Stochastic Processes

Problem sheet 1

1.1 Let $A, B, C_1, \ldots, C_n \subseteq \Omega$ be events in a probability space Ω such that $C_i \cap C_j = \emptyset$ for all $i \neq j$ and $\bigcup_{i=1}^{n} C_i = \Omega$. Using the law of total probability, show that

$$\mathbb{P}(A|B) = \sum_{i=1}^{n} \mathbb{P}(A|B \cap C_i)\, \mathbb{P}(C_i|B) \ . \qquad [2]$$

1.2 A dice is rolled repeatedly. Which of the following are Markov chains?
For those that are, supply the state space and the transition matrix.

(a) The largest number X_n shown up to the nth roll.

(b) The number N_n of sixes in n rolls.

(c) At time n, the time B_n since the most recent six. [3]

(d)* At time n, the time C_n until the next six.

1.3 Let $Z = (Z_n : n \in \mathbb{N})$ be a branching process, defined recursively by

$$Z_0 = 1\ , \quad Z_{n+1} = X_1^n + \ldots + X_{Z_n}^n \quad \text{for all } n \geq 0\ ,$$

where the $X_i^n \in \mathbb{N}$ are iidrv's denoting the offspring of individuum i in generation n.

(a) Consider a geometric offspring distribution $X_i^n \sim Geo(p)$, i.e.

$$p_k = \mathbb{P}(X_i^n = k) = (1-p)\, p^k\ , \quad p \in (0,1)\ .$$

Compute the probability generating function $G(s) = \sum_k p_k s^k$ as well as $\mathbb{E}(X_i^n)$ and $Var(X_i^n)$.
Sketch $G(s)$ for (at least) three (wisely chosen) values of p and compute the probability of extinction as a function of p.

(b) Consider a Poisson offspring distribution $X_i^n \sim Poi(\lambda)$, i.e.

$$p_k = \mathbb{P}(X_i^n = k) = \frac{\lambda^k}{k!}\, e^{-\lambda}\ , \quad \lambda > 0\ .$$

Repeat the same analysis as in (a). [5]

(c)* For geometric offspring with $p = 1/2$, show that $G_n(s) = \frac{n-(n-1)s}{n+1-ns}$ and compute $\mathbb{P}(Z_n = 0)$. If T is the (random) time of extinction, what is its distribution and its expected value?