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CO905
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Stochastic Processes

Problem sheet 1

1.1 Let $A, B, C_1, \ldots, C_n \subseteq \Omega$ be events in a probability space Ω such that $C_i \cap C_j = \emptyset$ for all $i \neq j$ and $\bigcup_{i=1}^{n} C_i = \Omega$. Using the law of total probability, show that

$$\mathbb{P}(A|B) = \sum_{i=1}^{n} \mathbb{P}(A|B \cap C_i)\, \mathbb{P}(C_i|B) \,. \qquad [3]$$

1.2 Let X and Y be two discrete random variables taking values in $\mathbb{N}$, with joint probability mass function $f(x, y) = \mathbb{P}(X = x, Y = y)$.
Using conditional probabilities we write $\quad \mathbb{E}(Y|X = x) = \sum_{y \in \mathbb{N}} y\, \mathbb{P}(Y = y|X = x)$.

Show that $\quad \mathbb{E}(Y) = \sum_{x \in \mathbb{N}} \mathbb{E}(Y|X = x)\, \mathbb{P}(X = x)$. [3]

This formula is often written as $\mathbb{E}(Y) = \mathbb{E}\big(\mathbb{E}(Y|X)\big)$ and $\mathbb{E}(Y|X)$ is called the *conditional expectation*. Note that in this notation $\mathbb{E}(Y|X)$ is a random quantity, depending on the value taken by X.
For example, let $Y = \sum_{k=1}^{X} Z_i$, where the Z_i are iidrvs with expected value $\mu_Z = \mathbb{E}(Z_i)$. Denote by $\mu_X = \mathbb{E}(X)$ and compute $\mathbb{E}(Y)$ in terms of μ_Z and μ_X. [2]

1.3 A dice is rolled repeatedly. Which of the following are Markov chains?
For those that are, supply the transition matrix.

(a) The largest number X_n shown up to the nth roll.
(b) The number N_n of sixes in n rolls.
(c) At time n, the time B_n since the most recent six.
(d) At time n, the time C_n until the next six. [4]

1.4 Ehrenfest urn model of diffusion
A total of M balls is distributed over two urns. In each time step, one ball is chosen uniformly at random and is transferred from its urn to the other urn.
Let $X_n \in \{0, \ldots, M\}$ be the number of balls in the left urn after n time steps. Clearly, $(X_n : n \in \mathbb{N})$ is a finite state, irreducible Markov chain.

(a) What are the transition probabilities of this process?
(b) What is the stationary distribution $\boldsymbol{\pi}^*$? (Think about (c) before you answer.)
(c) Is the process reversible with respect to $\boldsymbol{\pi}^*$?
(d) Show that $\quad \mathbb{E}\Big(X_n - \frac{M}{2}\,\Big|\, X_{n-1}\Big) = \Big(X_{n-1} - \frac{M}{2}\Big)\Big(1 - \frac{2}{M}\Big)$,

and deduce that for all initial conditions $X_0 = i$ [8]

$$\mathbb{E}\Big(X_n - \frac{M}{2}\Big) = \Big(i - \frac{M}{2}\Big)\Big(1 - \frac{2}{M}\Big)^n \to 0 \quad \text{as } n \to \infty \,.$$