TUTORIAL III:
One-parameter bifurcation analysis of limit cycles with MATCONT

Yu.A. Kuznetsov
Department of Mathematics
Utrecht University
Budapestlaan 6
3508 TA, Utrecht

September 20, 2011

This session is devoted to the numerical continuation of limit cycles in systems of autonomous ODEs depending on one parameter

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R},$$

and detection of their bifurcations. We will also switch to the continuation of the limit cycle at the Hopf bifurcation and to the continuation of the doubled cycle at the period-doubling bifurcation.

1 Fold and Neimark-Sacker bifurcations of cycles in a chemical model

Consider the following chemical model by *Steinmetz and Larter*:

$$\begin{cases} \dot{A} &= -k_1ABX - k_3ABY + k_7 - k_{-7}A, \\ \dot{B} &= -k_1ABX - k_3ABY + k_8, \\ \dot{X} &= k_1ABX - 2k_2X^2 + 2k_3ABY - k_4X + k_6, \\ \dot{Y} &= -k_3ABY + 2k_2X^2 - k_5Y. \end{cases} \tag{1}$$

We will fix all parameters but k_7 and study periodic solutions (limit cycles) of (1) when this parameter varies.

1.1 System specification

Specify a new ODE system – say `StLar` – in MATCONT

```
A'=-k1*A*B*X-k3*A*B*Y+k7-km7*A
B'=-k1*A*B*X-k3*A*B*Y+k8
X'=k1*A*B*X-2*k2*X^2+2*k3*A*B*Y-k4*X+k6
Y'=-k3*A*B*Y+2*k2*X^2-k5*Y
```

where (`A,B,X,Y`) are the coordinates and (`k1,k2,k3,k4,k5,k6,k7,km7,k8`) are the parameters. Use (default) `t` for time and generate symbolically partial derivatives of order 1,2, and 3.

1.2 Fold bifurcation of limit cycles

1.2.1 Continuation of an equilibrium

To begin with, continue an equilibrium of (1) and detect its Hopf bifurcation, from which the continuation of a limit cycle can be started.

Input **Type**|**Initial point**|**Equilibrium** in the main **MatCont** window.

Input the following numerical data in the appearing **Starter** window:

```
A                31.78997
B                 1.45468
X                 0.01524586
Y                 0.1776113
k1                0.1631021
k2             1250
k3                0.046875
k4               20
k5                1.104
k6                0.001
k7                4.235322
km7               0.1175
k8                0.5
```

These values correspond to an unstable equilibrium (A, B, X, Y) in the system. Activate the parameter `k7`.

Open **Window**|**Numeric** and change its appearance via the **Window**|**Layout** command. Namely, select `EIGENVALUES` to be shown in the window.

Use **Window**|**Graphic**|**2Dplot** to open the corresponding window and select the coordinates (`A,B`) as abscissa and ordinate, respectively, with the visibility limits

```
Abscissa:          32          38
Ordinate:           0           4
```

respectively.

Start **Compute**|**Forward**. The equilibrium curve will be continued and you get a *Hopf bifurcation point* labeled by `H`. The message in MATLAB Command Window

```
label = H , x = ( 34.808899 1.328517 0.015246 0.177611 4.590046 )
First Lyapunov coefficient = 1.527549e-02
```

at $k_7 = 4.590046\ldots$ indicates a *subcritical* Hopf bifurcation. Indeed, there are two eigenvalues of the equilibrium with Re $\lambda_{1,2} \approx 0$ at this parameter value visible in the **Numeric** window. The critical frequency Im $\lambda_1 \neq 0$, while the first Lyapunov coefficient is positive. Thus, there should exist an unstable limit cycle, bifurcating from the equilibrium. Resume computations and terminate them when the curve leaves the graphic window.

Rename the computed curve via **Select**|**Curve** and **Actions**|**Rename** into

```
Equilibrium(+)
```

1.2.2 Cycle continuation

Click **Select**|**Initial point** and select `H: Hopf` point in the `Equilibrium(+)` curve as initial. The **Starter** and **Continuer** windows for the continuation of the limit cycle from the Hopf point will appear.

Select `yes` in all `Monitor Singularities` fields of the **Starter** window. Increase the `MaxStepsize` to 1.0 in the **Continuer** window to allow larger steps along the curve and set `MaxNumPoints` to 50. The **Starter** and **Continuer** windows should look like in Figure 1.

Change the layout of the **Numeric** window via **Window**|**Layout** by selecting all `MULTIPLIERS` to be shown. The absolute values (*modulae*) and arguments in *angular grads* will be displayed.

Compute|**Forward** will produce a family of cycles with a *cycle limit point* labeled by `LPC` at $k_7 = 4.74838\ldots$. The **Numeric** window corresponding to `LPC` is shown in Figure 2.

In the MATLAB Command Window, the following message appears:

```
Limit point cycle (period = 1.036108e+01, parameter = 4.748384e+00)
Normal form coefficient = -2.714838e-01
```

The critical cycle has (approximately) a double multiplier $\mu = 1$ and the normal form coefficient is nonzero. Thus, the limit cycle manifold has a fold here. Resume the computations. The continuation algorithm will automatically follow the second (stable) cycle branch after the `LPC` point. Verify this by looking at the nontrivial multipliers in the **Numeric** window ($|\mu| < 1$ for all such multipliers). The computations produce Figure 3.

