Stochastic Models of Complex Systems

Problem sheet 2

Sheet counts $50/100$ homework marks, all questions carry equal weight.

2.1 Birth-death processes

A birth-death process X is a continuous-time Markov chain with state space $S = \mathbb{N} = \{0, 1, \ldots\}$ and jump rates

$$i \xrightarrow{\alpha_i} i+1 \quad \text{for all } i \in S\ , \quad i \xrightarrow{\beta_i} i-1 \quad \text{for all } i \geq 1\ .$$

(a) Write down the generator G. Under which conditions is X irreducible?
Using detailed balance, find a formula for the stationary probablities π^*_k in terms of π^*_0.

(b) Suppose $\alpha_i = \alpha$ for $i \geq 0$ and $\beta_i = \beta$ for $i > 0$. This is called an $M/M/1$ **queue**.
Under which conditions on α and β can the stationary distribution be normalized? Give a formula for π^*_k in that case. What kind of situation is this a good model for?

(c) Suppose $\alpha_i = \alpha$ and $\beta_i = i\beta$ for $i \geq 0$. This is called an $M/M/\infty$ **queue**.
Under which conditions on α and β can the stationary distribution be normalized? Give a formula for π^*_k in that case. What kind of situation is this a good model for?

(d) Suppose $\alpha_i = i\alpha$, $\beta_i = i\beta$ for $i \geq 0$ and $X_0 = 1$.
Discuss qualitatively the behaviour of X_t as $t \to \infty$.
What kind of situation is this a good model for?

2.2 Consider the **contact process** $(\eta_t : t \geq 0)$ on the complete graph $\Lambda = \{1, \ldots, L\}$ (all sites connected) with state space $S = \{0,1\}^L$ and transition rates

$$c(\eta, \eta^x) = \eta(x) + \lambda\big(1 - \eta(x)\big) \sum_{y \neq x} \eta(y)\ ,$$

where $\eta, \eta^x \in S$ are connected states such that $\eta^x(y) = \begin{cases} 1 - \eta(x) & , y = x \\ \eta(y) & , y \neq x \end{cases}$,

(η with site x flipped).

(a) Let $N_t = \sum_{x \in \Lambda_L} \eta_t(x) \in \{0, \ldots, L\}$ be the number of infected sites at time t. Show that $(N_t : t \geq 0)$ is a Markov chain with state space $\{0, \ldots, L\}$ by computing the transition rates $c(n, m)$ for $n, m \in \{0, \ldots, L\}$.
Write down the master equation for the process $(N_t : t \geq 0)$.

(b) Is the process $(N_t : t \geq 0)$ irreducible, does it have absorbing states?
What are the stationary distributions?

(c) Assume that $\mathbb{E}\big(N_t^k\big) = \mathbb{E}(N_t)^k$ for all $k \geq 1$. This is called a **mean-field assumption**, meaning basically that we replace the random variable N_t by its expected value.
Use this assumption to derive the **mean-field rate equation** for $\rho(t) := \mathbb{E}(N_t)/L$,

$$\frac{d}{dt}\rho(t) = f\big(\rho(t)\big) = -\rho(t) + L\lambda\big(1 - \rho(t)\big)\rho(t)\ .$$

(d) Analyze this equation by finding the stable and unstable stationary points via $f(\rho^*) = 0$.
What is the prediction for the stationary density ρ^* depending on λ?

Simple sample codes for the following questions is on the course webpage.

2.3 The totally asymmetric simple exclusion process **(TASEP)** with open boundaries is an exclusion process on the one-dimensional lattice $\Lambda = \{1, \ldots, L\}$ with transition rates

$$10 \xrightarrow{1} 01 \quad \text{in the bulk, and} \quad |0 \xrightarrow{\rho_l} |1 \,, \quad 1| \xrightarrow{1-\rho_r} 0| \quad \text{at the boundaries} .$$

So particles jump one site to the right with rate 1 if possible and are injected and ejected at the boundary, where the system is coupled to reservoirs with densities $\rho_l, \rho_r \in [0, 1]$. The state space is $S = \{0, 1\}^L$ and we denote a particle configuration by $\eta = (\eta(x) : x \in \Lambda)$.

