Flocking

self-propelled particles:

$$\mathbf{x}_i(t+\Delta t) = \mathbf{x}_i(t) + \mathbf{v}_i(t)\ \Delta t$$
$$\mathbf{v}_i(t) = v(\cos\theta_i(t)\ \hat{x} + \sin\theta_i(t)\ \hat{y})$$
$$\theta_i(t+\Delta t) = \langle\theta_j(t)\rangle_{j:|\mathbf{x}_j-\mathbf{x}_i|<R} + \underbrace{\Delta\theta}_{\text{uniform in } [\eta/2,\eta/2]}$$

Kinetic phase transition:
fixed ρ: $\Phi \sim (\eta_c(\rho) - \eta)^\beta$
fixed η: $\Phi \sim (\rho - \rho_c(\eta))^{\beta'}$
Scaling function:

$$\Phi(\eta,\rho) = \tilde{\Phi}\left(\frac{\eta}{\eta_c(\rho)}\right), \qquad \tilde{\Phi}(u) \sim \begin{cases} (1-u)^\beta, & \text{if } u<1 \\ 0 & \text{if } u>1 \end{cases}$$

scaling exponents:

$$\Phi(\eta,\rho_c(\eta)+\epsilon) = \tilde{\Phi}\left(\frac{\eta}{\eta_c(\rho_c(\eta)+\epsilon)}\right) \approx \tilde{\Phi}\left(\frac{\eta}{\eta + \epsilon\ \frac{d\eta_c}{d\rho}\Big|_{\rho=\rho_c(\eta)}}\right) \approx \tilde{\Phi}\left(1 - \frac{\epsilon}{\eta}\frac{d\eta_c}{d\rho}\right) \sim \epsilon^\beta$$

so $\beta = \beta'$.

Surface growth

average height:

$$\bar{h}(t) = \langle h(x,t) \rangle_x, \qquad \bar{h}(t) \sim t$$

surface width:

$$w(L,t) = \sqrt{\left\langle \left(h(x,t) - \bar{h}(t)\right)^2 \right\rangle_x}$$

scaling of width:

early times, $t \ll t_\times$:	$w(L,t) \sim t^\beta$	β: growth exponent
late times, $t \gg t_\times$:	$w(L,t) \sim w_{\text{sat}}(L) \sim L^\alpha$	α: roughness exponent
crossover time:	$t_\times \sim L^z$	z: dynamic exponent

Family-Vicsek scaling relation:

$$w(L,t) \sim L^\alpha f\left(\frac{t}{L^z}\right) \qquad f(u) \sim \begin{cases} u^\beta, & \text{if } u \ll 1 \\ \text{const}, & \text{if } u \gg 1 \end{cases}$$

scaling law:

$$\left.\begin{aligned} w(L,t_\times) \quad & \sim L^\alpha \\ & \sim t_\times^\beta \sim L^{z\beta} \end{aligned}\right\} \qquad z = \frac{\alpha}{\beta}$$

self-affine surface:

$$h(x) \qquad \text{similar to} \qquad b^\alpha h\left(\frac{x}{b}\right)$$

eg. graph of random walk:

$$\left\langle (y(t+\Delta t) - y(t))^2 \right\rangle_t \sim \Delta t \qquad \Rightarrow \qquad \alpha = \frac{1}{2}$$

Models:

- random deposition
- random deposition with surface relaxation
- restricted solid-on-solid (RSOS): maintain $|h(x) - h(x+1)| \leq 1$

Continuum equations:

$$\frac{\partial h}{\partial t} = G[h(\cdot), x, t] + \eta$$

noise:

$$\text{eg. } \eta(x,t): \qquad \langle \eta(x,t) \rangle = 0, \qquad \langle \eta(x,t)\eta(x',t') \rangle = 2D\delta(x-x')\delta(t-t')$$

desired symmetries:
$t \to t + \Delta t$
$h \to h + \Delta h$
$x \to x + \Delta x$
$x \to -x$, or rotation
$h \to -h$ (if equilibrium)

simplest, Edwards-Wilkinson:

$$\frac{\partial h}{\partial t} = \nabla^2 h + \eta(x,t) \qquad \alpha = 1 - \frac{d}{2}, \quad \beta = \frac{1}{2} - \frac{d}{4}, \quad z = 2$$

breaking $h \to -h$ symmetry, Kardar-Parisi-Zhang (KPZ):

$$\frac{\partial h}{\partial t} = \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(x,t) \qquad \text{for } d = 1: \quad \alpha = \frac{1}{2}, \quad \beta = \frac{1}{3}, \quad z = \frac{3}{2}$$