To visualize `LPC` in another way, open another **2Dplot** window and select parameter `k7` and `Period` of the cycle as abscissa and ordinate, respectively. Use the visibility limits

```
Abscissa:          4.5          4.8
Ordinate:          7           12
```

You will get a curve with a limit point, clearly indicating the presence of two limit cycles with different periods for $k_7 < 4.74838$ near `LPC` (see Figure 4). Close both **2Dplot** windows.

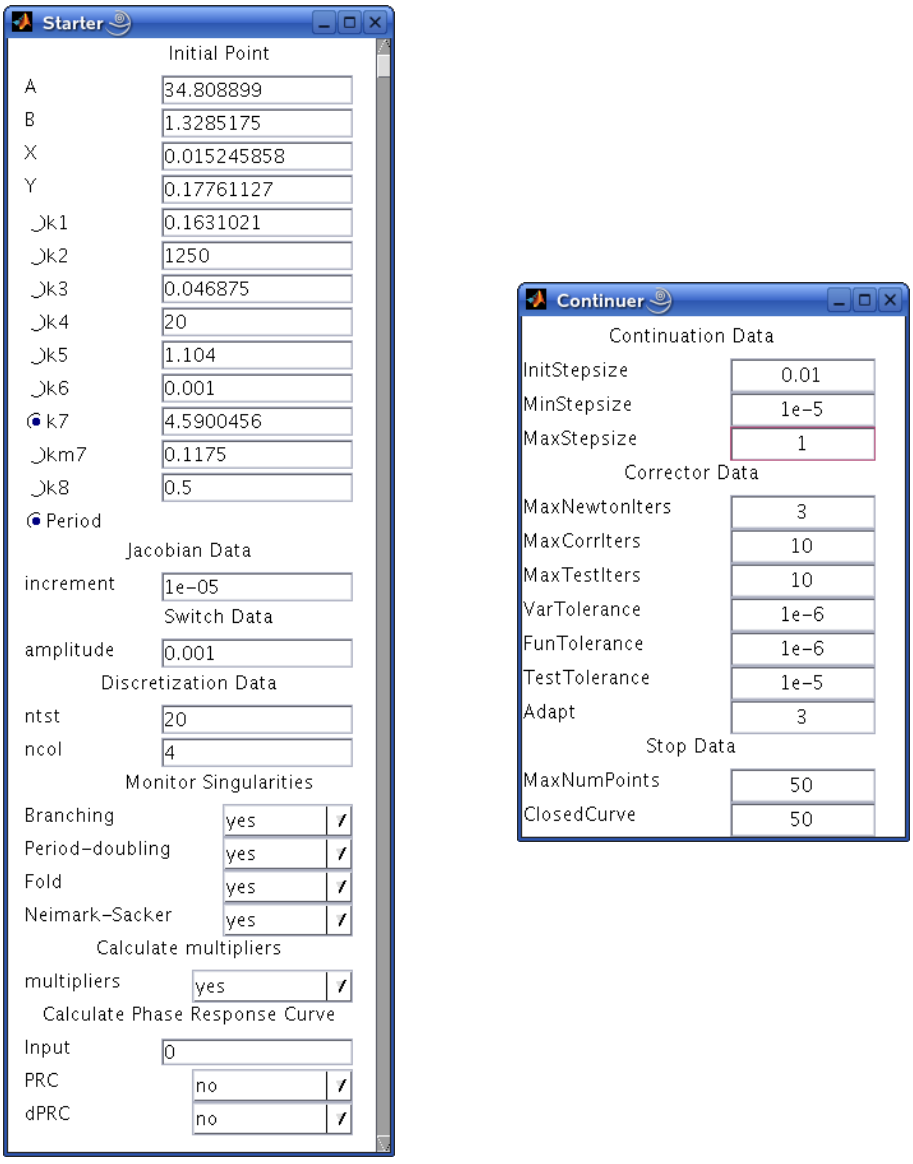

Figure 1: The **Starter** and **Continuer** windows for the cycle continuation.

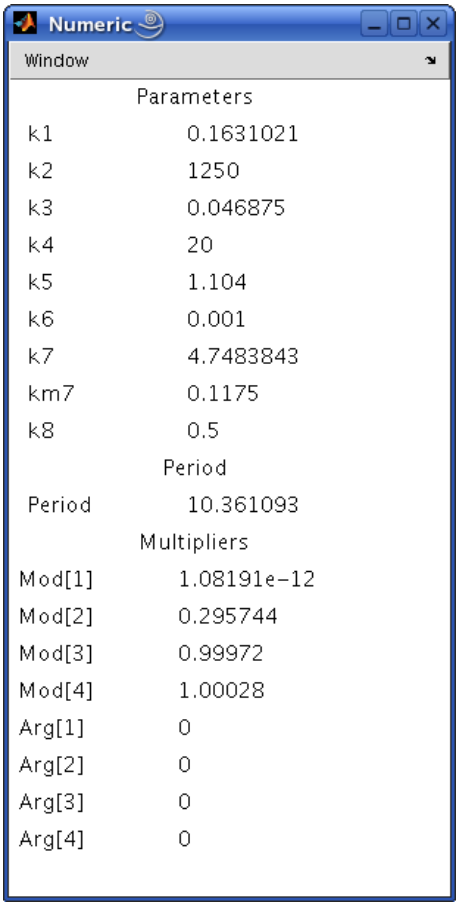

Figure 2: The **Numeric** window at the LPC-point.

