Stochastic models of complex systems

Problem sheet 1

Sheet counts $20/100$ homework marks, all questions carry equal weight.
* Questions do not enter the mark.

1.1 Generators and eigenvalues

The analysis of linear dynamical systems shares a lot of the structure with Markov chains.

(a) Consider the **harmonic oscillator** $\phi : \mathbb{R} \to \mathbb{R}$ given by the equation

$$\frac{d^2}{dt^2}\,\phi(t) = \ddot{\phi}(t) = -k\phi \quad \text{with} \quad k > 0\ .$$

Using the vector valued notation $\boldsymbol{x}(t) = \binom{\phi(t)}{\dot{\phi}(t)}$ write the system in the form

$$\frac{d}{dt}\,\boldsymbol{x}(t) = M\boldsymbol{x}(t) \quad \text{with} \quad M \in \mathbb{R}^{2\times 2}\ .$$

Compute the eigenvalues λ_1 and λ_2 of M and find a solution of the form

$$\phi(t) = a\,e^{\lambda_1 t} + b\,e^{\lambda_2 t}\ ,$$

fixing $a, b \in \mathbb{R}$ by the initial conditions $\phi(0) = 1$, $\dot{\phi}(0) = 0$.

(b) Consider the **Fibonacci numbers** $(F_n : n \in \mathbb{N}_0)$ defined by the recursion

$$F_n = F_{n-1} + F_{n-2} \quad (n \geq 2) \quad \text{with} \quad F_0 = 0,\ F_1 = 1\ .$$

Write $\binom{F_{n+1}}{F_n} = M\binom{F_n}{F_{n-1}}$ as a discrete-time dynamical system with $M \in \mathbb{R}^{2\times 2}$.
Compute the eigenvalues of M and show that

$$F_n = \frac{\eta^n - (1-\eta)^n}{\sqrt{5}} \quad \text{where} \quad \eta = \frac{1+\sqrt{5}}{2} \quad \text{is the Golden ratio}\ .$$

(c)* Derive a recursion relation for the generating function $G(s) = \sum_n F_n s^n$ of the Fibonacci numbers and solve it. Sketch $G(s)$. For which $s \geq 0$ is it well defined?

1.2 Branching processes

Let $Z = (Z_n : n \in \mathbb{N})$ be a branching process, defined recursively by

$$Z_0 = 1\ , \quad Z_{n+1} = X_1^n + \ldots + X_{Z_n}^n \quad \text{for all } n \geq 0\ ,$$

where the $X_i^n \in \mathbb{N}$ are iidrv's denoting the offspring of individuum i in generation n.

(a) Consider a geometric offspring distribution $X_i^n \sim Geo(p)$, i.e.

$$p_k = \mathbb{P}(X_i^n = k) = p\,(1-p)^k\ , \quad p \in (0,1)\ .$$

Compute the prob. generating function $G(s) = \sum_k p_k s^k$ as well as $\mathbb{E}(X_i^n)$ and $Var(X_i^n)$. Sketch $G(s)$ for (at least) three (wisely chosen) values of p and compute the probability of extinction as a function of p.

(b) Consider a Poisson offspring distribution $X_i^n \sim Poi(\lambda)$, i.e.

$$p_k = \mathbb{P}(X_i^n = k) = \frac{\lambda^k}{k!}\, e^{-\lambda} \,, \quad \lambda > 0 \,.$$

Repeat the same analysis as in (a). (The equation for the probability of extinction cannot be solved in this case, find an approximate solution.)

(c)* For geometric offspring with $p = 1/2$, show that $G_n(s) = \frac{n-(n-1)s}{n+1-ns}$ and compute $\mathbb{P}(Z_n = 0)$. If T is the (random) time of extinction, what is its distribution and its expected value?

1.3 (a) **Random walk**

Consider a simple symmetric random walk on $\{1, \dots, L\}$ with
- periodic boundary conditions, i.e. $p_{L,L-1} = p_{L,1} = p_{1,L} = p_{1,2} = 1/2$,
- closed boundary conditions, i.e. $p_{L,L-1} = p_{L,L} = p_{1,1} = p_{1,2} = 1/2$,
- reflecting boundary conditions, i.e. $p_{L,L-1} = p_{1,2} = 1$,
- absorbing boundary conditions, i.e. $p_{L,L} = p_{1,1} = 1$.

(All transition probabilities which are not specified above are 0.)
In each case, sketch the transition matrix $P = (p_{ij})_{ij}$ of the process, decide whether the process is irreducible, and give at least one stationary distribution $\boldsymbol{\pi}^*$.
(Hint: Use detailed balance.)

(b)* Consider a symmetric connected graph (G, E) without loops and double edges. A simple random walk on (G, E) has transition probabilities $p_{i,j} = e_{i,j}/c_i$, where c_i is the number of outgoing edges in vertex i, and $e_{i,j} \in \{0, 1\}$ denotes the presence of an edge (i, j). Find a formula for the stationary distribution $\boldsymbol{\pi}^*$.
Does your formula also hold on a non-symmetric, strongly connected graph?