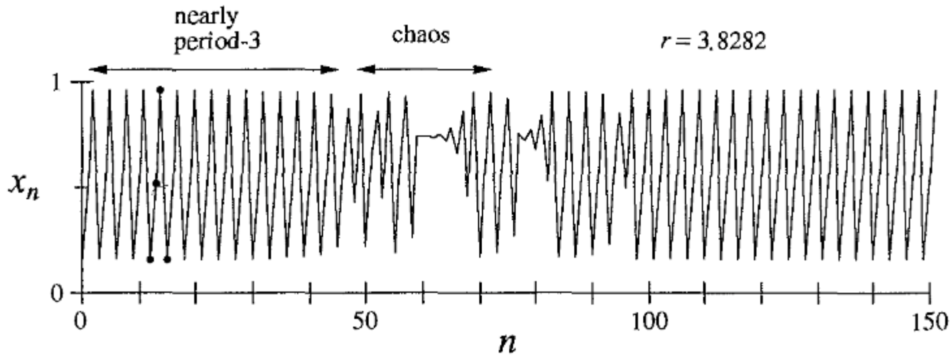
For small r, the attractor is always periodic and has period 2^n, with n increasing as r increases. Beyond some critical value $r = r_c$, with $r_c \approx 3.569946$, the attractor may be more complicated. This period-doubling cascade can be observed in many maps. In the logistic map if this bifurcation occurs with $r = r_n$ then $r_n \to r_c$ geometrically as $n \to \infty$, with

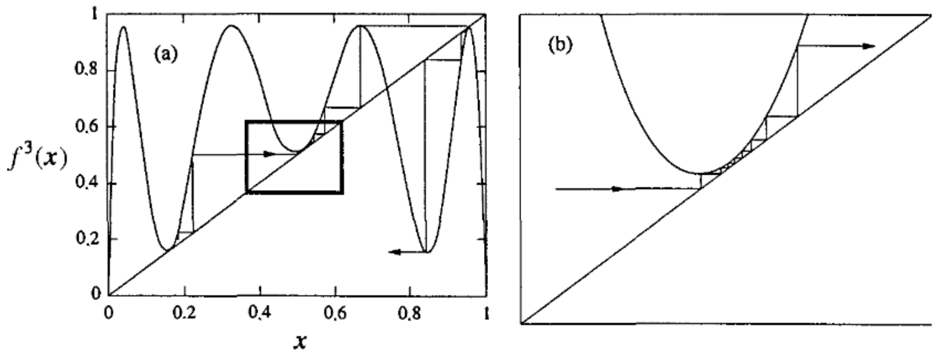
$$\lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = \delta \approx 4.66920.$$

Interestingly, for unimodal maps (smooth, concave down with single maxima, e.g. the logistic map, the sine map $r\sin(\pi x)$,...) this convergence rate is universal (although the constant r_c depends on the map).

Intermittency

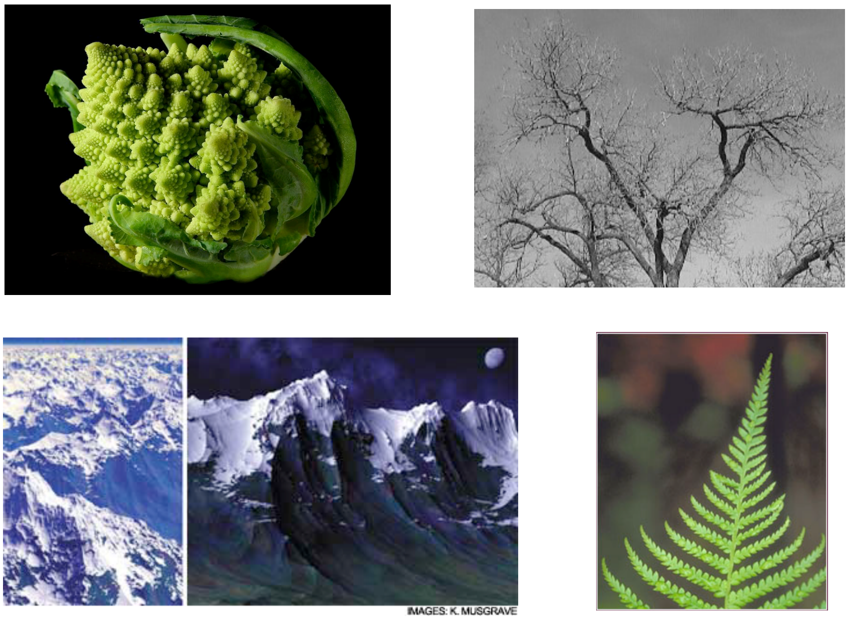
By intermittency we mean the occurrence of a signal which alternates randomly between long regular (laminar) phases (intermissions) and relatively short irregular bursts. A mechanism for this behaviour was first proposed by Pomeau and Manneville in 1979 (they observed such type of behaviour by solving numerically the Lorenz model).





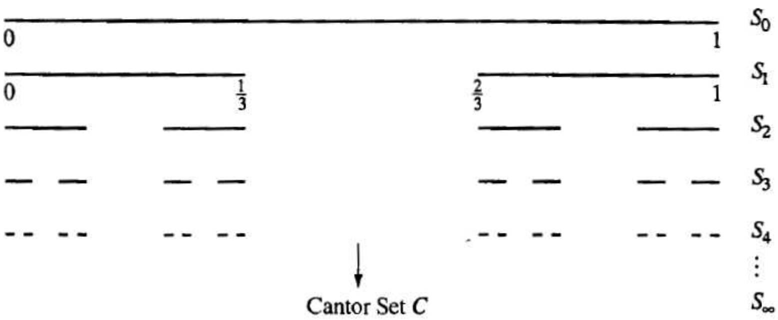
Fractals and fractal dimensions

A fractal is a complex geometric object with fine structure at arbitrarily small scales, perhaps with some degree of self-similarity.