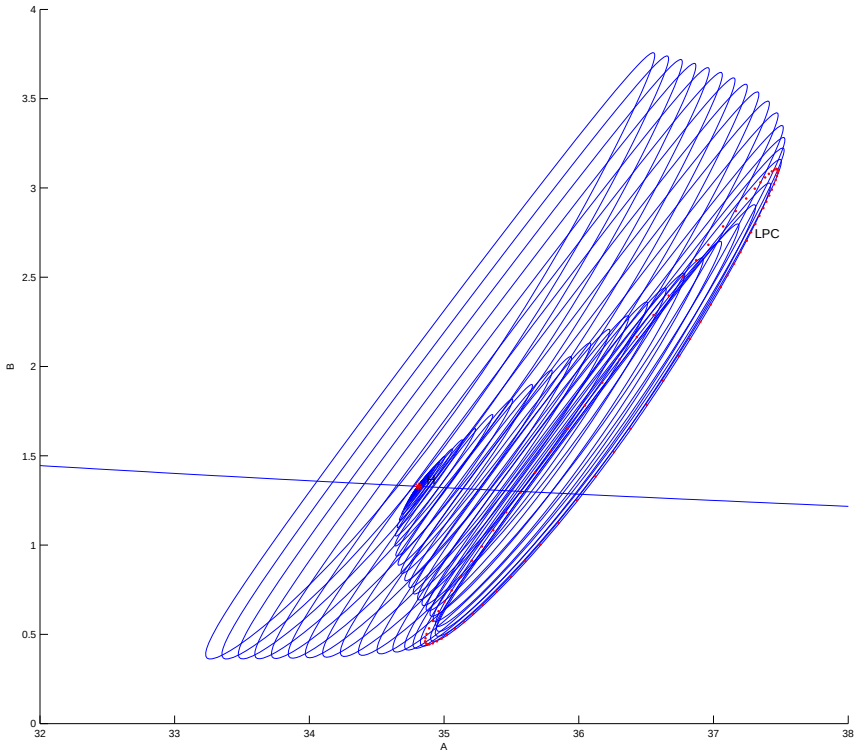

Figure 3: The family of limit cycles bifurcating from the Hopf point `H`: `LPC` is a fold bifurcation of the cycle.

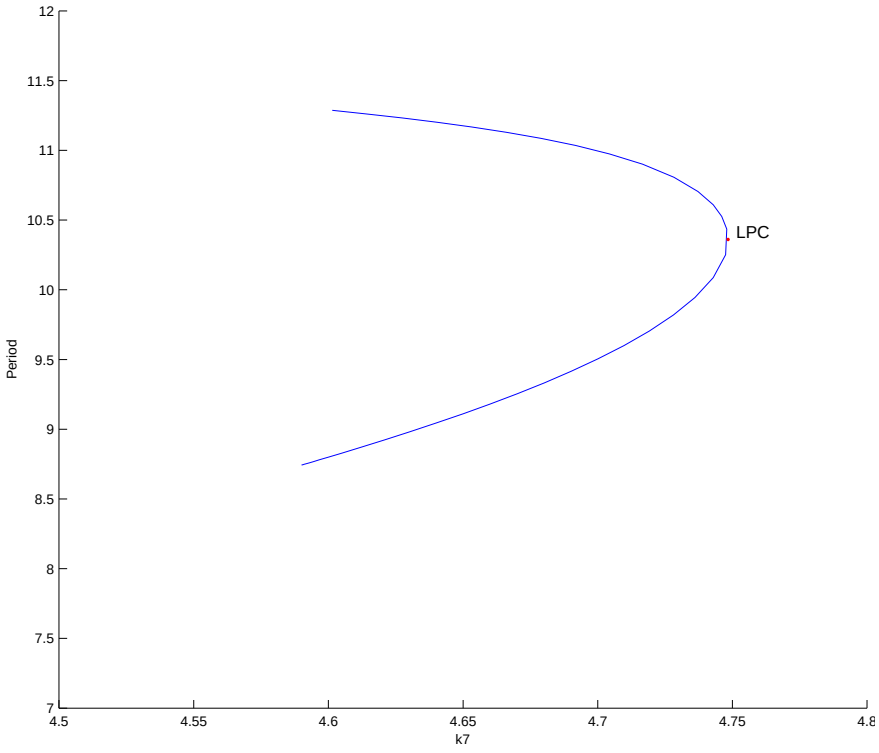

Figure 4: Period of the cycle versus k_7.

1.3 Neimark-Sacker bifurcation

1.3.1 Backward continuation of the equilibrium

Select the first computed equilibrium curve `Equilibrium(+)` in the dialog box appearing after the **Select**|**Curve** command.

Open a new **2Dplot** window but select now `B` and `X` as abscissa and ordinate, respectively, with the visibility limits

```
Abscissa:                25                27
Ordinate:                 0                 0.03
```

Compute|**Backward** and **Extend** once until you get another *Hopf bifurcation* at $k_7 = 0.712475\ldots$ with the message

```
label = H , x = ( 1.808301 25.573303 0.015246 0.177611 0.712475 )
First Lyapunov coefficient = -2.371880e-02
```

in the MATLAB Command Window. The first Lyapunov coefficient is negative now. This means that a *stable* limit cycle bifurcates from the equilibrium, when it looses stability. Resume computations and terminate them when the equilibrium curve leaves the graphic window. Rename the computed curve into `Equilibrium(-)`.