(a) Draw the initial occupation numbers $\eta(x)$ independently with $\eta(x) \sim Be(\rho_l)$ for $x < L/2$ and $\eta(x) \sim Be(\rho_r)$ for $x \geq L/2$. Then simulate the process using random sequential update, and record the configuration η in regular time intervals Δt up to time T. Visualize the time evolution (e.g. by using 'image' in MATLAB) for the following situations (three cases each)

$$\rho_l = 1,\ 0.8,\ 0.6 \quad \text{and} \quad \rho_r = 0 \qquad \text{(traffic light)}$$
$$\rho_l = 0.2, \quad \text{and} \quad \rho_r = 0.6,\ 0.8,\ 1 \qquad \text{(end of traffic jam)} .$$

Suggested parameter values are $L = 200$, $T = 400$, $\Delta t = 2$.
Interpret your findings in a few sentences, results get clearer if you averge 5 or 10 realizations.

(b) Initialize the system with $\eta(x) = 0$ for all $x \in \Lambda$ and measure the total density of particles $\rho(t) = \frac{1}{L} \sum_{x \in \Lambda} \eta_t(x)$ as a function of time for the parameter values

$$(\rho_l, \rho_r) = (0.2, 0.2),\ (0.8, 0.1) \quad \text{and} \quad (0.8, 0.8) .$$

Plot $\rho(t)$ for $t \leq T$ large enough to predict the limiting behaviour $\lim_{t\to\infty} \rho(t)$.
Interpret your findings in a few sentences, again results get clearer if you averge 5 or 10 realizations.

(c) Study the effect of a narrow road or a hill, by changing the jump rate in the bulk for $x \geq L/2$ from 1 to 0.8. Use $\rho_l = \rho_r = \rho$ and initialize $\eta(x) \sim Be(\rho)$ independently for all $x \in \Lambda$. Simulate the process for $\rho = 0.2,\ 0.4,\ 0.6,\ 0.8$ and visualize the profiles as in (a), for e.g. $L = 200$, $T = 400$ and $\Delta t = 2$. Interpret your findings in a few sentences.

2.4 Adapt your programme from Q2.3 to simulate a generalized TASEP, using now periodic boundary conditions on the lattice $\Lambda = \{1, \ldots, L\}$. The jump rates should depend on the neighbourhood configuration in the following way:

$$0100 \xrightarrow{1} 0010 \,, \quad 1101 \xrightarrow{\alpha} 1011 \,, \quad 0101 \xrightarrow{\beta} 0011 \,, \quad 1100 \xrightarrow{\gamma} 1010 .$$

For this model, the average stationary current is a function of the number N of particles, or the density $\rho = N/L$. It is defined by $j(\rho) = \mathbb{E}\big(c(\eta, \eta^{x,x+1})\big)$, where $c(\eta, \eta^{x,x+1})$ is the jump rate of a particle from x to $x + 1$ as given above.

(a) Making use of the ergodic theorem, measure the **fundamental diagram**, i.e. $j(\rho)$ as a function of the density. The easiest way is to just count all jumps up to a given time and normalize properly.
For fixed lattice size L (e.g. 500) vary the number of cars N to get j for $\rho = 0, 0.1, \ldots, 0.9, 1$.
Do this for $\alpha = \beta = \gamma = 1$ (usual TASEP) and at least two other choices of rates. Explain what your choices correspond to in terms of driver behaviour if you interpret this as a traffic model.

(b) Calculate the stationary current using a mean-field approximation for the parameters you chose in (a), and compare this to your measurement results in a plot. Discuss how the different shapes of the fundamental diagram are related to your choice of rates.