The Widget Problem

Setup: factory produces one colour of widgets every day, orders for different colours come during the day, want to fulfill them every evening. Task: decide which colour in the morning.

daily production: $D = 200$.

	R	G	B	decision?
(A) in stock:	$S_1 = 100$	$S_2 = 150$	$S_3 = 50$	B
(B) avg daily order total:	$A_1 = 50$	$A_2 = 100$	$A_3 = 10$	G
(C) avg daily number of orders:	$N_1 = 2/3$	$N_2 = 10$	$N_3 = 1/2$	R
(D) first order:			40	R(?)

(A): no knowledge about orders, so "best guess" is equal probability: B would run out first

(B): state: daily total volume: (n_1, n_2, n_3), $\quad n_i = 0, 1, \ldots$. Know: $\langle n_i \rangle = A_i$.

$$Z(\lambda_1, \lambda_2, \lambda_3) = \sum_{n_1, n_2, n_3 = 0}^{\infty} e^{-\lambda_1 n_1 - \lambda_2 n_2 - \lambda_3 n_3} = \underbrace{\left(\sum_{n_1=0}^{\infty} e^{-\lambda_1 n_1} \right)}_{\frac{1}{1-e^{-\lambda_1}}} \left(\sum_{n_2=0}^{\infty} e^{-\lambda_2 n_2} \right) \left(\sum_{n_3=0}^{\infty} e^{-\lambda_3 n_3} \right)$$

$$p(n_1, n_2, n_3) = \frac{1}{Z} e^{-\lambda_1 n_1 - \lambda_2 n_2 - \lambda_3 n_3} = p_1(n_1) p_2(n_2) p_3(n_3)$$

$$\langle n_i \rangle = -\frac{\partial \ln Z}{\partial \lambda_i} = \frac{1}{e^{\lambda_i} - 1}$$

$$Z_i(\lambda_i) = \frac{1}{1 - e^{-\lambda_i}} = A_i + 1$$

$$p_i(n_i) = \frac{1}{A_i + 1} \left(\frac{A_i}{A_i + 1} \right)^{n_i}$$

Loss function: volume of unfilled orders
If we have chosen R:

$$L_1(n_1, n_2, n_3) = R(n_1 - S_1 - D) + R(n_2 - S_2) + R(n_3 - S_3)$$

where ramp function:

$$R(x) = \begin{cases} x & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x \leq 0 \end{cases}$$

similarly

$$L_2(n_1, n_2, n_3) = R(n_1 - S_1) + R(n_2 - S_2 - D) + R(n_3 - S_3)$$
$$L_3(n_1, n_2, n_3) = R(n_1 - S_1) + R(n_2 - S_2) + R(n_3 - S_3 - D)$$

Expectation of loss:

$$
\begin{aligned}
\langle L_1 \rangle &= \sum_{n_1,n_2,n_3=0}^{\infty} L_1(n_1,n_2,n_3)p(n_1,n_2,n_3) \\
&= \sum_{n_1=0}^{\infty} L_{11}(n_1)p_1(n_1) \cdot \underbrace{\sum_{n_2=0}^{\infty} p_2(n_2)}_{1} \cdot \underbrace{\sum_{n_3=0}^{\infty} p_3(n_3)}_{1} + \big(\ldots L_{12} \ldots\big) + \big(\ldots L_{13} \ldots\big) \\
&= \underbrace{\frac{1}{A_1+1}(A_1+1)^2 \left(\frac{A_1}{A_1+1}\right)^{S_1+D+1}}_{A_1\left(\frac{A_1}{A_1+1}\right)^{S_1+D}} + A_2 \left(\frac{A_2}{A_2+1}\right)^{S_2} + A_3 \left(\frac{A_3}{A_3+1}\right)^{S_3}
\end{aligned}
$$

where we used

$$\sum_{i=n}^{\infty}(i-n)x^i = \frac{x^{n+1}}{(x-1)^2}$$

plugging in values:

$$
\begin{aligned}
\langle L_1 \rangle &\approx 0.131 + 22.48 + 0.085 \approx 22.70 \\
\langle L_2 \rangle &\approx 6.9 + 3.07 + 0.085 \approx 10.6 \\
\langle L_3 \rangle &\approx 6.9 + 22.48 + 4 \cdot 10^{-10} \approx 29.38
\end{aligned}
$$

Fluctuations

Central limit theorem: suppose X_i are iid (independent identically distributed) random variables, with $\langle X_i \rangle = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$. Then

$$Z_n \stackrel{\text{def}}{=} \frac{\overbrace{X_1 + \cdots + X_n}^{S_n} - n\mu}{\sqrt{n}\sigma} \xrightarrow{D} \mathcal{N}(0,1)$$

where convergence in distribution:

$$\lim_{n\to\infty} P(Z_n < z) = P(\zeta < z) \qquad \text{where } \zeta \text{ is std normal}$$

Cauchy distribution:

$$f(x) = \frac{1}{\pi(1+x^2)} \qquad \text{or} \qquad f(x) = \frac{1}{\pi\gamma\left(1 + \left(\frac{x-x_0}{\gamma}\right)^2\right)}$$

Stable distributions:

$$X_1 \sim \mathrm{Fam}(\Theta_1),\ \ X_2 \sim \mathrm{Fam}(\Theta_2) \ \ \Rightarrow \ \ aX_1 + bX_2 \sim \mathrm{Fam}(\Theta_3) + c$$

then Fam is stable

Levy:

$$f(x) = \sqrt{\frac{c}{2\pi}}\frac{e^{-c/(2x)}}{x^{3/2}}$$

Levy flight increments: $f(x) \sim 1/|x|^{\alpha+1}$, where $0 < \alpha < 2$.