1.3.2 Cycle continuation

Select the Hopf point in the curve `Equilibrium(-)` as initial. MATCONT will prepare to continue a limit cycle curve from the Hopf point (curve type `H_LC`). Select `yes` in all `Monitor Singularities` fields of the **Starter** window. Activate k_7 and the Period in the **Starter** window. Set

```
MaxStepsize             1
MaxNumPoints           25
```

in the **Continuer** window. Notice that in the **Numeric** window the eigenvalues were replaced by multipliers of the cycle.

Click **Compute**|**Forward** to start the continuation of the limit cycle. At $k_7 = 0.716434\ldots$ the message `Neimark-Sacker` indicates a *torus* bifurcation. In the MATLAB Command Window, the following message appears:

```
Neimark-Sacker (period = 1.091213e+01, parameter = 7.164336e-01)
Normal form coefficient = -4.912065e-08.
```

Indeed, there are two complex multipliers with (approximately) $|\mu| = 1$. This can be seen in the **Numeric** window that stays open, see Figure 5. The normal form coefficient is small but nonzero, indicating that a stable two-dimensional *invariant torus* bifurcates from the limit cycle.

Resume computations further to see that after the `NS`-point the cycle becomes unstable (with two multipliers satisfying $|\mu| > 1$. You should get get a family of limit cycles bifurcating from the Hopf point as in Figure 6.

Close the **Numeric** window.

1.3.3 Dynamics on a stable torus

Click **Select**|**Initial point...** and select the `NS: Neimark-Sacker` point in the computed limit cycle curve. The critical parameter values will be read in from the archive. Immediately after that select **Type**|**Initial point**|**Point**. A point on the critical cycle is automatically selected as initial for orbit integration.

Increase slightly the value of the parameter k_7 and perturb the initial point, namely set:

```
k7              0.7165
B              26.2
```

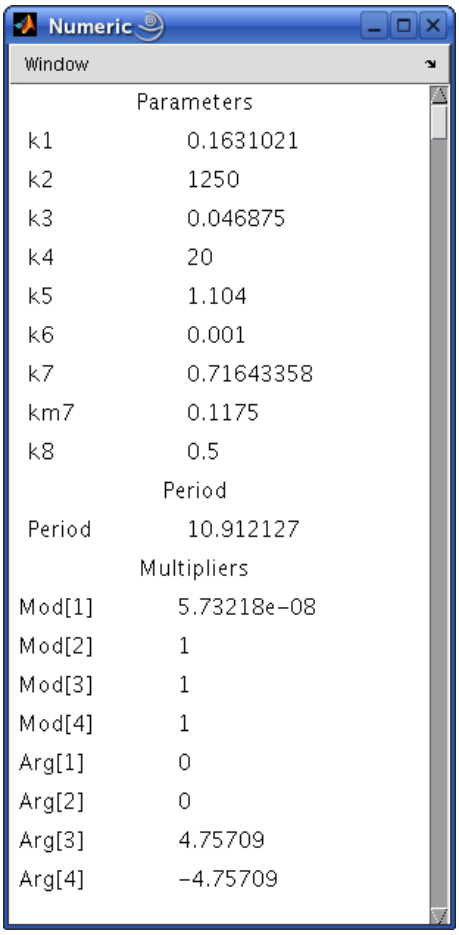

Figure 5: The **Numeric** window at the `NS`-point.

