CO904: Problem set 1

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Solutions are due by 1300 on Tuesday of 6th week (10th February). You may hand them to me in person, or place them in my pidge in Complexity.

QUESTION 1

Decide whether the following statements are true or false. If true, explain why; if false, explain why or provide a counterexample.

(i) $S(X \cap Y) = S(X) \cdot S(Y)$, where X and Y are independent random variables and $S(X \cap Y)$ is their joint information entropy.

(ii) $S(X|X) = S(X)$, where $S(X|Y)$ is the conditional information entropy of X and Y.

(iii) $I(X, X) = S(X)$, where $I(X, Y)$ is the mutual information between X and Y.

(iv) $S\big(f(X)\big) = S(X)$, where $f(X)$ is an arbitrary function of X.

QUESTION 2

Consider a fair die: each of the outcomes $1, 2, \ldots, 6$ have equal probability. Let P measure whether the outcome is prime or non-prime, and Q measure whether the outcome is even or odd. Measuring the Shannon entropy in units where the constant k (Boltzmann's constant) is equal to 1, calculate (from the appropriate probability distributions)

(i) The Shannon entropies $S(P)$ and $S(Q)$.

(ii) The joint entropy $S(P, Q)$.

(iii) The conditional entropies $S(P|Q)$ and $S(Q|P)$.

(iv) The mutual information $I(P, Q)$.

Verify that any general relations between these entropies do indeed hold in this case.

Suppose an unfair die on average rolls a 4. Give an estimate for the probability of getting a 6.

QUESTION 3

Consider a q-state Potts model with spins s_i on a one-dimensional lattice with N sites and periodic boundary conditions. The Hamiltonian may be written

$$H = -J \sum_{i=1}^{N} \delta_{s_i, s_{i+1}},$$

where δ_{ij} is the Kronecker delta, equal to one if i, j are the same and zero otherwise.

(i) Show that the partition function is

$$Z = \Big(\mathrm{e}^{\beta J} + q - 1\Big)^N + (q-1)\Big(\mathrm{e}^{\beta J} - 1\Big)^N.$$

(ii) What is the free energy per site in the thermodynamic limit?

(iii) Find an expression for the correlation length.

QUESTION 4

Onsager's exact expression for the partition function of the two-dimensional Ising model, in the thermodynamic limit, can be written in the form

$$\frac{1}{N}\ln Z = \ln\bigl(2\cosh^2(\beta J)\bigr) + \frac{1}{2}\int_{-\pi}^{\pi}\frac{dq_1}{2\pi}\int_{-\pi}^{\pi}\frac{dq_2}{2\pi}\ln\Bigl[(1+x^2)^2 - 2x(1-x^2)\bigl(\cos(q_1)+\cos(q_2)\bigr)\Bigr],$$

where $x \equiv \tanh(\beta J)$ and the expression assumes equal coupling constants between horizontal and vertical nearest neighbour spins.

(i) Show that the argument of the logarithm in the integral is non-negative.

(ii) Show that it vanishes at $q_1 = q_2 = 0$ at a critical value of x and compare this with the expression found in the lectures by Kramers-Wannier duality.

(iii) Expand the logarithm about both $q_1 = q_2 = 0$ and this critical value of x and evaluate the resulting integral to extract the leading singular behaviour of $\ln Z$ (and hence the free energy) near the transition. What is the nature of the singularity in the heat capacity?