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CO905
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Stochastic Processes

Problem sheet 2

2.1 Consider the continuous-time Markov chain X with generator $G = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{pmatrix}$.

(a) Draw a diagram for X (i.e. connect the three states by their jump rates), and give the transition matrix P^Y of the corresponding jump chain Y.

(b) Show that for some matrix B,

$$P(t) = \exp(tG) = B^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-4t} \end{pmatrix} B \, .$$

(You may assume the result (1.20) in the notes.)

(c) Convince yourself that $P'(0) = G$, $P''(0) = G^2$ etc., and use this and the result of (b) to compute $p_{11}(t)$.

(d) What is the stationary distribution $\boldsymbol{\pi}^*$ of X? [8]

2.2 Each bacterium in a colony splits into two identical bacteria after an exponential time of parameter λ, the two then split independently in the same way, and so on. Let X_t be the number of bacteria at time t with $X_0 = 1$.
The chain $X = (X_t : t \geq 0)$ is also called a **simple birth process**.

(a) Give the generator of X and draw a diagram.

(b) Is X irreducible? Is X recurrent? Does it have a stationary distribution? (Justify your answers.)

(c) Let $\mu(t) = \mathbb{E}(X_t)$ and J_1 be the time of the first split.
Convince yourselve that $E(X_t | J_1 = s) = 2\mu(t-s)$. Then use

$$\mu(t) = \mathbb{E}(X_t | J_1 > t)\, \mathbb{P}(J_1 > t) + \mathbb{E}(X_t | J_1 \leq t)\, \mathbb{P}(J_1 \leq t)$$

(this is a socalled *renewal argument*) to show that

$$\mu(t) = e^{-\lambda t} + \int_0^t 2\lambda e^{-\lambda s} \mu(t-s)\, ds \, .$$

Substitute $u = t - s$ in the integral and show that $\mu'(t) = \lambda\, \mu(t)$.
Calculate $\mu(t)$. [6]

*(d)** Let $\phi(t,z) = \mathbb{E}(z^{X_t})$ be the probability generating function of X_t. Show with a similar argument as in (c) that

$$\frac{\partial}{\partial t} \phi(t,z) = \lambda \phi(t,z) \big(\phi(t,z) - 1 \big) \quad \text{with} \quad \phi(0,z) = z \, ,$$

and deduce that $\mathbb{P}(X_t = n) = \dfrac{\partial^n}{\partial z^n} \phi(t,z)\big|_{z=0} = e^{-\lambda t} \big(1 - e^{-\lambda t}\big)^{n-1}$.

2.3 Birth-death processes

A birth-death process X is a generalized $M/M/1$ queue with state-dependent rates, i.e. a continuous-time Markov chain with state space $S = \mathbb{N}$ and jump rates

$$i \xrightarrow{\alpha_i} i+1 \quad \text{for all } i \in S \ , \quad i \xrightarrow{\beta_i} i-1 \quad \text{for all } i \geq 1 \ .$$

(a) Write down the generator G. Under which conditions is X irreducible?
Using detailed balance, find a formula for the stationary probablities π_k^* in terms of π_0^*.

(b) Suppose $\alpha_i = \alpha$ and $\beta_i = i\beta$, so customers arrive at rate α and each of them is served separately with rate β. This is called an $M/M/\infty$ queue.
Is X positive recurrent? If yes compute the stationary distribution.

(c) Let X_t be the size of a population at time t, with $X_0 = 1$. Each member independently gets a child with rate α and dies with rate β, except if $X_t = 1$, the last remaining member does not die and the state space is $S = \{1, 2, \ldots\}$.
Determine the generator of X. Is X irreducible? Under which condition on α and β does X have a stationary distribution? [6]

2.4 Simulation of the Moran model

We consider the Moran model in continuous time: In a population of size N each individuum can be of type A or B. At rate 1, each individuum gets an identical child. When this happens, one of the now $N+1$ individuums is chosen uniformly at random and dies instantaneously, to keep the population size constant to N.

Let X_t be the number of type A individuals at time t. Then $X = (X_t : t \geq 0)$ is a continuous-time Markov chain with two absorbing states, 0 and N.

(a) What is the generator of X and what is the transition matrix for the jump chain Y?
What are the stationary distributions of X?

(b) Let $h_i^0 = \inf\{t \geq 0 : X_t = 0,\ X_0 = i\}$ be the absorption time at 0 and h_i^N the one at N, respectively. Then starting with $X_0 = i$, the chain stops at time $\bar{h}_i = \min\{h_i^0, h_i^N\}$ and A dies out if and only if $h_i^0 < h_i^N$.
Simulate X for $N = 1024$, measure and plot

$$\mathbb{P}(B \text{ dies out}) \quad \text{and} \quad \mathbb{E}(h_i^N | B \text{ dies out}) \quad \text{as a function of } X_0 = i \ .$$

(c) Measure and plot $\mathbb{E}(\bar{h}_{N/2})$ for $N = 100, 200, \ldots, 1000$. How does it grow with N?
For $N = 512$, measure the distribution function of the normalized absorption time $\bar{h}_{N/2}/\mathbb{E}(\bar{h}_{N/2})$ using quantiles and compare with e.g. the exponential distribution.

(d) Introduce a bias in the model (**selection**): Type A individuals give birth with rate $\alpha \geq 1$ and type B individuals with rate 1. Do the same measurement as in (b) for $\alpha = 1 + 1/N$ and $\alpha = 1 + 10/N$.

(e) Introduce **mutation**: Each born individuum instantaneously changes its species with probability ϵ. What happens to the stationary distributions $\boldsymbol{\pi}^*$ of X?
For $N = 1024$ measure $\boldsymbol{\pi}^*$ and plot a histogram with appropriate spacing. Do this for several values of ϵ, e.g. $0.0001, 0.001, 0.01$ and 0.1 and for $\alpha = 1$ and $1 + 1/N$.
Or better: Solve the detailed balance recursions (e.g. in MATLAB) and look at $\boldsymbol{\pi}^*$ to find interesting parameter values. Then do the simulations and compare with $\boldsymbol{\pi}^*$.

[20]

These are only suggestions of what can be done. Use your own wisdom to study this model!