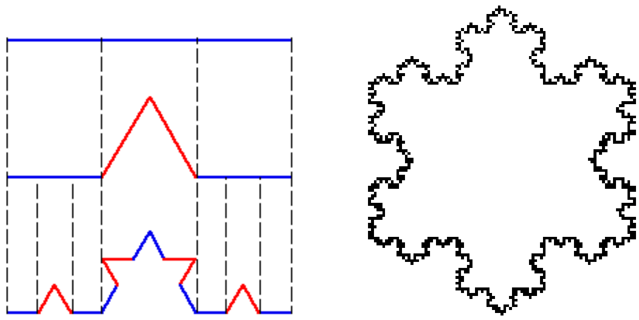
Fractals in nature

Consider the example of the Cantor set:
Start with the closed interval $S_0 = [0, 1]$ and remove the (middle-third) interval $(1/2, 1/3)$. This leaves a pair of closed intervals, which we call S_1. Repeated middle thirds removals gives rise to the Cantor set $C = S_\infty$.



1. C has structure at arbitrarily small scales.
2. C is self-similar (eg. the left half of S_2 is a scaled version of S_1).
3. C has noninteger dimension ($\ln 2/\ln 3 \approx 0.63$).

Consider the Koch curve: Start with a line segment S_0. To generate S_1 delete the middle third of S_0 and replace it with the other two sides of an equilateral triangle. Iterate the process to obtain the Koch curve $K = S_\infty$. (Figure on right is a Koch snowflake, everywhere continuous and nowhere differentiable - it is all *corners*!).



The length of the K is infinite. To see this note that $L_1 = 4/3L_0$ (because S_1 contains four segments each of length $L_0/3$). The length increases by a factor 4/3 at each stage so that $L_n = (4/3)^n L_0 \to \infty$ as $n \to \infty$. Hence, every point is infinitely far from every other. This suggests that K is more than one-dimensional, possibly between 1 and 2. There are many definitions of fractal dimension (similarity, box, Hausdorff, ...).

Similarity dimension (for self-similar fractals)

Suppose that a self-similar set is composed of m copies of itself scaled down by a factor of r. Then the similarity dimension d is the exponent defined by $m = r^d$, or equivalently,

$$d = \frac{\ln m}{\ln r}$$

Box dimension

Let S be a subset of $\mathbb{R}^D$ and let $N(\epsilon)$ be the minimum number of D-dimensional boxes of side ϵ needed to cover S.

$$d_{\text{box}} = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)}$$

For a smooth curve of length L, $N(\epsilon) \propto L/\epsilon$, so $d_{\text{box}} = 1$ as expected. For a planar region of area A bounded by a smooth curve, $N(\epsilon) \propto A/\epsilon^2$ and $d_{\text{box}} = 2$.

Example 1. Show that the box dimension of the Cantor set is $\ln 2/\ln 3 \approx 0.63$.

Each S_n consists of 2^n intervals of length $(1/3)^n$, so if we pick $\epsilon = (1/3)^n$ we need 2^n of these intervals to cover the Cantor set. Hence, $N = 2^n$ when $\epsilon = (1/3)^n$. Since $\epsilon \to 0$ as $n \to \infty$ we find

$$d_{\text{box}} = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln 1/\epsilon} = \frac{\ln 2^n}{\ln 3^n} = \frac{n \ln 2}{n \ln 3} = \frac{\ln 2}{\ln 3}$$

Example 2. Show that the box dimension of the Koch curve is $\ln 4/\ln 3 \approx 1.26$.

Each S_n consists of 4^n pieces of length $(1/3)^n L_0$. Ignoring the scale set by L_0 we have that

$$d_{\text{box}} = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln 1/\epsilon} = \frac{\ln 4^n}{\ln 3^n} = \frac{n \ln 4}{n \ln 3} = \frac{\ln 4}{\ln 3}$$

Correlation dimension

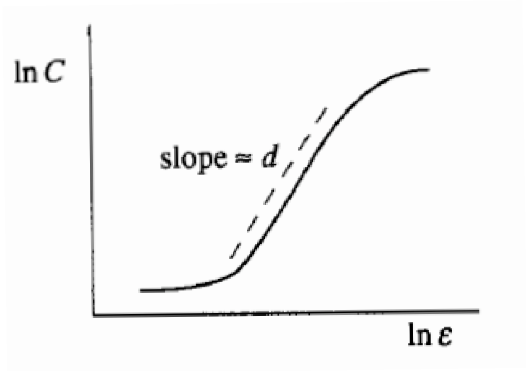
First we generate a set of very many points $\mathbf{x}_i$, $i = 1, ..., n$, on the attractor by letting the system evolve for a long time. Then fix a point $\mathbf{x}$ on the attractor A and let $N_{\mathbf{x}}(\epsilon)$ denote the number of points on A inside a ball of radius ϵ about $\mathbf{x}$. $N_{\mathbf{x}}(\epsilon)$ measures how frequently a typical trajectory visits an ϵ-neighborhood of $\mathbf{x}$. We vary ϵ and the number of points typically grows as a power law

$$N_{\mathbf{x}}(\epsilon) \propto \epsilon^d.$$

We can average $N_{\mathbf{x}}(\epsilon)$ over many $\mathbf{x}$

$$C(\epsilon) \propto \epsilon^d,$$

where d is called the correlation dimension.



For example, the correlation dimension of the Lorenz attractor (for the standard parameter value $r = 28$, $\sigma = 10$, $b = 8/3$) can be found as $d_{\text{corr}} = 2.05 \pm 0.01$.