Legendre transform

Consider a convex function $f(x)$, and define the following function

$$f^*(p) := \max_x \big(px - f(x)\big) \tag{9}$$

We call this[1] the *Legendre tranform* of $f(x)$. If f is differentiable as well, we can calculate the maximum as

$$0 = \frac{d}{dx}(px - f(x)) = p - \frac{df(x)}{dx}$$

Its solution for x depends on p, which we call $x(p)$:

$$\left.\frac{df(x)}{dx}\right|_{x=x(p)} = p$$

which plugged into (9) gives

$$f^*(p) = px(p) - f\big(x(p)\big)$$

Now let's calculate the Legendre transform of f^*:

$$(f^*)^*(y) = \max_p \big(yp - f^*(p)\big)$$

Again, if f^* is differentiable then

$$\left.\frac{df^*(p)}{dp}\right|_{p=p(y)} = y$$

However,

$$\frac{df^*(p)}{dp} = \frac{px(p) - f(x(p))}{dp} = x(p) + p\frac{dx(p)}{dp} - \underbrace{\left.\frac{df(x)}{dx}\right|_{x(p)}}_{p} \frac{dx(p)}{dp} = x(p)$$

so

$$y = \left.\frac{df^*(p)}{dp}\right|_{p=p(y)} = x(p(y))$$

thus

$$f^{**}(y) = yp(y) - f^*(p(y)) = yp(y) - p(y)x(p(y)) + f(x(p(y))) = f(y)$$

Thus the function $f^{**}(\cdot)$ and $f(\cdot)$ are equal, or in other way to say the Legendre transform is its own inverse.

The Legendre transform can easily generalised to concave functions: the $\max$ needs to be replaced by $\min$.

The other generalisation is functions of multiple variables:

$$f^*(p_1, \dots, p_k) = \sum_{k=1}^{m} x_k p_k - f(x_1, \dots, x_m)\ , \qquad \text{where } \ p_k = \frac{\partial f}{\partial x_k}$$

In the previous section we have seen that $S(F_1, \dots, F_m)$ and $-\log Z(\lambda_1, \dots, \lambda_m)$ are Legendre transforms of each other, either one of them provides a full description of the system.

[1] The Legendre transform is often defined with a sign difference: $f^*(p) = \max(f(x) - px)$. The advantage of our notation is that the inverse is completely symmetric.