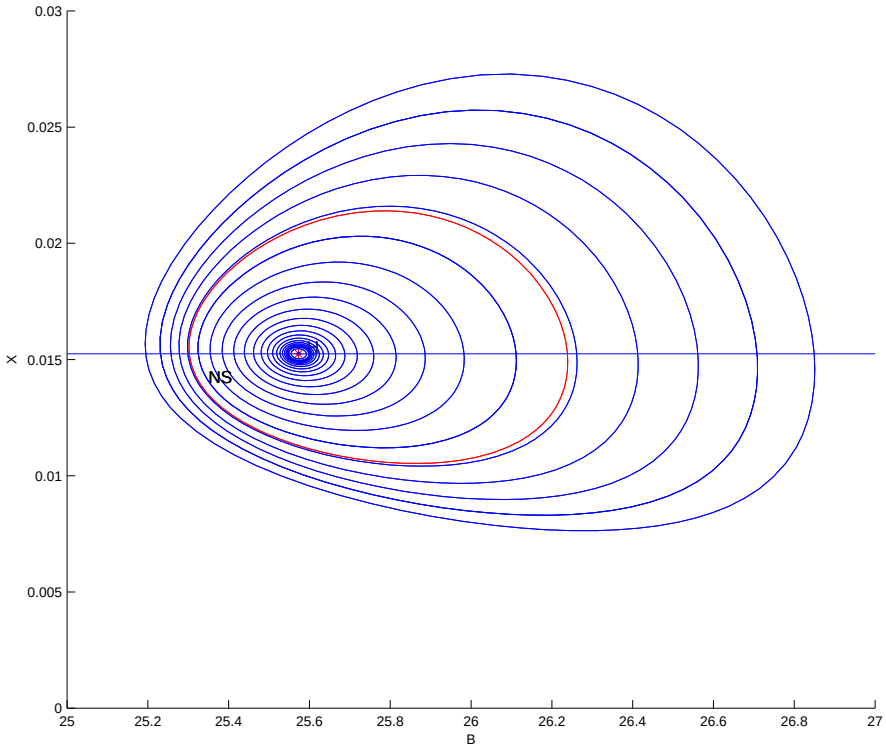

Figure 6: Family of limit cycles bifurcating from the second Hopf point

(the set value of B should be very close to the one obtained from loading the **NS** point). Change **Method** of integration to `ode23s` and alter

```
Interval            1000
Rel.Toletance          1e-6
Abs.Tolerance          1e-9
```

in the **Integrator** window.

To speed up visualization, click **Options**|**Output** and set `Plot after` 1000 points, see Figure 7

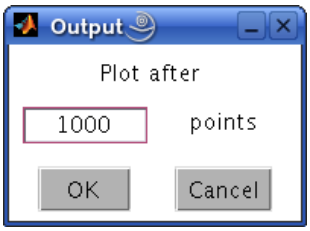

Figure 7: **Output** window.

Clear the graphic window and input **Compute**|**Forward** followed by **Compute**|**Extend**. The integration will take some time. After a transient, the orbit will exhibit modulated oscillations with two frequencies near the limit cycle. This is a motion on a stable two-dimensional torus born via the Neimark-Sacker bifurcation. See Figure 8. Open another **2Dplot** window with `t` and `B`

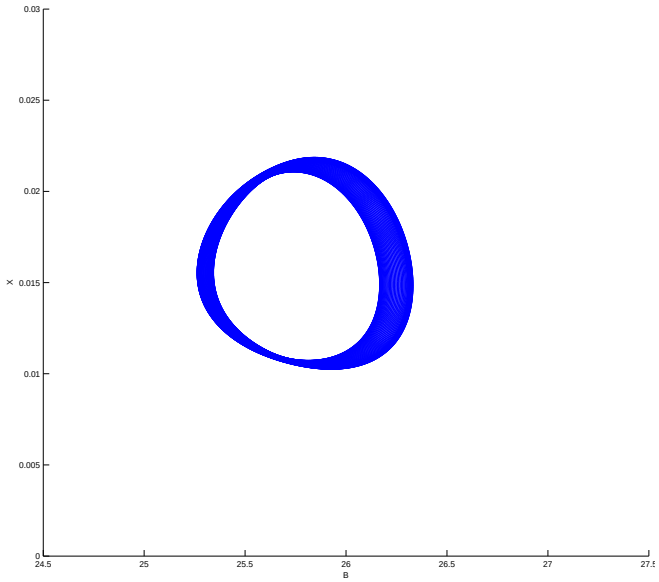

Figure 8: Dynamics on a stable 2-torus

as abscissa and ordinate, with the visibility limits

```
Abscissa:          1000          2000
Ordinate:            26.0          26.5
```

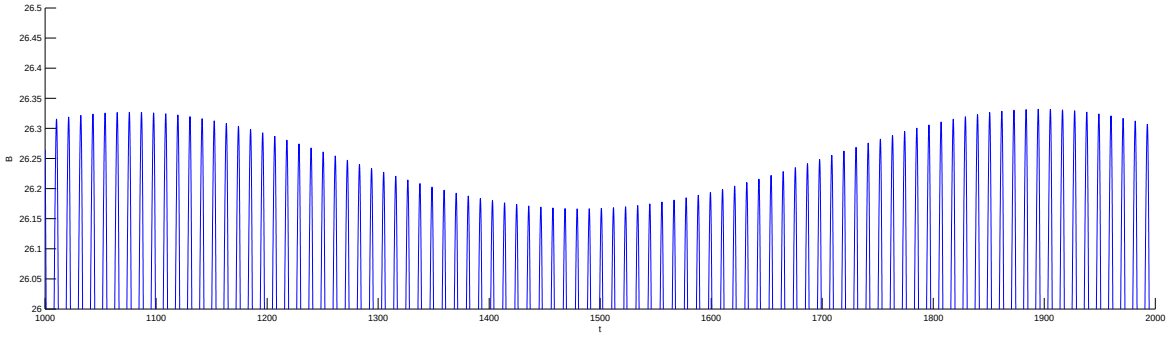

Figure 9: Modulated chemical oscillations

You will see the high-frequency oscillations with low-frequency modulation as in Figure 9.

Restore the original `Plot after` 1 point setting using **Options**|**Plot** menu of the main **MatCont** window.

2 Period-doubling bifurcation in an adaptive control model

Consider the following *adaptive control system of Lur'e type*

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -\alpha z - \beta y - x + x^2. \end{cases} \tag{2}$$

We will fix $\alpha = 1$ and continue limit cycles of (2) using β as a bifurcation parameter.

2.1 System specification

As usual, define a new system in MATCONT with the coordinates `x,y,z`, parameters `alpha,beta`, and select derivatives of order 1, 2, and 3 to be generated symbolically.

2.2 Equilibrium continuation

Select **Type**|**Initial point**|**Equilibrium**. In the **Starter** window, set `alpha` equal to 1 and activate `beta`.

