Stochastic Processes

Problem sheet 2

2.1 Consider the continuous-time Markov chain X with generator $G = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -4 & 3 \\ 0 & 1 & -1 \end{pmatrix}$.

(a) Draw a diagram for X (i.e. connect the three states by their jump rates), and give the transition matrix P^Y of the corresponding jump chain Y.

(b) Using the same strategy as in the proof of Theorem 1.7, show that for some matrix B,

$$P(t) = \exp(tG) = B^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-5t} \end{pmatrix} B\ .$$

(c) Convince yourself that $P'(0) = G$, $P''(0) = G^2$ etc., and use this and the result of (b) to compute $p_{11}(t)$, i.e. determine the coefficients in

$$p_{11}(t) = a + b\, e^{-2t} + c\, e^{-5t}\ .$$

(d) What is the stationary distribution $\boldsymbol{\pi}^*$ of X? [5]

2.2 Each bacterium in a colony splits into two identical bacteria after an exponential time of parameter λ, the two then split independently in the same way, and so on. Let X_t be the number of bacteria at time t with $X_0 = 1$.
The chain $X = (X_t : t \geq 0)$ is also called a **simple birth process**.

(a) Give the generator of X and draw a diagram.

(b) Is X irreducible? Is X recurrent? Does it have a stationary distribution?
(Justify your answers concisely.)

(c) Let $\mu(t) = \mathbb{E}(X_t)$ and J_1 be the time of the first split.
Convince yourselve that $E(X_t|J_1 = s) = 2\mu(t-s)$. Then use

$$\mu(t) = \mathbb{E}(X_t|J_1 > t)\,\mathbb{P}(J_1 > t) + \mathbb{E}(X_t|J_1 \leq t)\,\mathbb{P}(J_1 \leq t)$$

(this is a socalled *renewal argument*) to show that

$$\mu(t) = e^{-\lambda t} + \int_0^t 2\lambda e^{-\lambda s} \mu(t-s)\, ds\ .$$

Substitute $u = t - s$ in the integral and show that $\mu'(t) = \lambda\, \mu(t)$.
Calculate $\mu(t)$. [4]

*(d)** Let $\phi(t,z) = \mathbb{E}(z^{X_t})$ be the probability generating function of X_t. Show with a similar argument as in (c) that

$$\frac{\partial}{\partial t}\phi(t,z) = \lambda\phi(t,z)\big(\phi(t,z) - 1\big) \quad \text{with} \quad \phi(0,z) = z\ ,$$

and deduce that $\mathbb{P}(X_t = n) = \dfrac{\partial^n}{\partial z^n}\phi(t,z)\big|_{z=0} = e^{-\lambda t}\big(1 - e^{-\lambda t}\big)^{n-1}$.

2.3 Birth-death processes

A birth-death process X is a generalized $M/M/1$ queue with state-dependent rates, i.e. a continuous-time Markov chain with state space $S = \mathbb{N}$ and jump rates

$$i \xrightarrow{\alpha_i} i+1 \quad \text{for all } i \in S\ , \quad i \xrightarrow{\beta_i} i-1 \quad \text{for all } i \geq 1\ .$$

(a) Write down the generator G. Under which conditions is X irreducible?
Using detailed balance, find a formula for the stationary probablities π_k^* in terms of π_0^*.

(b) Suppose $\alpha_i = \alpha$ and $\beta_i = i\beta$. This is called an $M/M/\infty$ queue.
Is X positive recurrent? If yes compute the stationary distribution.
What kind of situation is this a good model for?

(c) Suppose $\alpha_i = i\alpha$, $\beta_i = i\beta$ and $X_0 = 1$.
Discuss qualitatively the behaviour of X_t as $t \to \infty$.
What kind of situation is this a good model for? [5]

2.4 Maximum Entropy

The entropy S assigns a number $S(\pi)$ to a probability distribution, interpreted as the '*uncertainty*' (degree randomness) of π. Let f be the PDF for a continuous random variable X, then the entropy is given by

$$S(f) := -\int_D f(x) \ln f(x)\, dx\ , \quad \text{where } X \text{ is taking values in } D \subseteq \mathbb{R}\ .$$

We want to maximize the entropy over PDFs f, i.e. find the PDF with maximal uncertainty. We 'Taylor-expand' S around some PDF f using the variational formula

$$S(f + \epsilon f_1) = S(f) + \epsilon \frac{\delta S(f)}{\delta f}(f_1) + o(\epsilon)$$

for some fixed function $f_1 : D \to \mathbb{R}$ and $\epsilon \searrow 0$. The variational derivative is just given by the partial derivative (fixing everything except ϵ)

$$\frac{\delta S(f)}{\delta f}(f_1) := \frac{\partial}{\partial \epsilon} S(f + \epsilon f_1)\Big|_{\epsilon=0}\ .$$

(a) Show that

$$\frac{\delta S(f)}{\delta f}(f_1) = -\int_D f_1(x)\big(1 + \ln f(x)\big) dx$$

(b) Let $D = [0, 1]$. Use the method of Lagrange multipliers to maximize $S(f)$ where variations of f should fulfill the constraint

$$\int_D \big(f(x) + \epsilon f_1(x)\big) dx = 1\ .$$

What is the resulting PDF of maximal entropy? Why do we need the constraint?

(c) Now let $D = [0, \infty]$ and maximize S under the additional constraint

$$\int_D x\big(f(x) + \epsilon f_1(x)\big) dx = \mu\ .$$

So we maximize among PDFs with a given mean $\mu > 0$. What is the resulting PDF?

(d) Now let $D = \mathbb{R}$ and maximze S among distributions with given mean $\mu = 0$ and variance σ^2. What is the resulting PDF? [6]