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CO905
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Stochastic Processes

Problem sheet 3 – Part 2

3.4 Let $X = (X_n : n \in \mathbb{N})$ be a stochastic process. Prove or disprove:

(a) X is a martingale $\Rightarrow$ X is a Markov process.

(b) X is a Markov process $\Rightarrow$ X is a martingale.

[2]

3.5 Let $X \sim PP(\lambda)$ be a Poisson process with intensity $\lambda > 0$.

(a) Is X a martingale? (Justify your answer.)

(b) Find two nontrivial martingales associated to X and show that they fulfill the definition.

[3]

3.6 Let $X = (X_n : n \in \mathbb{N})$ be a simple symmetric random walk on $\mathbb{Z}$ starting in the origin.
Define the stopping time $T = \inf\{n \in \mathbb{N} : X_n^2 \geq N^2\}$ for some $N \in \{1, 2, \ldots\}$.
Use the optional stopping theorem for an appropriate martingale associated to X to find $\mathbb{E}(T)$.
[4]

3.7 Let $B = (B_t : t \geq 0)$ be a standard Brownian motion in $\mathbb{R}$.

(a) Define $B' = (B'_t : t \geq 0)$ by $B'_t = \begin{cases} t\, B_{1/t} & , t > 0 \\ 0 & , t = 0 \end{cases}$.
Show that B' is a standard Brownian motion.
(Hint: Show that it is a Gaussian process with the right covariances.)

(b) Define $S = \sup\{t \leq 1 : B_t = 0\}$ and $T = \inf\{t \geq 1 : B_t = 0\}$.
Are these random times stopping times? Justify your answers.
Show that S has the same distribution as T^{-1}.

[5]

3.8 Brownian bridge
Let $B = (B_t : 0 \leq t \leq 1)$ be a standard Brownian motion in $\mathbb{R}$. For each $y \in \mathbb{R}$ we define

$$Z^y = (Z_t^y : 0 \leq t \leq 1)\ , \quad Z_t^y := y\, t + B_t - t\, B_1\ .$$

(a) Compute $\mathbb{E}(Z_t^y)$ and $\mathrm{Var}(Z_t^y)$. (Note that B_t and B_1 are not independent!)

(b) What is the pdf $f_{Z^y}(t, x)$ for the process Z^y? (Justify your answer.)

(c) Depending on y, can Z^y be a martingale with respect to B?

(d) Why is Z^y called a *Brownian bridge*?

[6]