## Functions of many variables, partial derivatives

Suppose $f$ is a function of two variables, $x$ and $y$: $f(x,y)$. Its differential, $df$ can be written as

$$df = \left.\frac{\partial f}{\partial x}\right|_y dx + \left.\frac{\partial f}{\partial y}\right|_x dy \tag{1}$$

Now we change variables: instead of $x, y$ we will use $u, v$. The old variables are expressed in the new variables: $x = x(u,v)$ and $y = y(u,v)$, and through this $f = f\big(x(u,v), y(u,v)\big) = \hat{f}(u,v)$, where the second equation sign can be thought of the definition of $\hat{f}$. We often simply write $f(u,v)$, with the understanding that $f(x,y)$ and $f(u,v)$ are really two different functions (eg. they don't depend the same way on ther first argument), *not* the same function evaluated with different arguments.

We can now evaluate quantities like $\left.\frac{\partial f}{\partial u}\right|_v$. Depending taste, we can either use the differential form (1) above to get

$$\left.\frac{\partial f}{\partial u}\right|_v = \left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial u}\right|_v + \left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial u}\right|_v$$

(noting that the prefactors $\left.\frac{\partial f}{\partial x}\right|_y$ and $\left.\frac{\partial f}{\partial y}\right|_x$ remain there). Or alternatively, apply the $\left.\frac{\partial}{\partial u}\right|_v$ operator on $f(x,y)$, and use the chain rule:

$$\left.\frac{\partial f}{\partial u}\right|_v = \left.\frac{\partial f\big(x(u,v), y(u,v)\big)}{\partial u}\right|_v = \left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial u}\right|_v + \left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial u}\right|_v$$

For many variables, like $f(x_1, \ldots, x_n)$ this becomes

$$\left.\frac{\partial f}{\partial u_k}\right|_{\{u\}} = \sum_{i=1}^{n} \left.\frac{\partial f}{\partial x_i}\right|_{\{x\}} \left.\frac{\partial x_i}{\partial u_k}\right|_{\{u\}} ,$$

where the notation $\left.\frac{\partial f}{\partial u_k}\right|_{\{u\}}$ means keeping all $u_{k'}$ fixed except $u_k$.

These general results are used throughout the MaxEnt derivations. It is fairly straightforward, but can become a bit cumbersome, as well as somewhat tricky at the grand canonical calculations where we keep changing between $\beta, \mu$ and $\beta, -\mu\beta$. When a product is kept fixed, one can make the calculations more straightforward to introduce a symbol for it. For example:

$$\left.\frac{\partial \mu}{\partial \beta}\right|_{-\mu\beta} = \left.\frac{\partial \frac{C}{\beta}}{\partial \beta}\right|_C = -\frac{C}{\beta^2} = -\frac{\mu}{\beta}$$

where $C = \mu\beta$, and keeping $\mu\beta$ or $-\mu\beta$ fixed is the same.