Stochastic Processes

Problem sheet 3

3.1 Consider the contact process $(\eta_t : t \geq 0)$ on the complete graph $\Lambda = \{1, \ldots, L\}$ (all sites connected) with state space $S = \{0,1\}^L$ and transition rates

$$c(\eta, \eta^x) = \eta(x) + \lambda\big(1 - \eta(x)\big) \sum_{y \neq x} \eta(y) ,$$

where $\eta, \eta^x \in S$ are connected states such that $\eta^x(y) = \left\{ \begin{array}{cl} 1 - \eta(x) & , y = x \\ \eta(y) & , y \neq x \end{array} \right. ,$

(η with site x flipped).

(a) Let $N(t) = \sum_{x \in \Lambda_L} \eta_t(x) \in \{0, \ldots, L\}$ be the number of infected sites at time t. Show that $(N(t) : t \geq 0)$ is a Markov chain with state space $\{0, \ldots, L\}$ by computing the transition rates $c(n, m)$ for $n, m \in \{0, \ldots, L\}$.
Write down the master equation for the process.

(b) Is the process $(N(t) : t \geq 0)$ irreducible, does it have absorbing states?
What are the stationary distributions?

(c) Assume that $\mathbb{E}\big(N(t)^k\big) = \mathbb{E}(N(t))^k$ for all $k \geq 1$. This is called a *mean-field assumption*, meaning basically that we replace the random variable $N(t)$ by its expected value.
Use this assumption and the master equation to derive the *mean-field rate equation* for $\rho(t) := \mathbb{E}(N(t))/L$,

$$\frac{d}{dt}\,\rho(t) = f\big(\rho(t)\big) = -\rho(t) + L\lambda\big(1 - \rho(t)\big)\rho(t) .$$

(d) Analyze this equation by finding the stable and unstable stationary points via $f(\rho^*) = 0$.
What is the prediction for the stationary density ρ^* depending on λ?

[11]

3.2 Consider the contact process on the lattice $\Lambda_L = \{1, \ldots, L\}$ with connections only between nearest neighbours and periodic boundary conditions as in Q2.2.
Simulate the process for $L = 200$ with initial condition $\eta(x) = 1$ for all $x \in \Lambda$ and several values of $\lambda \in [1.6, 1.7]$. After an equilibration time $\tau_{equ} = 10^3$, use the ergodic theorem to sample from the stationary distribution of the observable $N(t) = \sum_{x \in \Lambda_L} \eta_t(x)$, i.e. over a time interval of length $\tau_{meas} = 10^3$ count the fraction of time $N(t)$ spent in n for each $n \in \{0, \ldots, L\}$. Average this measurement over 100 realizations and plot your estimate of the stationary distribution for all values of λ in a single plot.
Explain the form of the observed curves.

[10]

3.3 Let B be a standard Brownian motion in $\mathbb{R}$. Show the following:

(a) Scaling property:
If $\lambda > 0$, then $B^\lambda := \big(\lambda^{-1/2} B_{\lambda t} : t \geq 0\big)$ is a standard Brownian motion.

(b) Define $B' = (B'_t : t \geq 0)$ by $B'_t = \begin{cases} t\, B_{1/t} & , t > 0 \\ 0 & , t = 0 \end{cases}$,
then B' is a standard Brownian motion.
Hint: Use that a process $(X_t : t \geq 0)$ is a standard BM iff it has continuous paths, $X_t \sim N(0,t)$ and the right covariances, i.e. $\mathbb{E}(X_t X_s) = \min\{s,t\}$.

[4]