Open a **3Dplot** window and select `beta,x` and `y` as variables along the coordinate axes with the visibility limits

```
Abscissa:          -0.4       0.8
Ordinate:          -1.0       0.3
Applicate:         -0.2       0.8
```

respectively.

Compute|**Forward** results in an equilibrium curve with a Hopf bifurcation (labeled `H`) detected with the following message in the MATLAB Command Window:

```
label = H , x = ( 0.000000 0.000000 0.000000 1.000000 )
First Lyapunov coefficient = -3.000000e-01
```

The Hopf bifurcation occurs at $\beta = 1$ and is supercritical, thus generates a stable limit cycle. Resume the continuation and eventually stop it, when the curve leaves the plotting region.

2.3 Cycle continuation starting from the H-point

Click **Select**|**Initial point** and choose `H: Hopf` point in the computed `EP_EP(1)` curve. The **Starter** and **Continuer** windows appear, corresponding to the limit cycle curve `H_LC`.

In the **Starter** window, increase the number of mesh points `ntst` to 40 and select `yes` in all `Minotor Singularities` fields. In the **Continuer** window, set `MaxStepsize` to 0.7 and `MaxNumPoints` to 80.

With **Window**|**Numeric**, open a **Numeric** window and make the cycle `MULTIPLIERS` visible using the **Window**|**Layout** command there.

Compute|**Forward** will produce a branch of limit cycles of increasing amplitude with two *period-doubling bifurcations* (labeled `PD`). The **Numeric** windows at both `PD` bifurcations are shown in Figure 10.

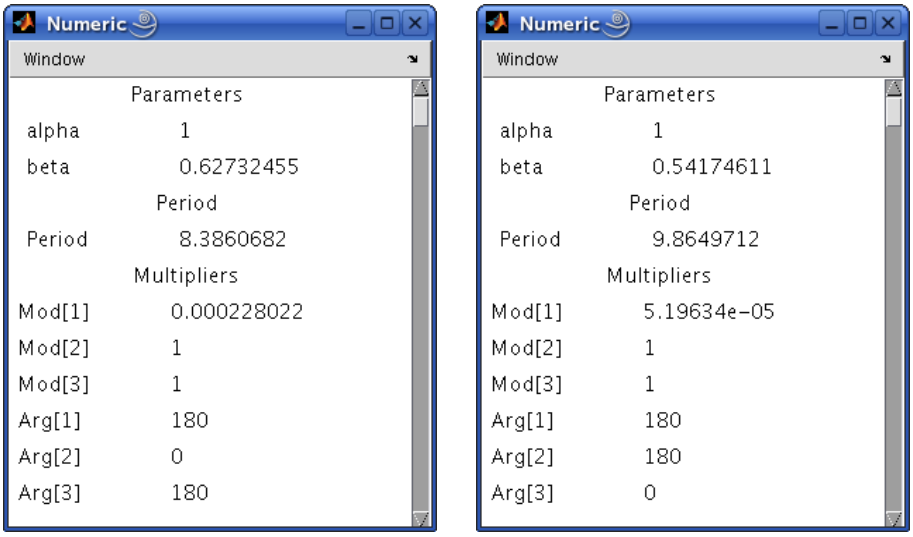

Figure 10: The **Numeric** windows at both period-doubling bifurcations. The bifurcation parameter values are $\beta_{\mathrm{PD},1} = 0.6273246\ldots$ and $\beta_{\mathrm{PD},2} = 0.5417461$.

Resume the computations at both `PD` points. The MATLAB Command Window will contain the messages

```
Period Doubling (period = 8.386067e+00, parameter = 6.273246e-01)
Normal form coefficient = -3.405335e-03
Period Doubling (period = 9.864971e+00, parameter = 5.417461e-01)
Normal form coefficient = -7.655597e-04
```

indicating that in both cases the normal form coefficients are negative, so that stable double-period cycles are involved.

You get a cycle manifold shown in Figure 11. The period of the cycle rapidly increases when β approaches $\beta_{\mathrm{hom}} = 0.515489\ldots$. To see this, open a **2Dplot** window (see Figure 12) with the axes `beta` and `Period` and the plotting region

```
Abscissa:       0.51        0.53
Ordinate:       0.0        35.0
```

Actually, the limit cycle approaches a *homoclinic orbit* to the second equilibrium $(x, , y, z) = (1, 0, 0)$ of (2), which is a global bifurcation[1]. Below β_{hom} no cycle exists. Near the homoclinic

[1] At $\beta = \beta_{\mathrm{hom}}$, this equilibrium is a saddle-focus with the *saddle quantity* $\sigma < 0$. Thus, acccording to Shilnikov's Theorem, from this homoclinic orbit bifurcates only one limit cycle - the one we computed.

