CO905 Hand-out

Linear Algebra

- Matrix $\underline{\underline{A}} \in \mathbb{R}^{n\times m}$, $\underline{\underline{A}} = \begin{array}{c} \overset{\leftarrow m \rightarrow}{} \\ n\updownarrow\begin{pmatrix} a_{ij} \end{pmatrix}\end{array}$
  - $\underline{v} \in \mathbb{R}^m$ is right __eigenvector__ with __eigenvalue__ $\lambda \in \mathbb{R}$
    if $\underline{\underline{A}}\,\underline{v} = \lambda \underline{v}$ (column v)
  - $\underline{u}^T \in \mathbb{R}^n$ is left eigenvector with eigenvalue $\lambda \in \mathbb{R}$
    if $\underline{u}^T \underline{\underline{A}} = \lambda \underline{u}^T$ (row v)

Only true for n = m

- Solutions to hom. linear equations $(\underline{\underline{A}} - \lambda \underline{\underline{Id}})\underline{v} = \underline{0}$
  $\underline{u}^T(\underline{\underline{A}} - \lambda \underline{\underline{Id}}) = \underline{0}$

$\Rightarrow$ If $\underline{v}_1, \underline{v}_2$ are e.ve.s to e.va. λ, then all linear combinations $\alpha \underline{v}_1 + \beta \underline{v}_2$ are e.ve.s to e.va. λ
(the same holds for left e.ve.s $\underline{u}_1^T, \underline{u}_2^T \ldots$)

- For __square matrices__ $\underline{\underline{A}} \in \mathbb{R}^{n\times n}$ the eigenvalues are the roots of the __characteristic polynomial__:

$$\chi_A(\lambda) = \det(\underline{\underline{A}} - \lambda \underline{\underline{Id}}) = 0 \Leftrightarrow \lambda \text{ is e.va. of } \underline{\underline{A}}$$

$\chi_A(\lambda)$ is a polynomial of degree n
$\Rightarrow$ has n complex roots $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$.

$\Rightarrow \chi_A(\lambda) = \prod_{i=1}^{n} (\lambda_i - \lambda)$ and
if λ_i is e.va. $\Rightarrow \overline{\lambda_i}$ is also e.va.

- If $\lambda \in \mathbb{C}$ is e.va. of $A \in \mathbb{R}^{n\times n}$
  $\Rightarrow$ there exist left and right e.ves $\underline{u}^T, \underline{v} \in \mathbb{C}^n$
  (if $\lambda \in \mathbb{R}$ then $\underline{u}^T, \underline{v} \in \mathbb{R}^n$)
  __and:__ $\left.\begin{array}{l}\text{if } \underline{u}^T \text{ is e.ve. wrt } \lambda_1 \\ \underline{v} \text{ is e.ve. wrt } \lambda_2 \neq \lambda_1\end{array}\right\} \Rightarrow \underline{u}^T \perp \underline{v}$ i.e. $\underline{u}^T \underline{v} = 0$

- If $A \in \mathbb{R}^{n\times n}$ is symmetric, i.e. $A^T = A$
  $\Rightarrow$ all e.va.s $\lambda_i \in \mathbb{R}$
  $\underline{u}^T, \underline{v}$ e.ve.s wrt λ (single) $\Rightarrow \underline{u} = \underline{v}$
  __and:__ there exists an ONB $\underline{v}_1, \ldots, \underline{v}_n$ of e.ve.s in $\mathbb{R}^n$

Determinants:

$M \in \mathbb{R}^{n\times n}$ $\quad M = (m_{ij})_{i,j=1,\ldots,n}$

$$\det M = \sum_{\pi \in S_n} \prod_{i=1}^{n} m_{i,\pi(i)}$$

Sum over all permutations of indices ($S_n \hat{=}$ symmetric group)