Stochastic Processes

Problem sheet 3

3.1 Let $\mathbf{B}$ be a standard Brownian motion in $\mathbb{R}^d$. Show the following:

(a) Scaling property:
If $\lambda > 0$, then $\mathbf{B}_\lambda = \big(\lambda^{-1/2} B_{\lambda t} : t \geq 0\big)$ is a standard Brownian motion in $\mathbb{R}^d$.

(b) Orthogonal transformations:
If $U \in O(d)$ is an *orthogonal* $d \times d$ matrix (i.e. $U^{-1} = U^T$), then $U\,\mathbf{B} = \big(U\,\mathbf{B}_t : t \geq 0\big)$ is a standard Brownian motion. In particular $-\mathbf{B}$ is a standard Brownian motion.

(c) For $d = 1$ define $B' = (B'_t : t \geq 0)$ by $B'_t = \begin{cases} t\, B_{1/t} & , t > 0 \\ 0 & , t = 0 \end{cases}$,
then B' is a standard Brownian motion.
(Hint: Show that the joint distributions are Gaussian with the right covariances.)

[5]

3.2 Consider a continuous time traffic model $(\boldsymbol{\eta}_t : t \geq 0)$ given by jump rates $c(x, x+1, \boldsymbol{\eta})$ on the lattice $\Lambda_L = \{1, \ldots, L\}$ with periodic boundary conditions and state space $S_L = \{0,1\}^L$.

(a) For the average density and current

$$\rho(x,t) := \mathbb{E}\big(\eta_t(x)\big) \quad \text{and} \quad j(x,t) := \mathbb{E}\big(c(x, x+1, \boldsymbol{\eta})\big)$$

derive the **lattice continuity equation**

$$\frac{\partial}{\partial t}\rho(x,t) + \nabla_x j(x,t) = 0 \quad \text{where} \quad \nabla_x j(x,t) = j(x,t) - j(x-1,t) \qquad (1)$$

Hint: Write $\eta_{t+\Delta t}(x)$ in terms of $\eta_t(x)$ and the total currents $J_{x-1,x}(t, t+\Delta t)$ and $J_{x,x+1}(t, t+\Delta t)$, then send $\Delta t \to 0$.

(b) Rescale space $y = x/L$ by the size of the lattice. Derive a continuous continuity equation (also called **conservation law**)

$$\frac{\partial}{\partial s}\tilde\rho(y,s) + \frac{\partial}{\partial y}\tilde j(y,s) = 0$$

as a scaling limit of the lattice equation (1) as $L \to \infty$. What is the appropriate time scaling $s = t/L^\delta$ to arrive at this equation?
Note that we used the notation $\rho(x,t) = \tilde\rho(x/L, t/L^\delta)$ and analogous for j. Plug this into (1) and use a Taylor expansion as $L \to \infty$.

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3.3 Simulation of a traffic model

Simulate your favorite continuous-time traffic model on a one-dimensional lattice $\Lambda = \{1, \dots L\}$. You can use any software to do this, please attach a printout of your code.
As an example you can use

$$0100 \xrightarrow{1} 0010\ , \quad 1101 \xrightarrow{\alpha} 1011\ , \quad 0101 \xrightarrow{\beta} 0011\ , \quad 1100 \xrightarrow{\gamma} 1010\ . \tag{2}$$

You can use any other model, but please always specify your parameters and give all the relevant information.

(a) For periodic boundary conditions, measure the *fundamental diagram*, i.e. the stationary average current $j(\rho)$ as a function of the density $\rho = N/L$. For fixed lattice size L (e.g. 500) vary the number of cars N to get j for $\rho = 0, 0.1, \dots, 0.9, 1$. Do this for at least two different choices of rates.
(If you use model (2) you can check your program: for $\alpha = \beta = \gamma = 1$ you should have $j(\rho) = \rho(1-\rho)$.)

(b) Calculate the stationary current using a mean-field approximation for the parameters you chose in (a), and compare this to your measurement results in a plot.

(c) Do the same measurement as in (a) (can be done simultaneously) for the average stationary speed of a particle $v(\rho)$ and present your results in a plot.
How is this related to $j(\rho)$?

(d) Alter your programme to do **one** of the following or any other interesting study you might think of (for a fixed set of bulk rates of your choice):

- Introduce slow drivers or different kinds of drivers and measure how this affects the average currents and velocities j and v.
- Create stop & go waves by choosing appropriate rates and visualize them in a plot.
- Measure the statistics of jump intervals: Record the times between jumps over a bond $(x, x+1)$ and also times between jumps of a particle. Use the 'quantile' function in Matlab to plot the tail distribution
  $$\bar{F}(x) = \text{fraction of measurements above } x\ .$$
  What are the distributions, are they exponential?
- Study the system with open boundaries, i.e. particle input at the left with rate δ_1 and output at the right with rate δ_2. Do e.g. 4 measurements of the stationary density ρ and the current j with each of the two rates δ_1 and δ_2 being above and below all the bulk rates. In which cases do you see traffic jams, in which cases free flow?

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