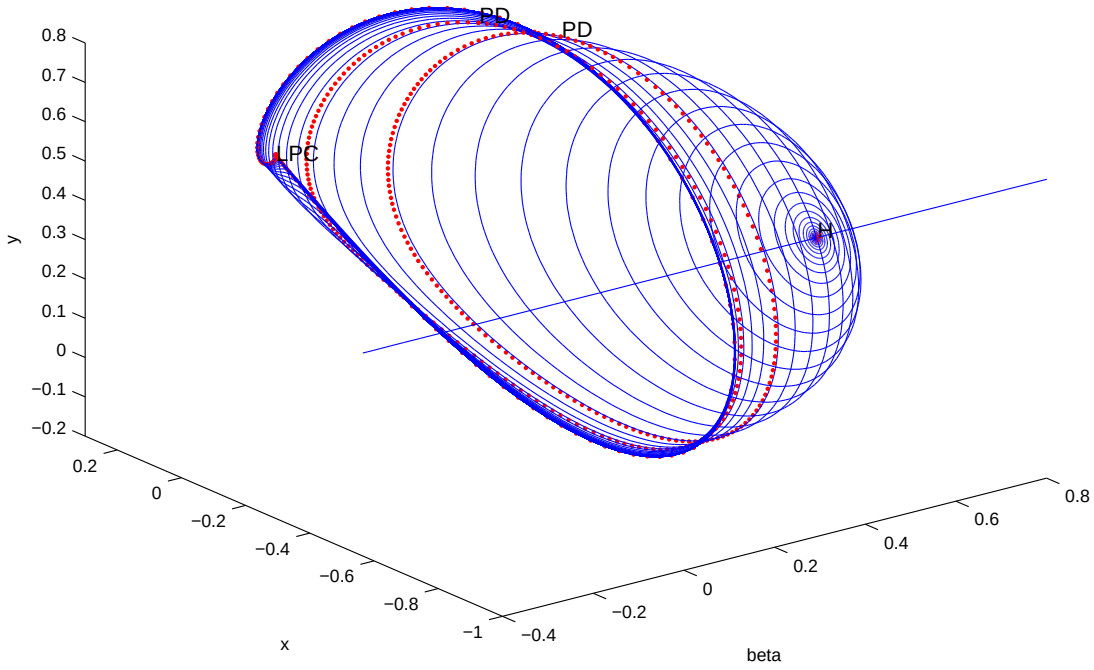

Figure 11: The cycle branch in (2) system rooted at the Hopf point. Two critical cycles undergoing the period-doubling bifurcations are labeled by PD.

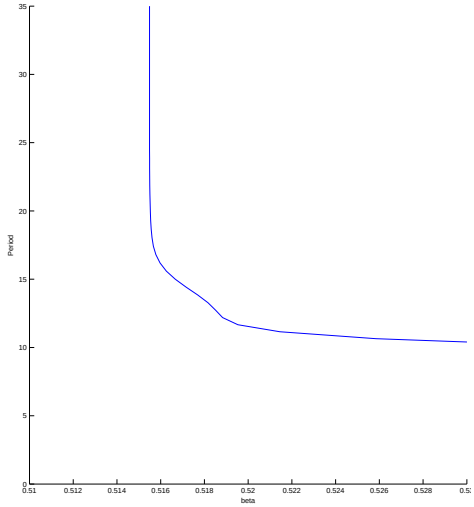

Figure 12: The cycle period near the homoclinic bifurcation.

bifurcation, the computation of the cycle and its multipliers becomes inaccurate. The loss of accuracy can be seen in the **Numeric** window, where no multiplier close to 1 is present at the last computed point.

Close the **2Dplot** window.

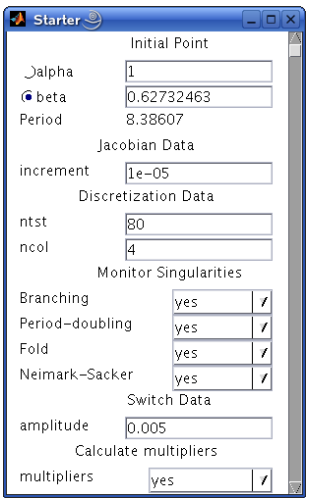

Figure 13: **Starter** window to initialize the continuation of the doubled cycle from the PD-point.

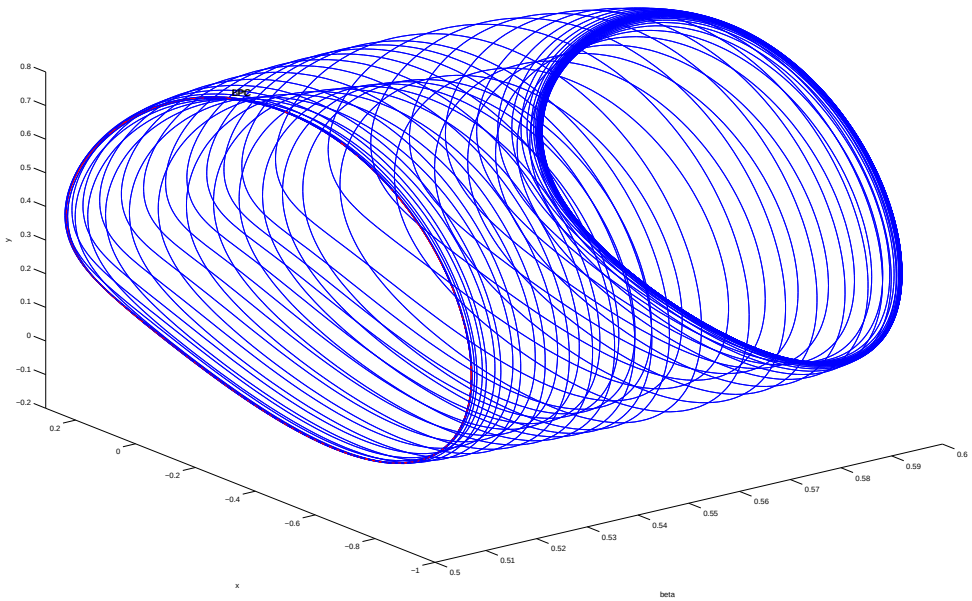

Figure 14: The branch of stable doubled cycles connects two PD points.

2.4 Continuation of the double cycle from the PD-point

Using **Select|Initial point**, take the first `PD: Period Doubling` point in the computed above curve as initial.

Adjust the **3Dplot** window by setting the visibility limits for `beta` as following:

```
Abscissa:          0.5          0.6
```

In the **Starter** window, set the number of mesh points `ntst` to 80, increase the `amplitude` to 0.005, and select `yes` in all `Monitor Singularities` fields as shown in Figure 13. Also, increase `MaxTestIters` to 20 in the **Continuer** window.

Click **Compute|Forward** and continue the doubled cycle bifurcating from the first `PD`-point at $\beta = \beta_{\mathrm{PD},1}$, see Figure 14. This cycle remains stable until at (approximately) $\beta = \beta_{\mathrm{PD},2}$ a *branching point cycle* (`BPC`) is detected. This is not surprising, since the `PD`-bifurcation for the original limit cycles corresponds to a branching point for the branch of the doubled cycles.

Stop the computations and exit MATCONT.

3 Additional Problems

A. Study numerically limit cycles in the system

$$\begin{cases} \dot{x} &= \mu x - y - xz, \\ \dot{y} &= x + \mu y, \\ \dot{z} &= -z + y^2 + x^2 z, \end{cases}$$

that is another model of a feedback control system.

B. Consider the following *predator-prey* model

$$\begin{cases} \dot{x} &= x - \dfrac{xy}{1+\alpha x}, \\ \dot{y} &= -y + \dfrac{xy}{1+\alpha x} - \delta y^2. \end{cases}$$

1. Prove that the following polynomial system

$$\begin{cases} \dot{x} &= x(1+\alpha x) - xy, \\ \dot{y} &= -(y+\delta y^2)(1+\alpha x) + xy, \end{cases} \tag{3}$$

has the same orbits in the positive quadrant as the original model. Derive an explicit condition on (α, δ) for system (3) to have a positive equilibrium with one zero eigenvalue. Derive a similar condition for (3) to have a positive equilibrium with a pair of purely imaginary eigenvalues.

2. Introduce new variables

$$\begin{cases} \xi &= \ln x, \\ \eta &= \ln y, \end{cases}$$

in which the original system becomes:

$$\begin{cases} \dot{\xi} &= 1 - \dfrac{\exp(\eta)}{1+\alpha\exp(\xi)}, \\ \dot{\eta} &= -1 + \dfrac{\exp(\xi)}{1+\alpha\exp(\xi)} - \delta\exp(\eta). \end{cases} \tag{4}$$

Fix $\alpha = 0.3$ and study numerically limit cycles of (4), when the parameter δ varies. *Hint*: Begin with finding an equilibrium at $\delta = 0.35$ by integration.

3. Relate your analytical and numerical results.

C. Consider the following *predator-double prey* system:

$$\begin{cases} \dot{x} &= x(2.4 - x - 6y - 4z), \\ \dot{y} &= y(\beta - x - y - 10z), \\ \dot{z} &= -z(1 - 0.25x - 4y + z), \end{cases}$$

modelling dynamics of two prey populations affected by a predator.

1. Starting from $\beta = 1.77$, find its *positive* equilibrium by orbit integration.
2. Continue the equilibrium with respect to β until it exhibits a Hopf bifurcation.
3. Continue a limit cycle from the Hopf point and monitor the dependence of its period upon β. *Hint*: Use rather big `ntst`.
4. Plot the cycle at different period values and try to understand its asymptotic shape. *Hint*: Compute and analyze system's equilibria located in the coordinate planes.

D. Consider the famous *Lorenz system*

$$\begin{cases} \dot{x} &= \sigma(-x + y), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= -bz + xy, \end{cases}$$

with the standard parameter values $\sigma = 10, b = \frac{8}{3}$, and $r > 0$. Use MATCONT to analyse its simplest limit cycles numerically.

1. Compute a critical parameter value $r_{\mathtt{BP}}$, at which the trivial equilibrium $(0, 0, 0)$ undergoes a *symmetric* pitchfork bifurcation.
2. Compute a critical parameter value $r_{\mathtt{H}}$, corresponding to Hopf bifurcation of *nontrivial* equilibria.
3. Continue the limit cycle born at the Hopf bifurcation with respect to r and find out what happens to it. Plot the cycle period as function of r.
4. Demonstrate by numerical integration that for $r = 400$ the system has a stable symmetric limit cycle and determine its period approximately. Continue this cycle with respect to r and find the critical parameter value $r_{\mathtt{BPC}}$ corresponding to a *branch point of cycles* (*symmetry breaking*). *Hint*: Use **Select cycle** button in the `P_O` **Starter** window.

Derive explicit formulas for $r_{\mathtt{BP}}$ and $r_{\mathtt{H}}$ as functions of (σ, b) and verify the numerical values obtained with MATCONT.